Discrete-Time Fourier Magnitude and Phase

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Complex Nature of $X(\omega)$

Recall, Fourier Transform:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \in \mathbb{C}$$

and Inverse Fourier Transform:

$$\begin{aligned} x(n) &= \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{0} X(\omega)e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{0}^{\pi} X(\omega)e^{j\omega n}d\omega \end{aligned}$$

Note: If $x(n)$ is real, then the imaginary part of the negative frequency sinusoids (i.e., $e^{j\omega n}$ for $\omega<0$) cancel out the imaginary part of the positive frequency sinusoids (i.e., $e^{j\omega n}$ for $\omega>0$)

Complex Nature of $X(\omega)$

- Rectangular coordinates: rarely used in signal processing

$$X(\omega) = X_R(\omega) + j\, X_I(\omega)$$

where $X_R(\omega), X_I(\omega) \in \mathbb{R}$.

- Polar coordinates: more intuitive way to represent frequency content

$$X(\omega) = |X(\omega)|\; e^{j\Theta(\omega)}$$

where $|X(\omega)|, \Theta(\omega) = \angle X(\omega) \in \mathbb{R}$.

Magnitude and Phase of $X(\omega)$

- $|X(\omega)|$: determines the relative presence of a sinusoid $e^{j\omega n}$ in $x(n)$
- $\Theta(\omega) = \angle X(\omega)$: determines how the sinusoids line up relative to one another to form $x(n)$

Magnitude and Phase of $X(\omega)$

$$\begin{aligned} x(n) &= \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{2\pi} |X(\omega)|\ e^{j\Theta(\omega)}e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{2\pi} |X(\omega)|\ e^{j(\omega n+\Theta(\omega))}d\omega \end{aligned}$$

- Recall, $e^{j(\omega n+\Theta(\omega))} = \cos(\omega n + \Theta(\omega)) + j\sin(\omega n + \Theta(\omega))$.
- The larger $|X(\omega)|$ is, the more prominent $e^{j\omega n}$ is in forming $x(n)$.
- $\Theta(\omega) = \angle X(\omega)$ determines the relative phases of the sinusoids (i.e. how they line up with respect to one another).

Magnitude versus Phase

Q: Which is more important for a given signal?

- Does one component (magnitude or phase) contain more information than another?
- When filtering, if we had to preserve on component (magnitude or phase) more, which one would it be?

Example: audio information signal

- An audio signal is represented by a real function $x(n)$.
- The function $x(-n)$ represents playing the audio signal backwards.
- Since $x(n)$ is real:

$$\begin{aligned} X(\omega) &= X^*(-\omega) \\ |X(\omega)| &= |X^*(-\omega)| = |X(-\omega)| \quad \text{since } |c| = |c^*| \text{ for } c \in \mathbb{C} \end{aligned}$$

- Therefore,

$$|X(\omega)| = |X(-\omega)|$$

That is, the FT magnitude is even for $x(n)$ real.

Example: audio information signal

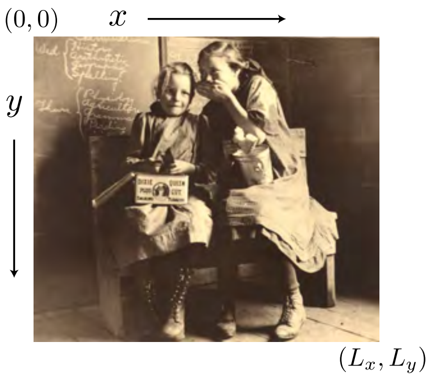
- Recall, $x(n) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega) \qquad x(-n) \stackrel{\mathcal{F}}{\longleftrightarrow} X(-\omega)$
- Therefore,

$$\underbrace{|X(\omega)|}_{\text{spectrum magnitude of } x(n)} = \overbrace{|X(-\omega)|}^{\text{spectrum magnitude of } x(-n)}$$

Therefore, the magnitude of the FT of an audio signal played forward and backward is the same!

Example: grayscale still images

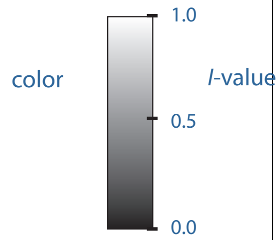
- A still image can be considered a two-dimensional signal: $x(n_1, n_2)$ where n_1 represents the horizontal dimension and n_2 represents the vertical dimension.



The image shown is "Dixie Queens" (two schoolgirls at lunch from Hadleyville, Oregon, circa 1911), Roy C. Andrews collection, PH003-P954, Special Collections and University Archives, University of Oregon, Eugene, Oregon 97403-1299.

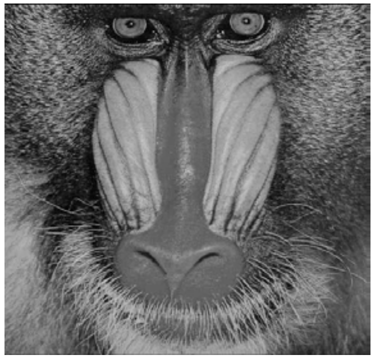
Intensity Images

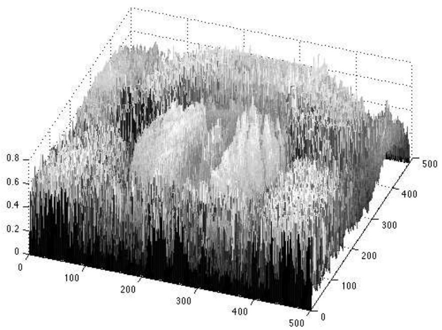
- discrete-space and continuous-amplitude image consisting of intensity (grayscale) values
- $x(n_1, n_2)$ is a two-dimensional signal representing the grayscale value at location (n_1, n_2) where:
  - $0 \leq n_1 \leq N_1$ and $0 \leq n_2 \leq N_2$
  - $x(n_1, n_2) = 0$ represents black
  - $x(n_1, n_2) = 1$ represents white
  - $0 < x(n_1, n_2) < 1$ represents proportional gray-value



Analog Intensity Images

- $x(n_1, n_2)$ can be displayed as an intensity image or as a mesh graph





Example: grayscale still images

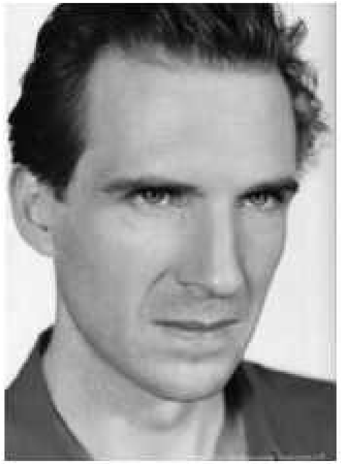
- The Fourier transform $x(n_1, n_2)$ has two frequency variables: ω_1 and ω_2 and is given by:

$$X(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2) e^{-j(\omega_1 n_1 + \omega_2 n_2)} \quad \in \quad \mathbb{C}$$

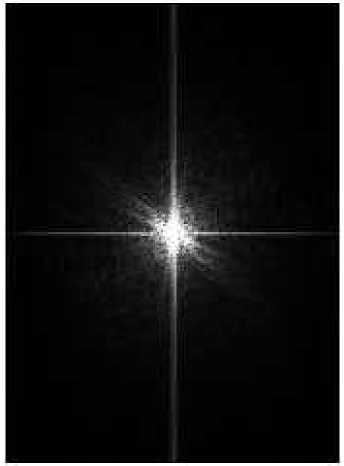
- Typically, we consider the magntiude and phase of $X(\omega_1, \omega_2)$:

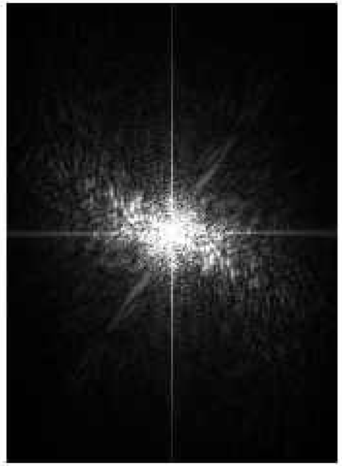
$$|X(\omega_1, \omega_2)| \quad \text{and} \quad \angle X(\omega_1, \omega_2)$$

Example: $x(n_1, n_2)$

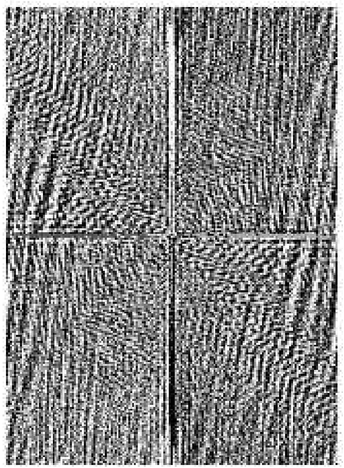


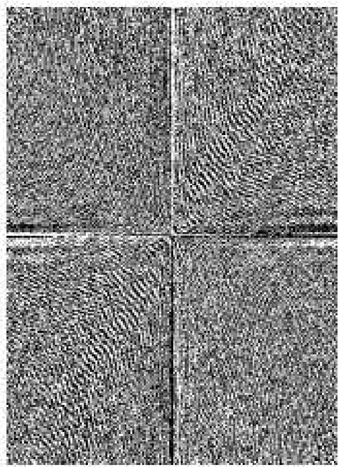
Example: $|X(\omega_1, \omega_2)|$

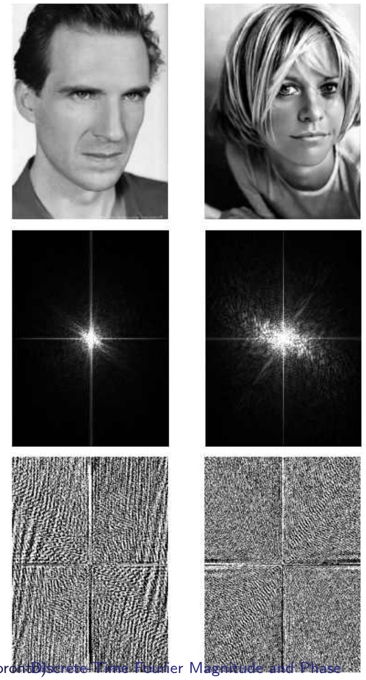




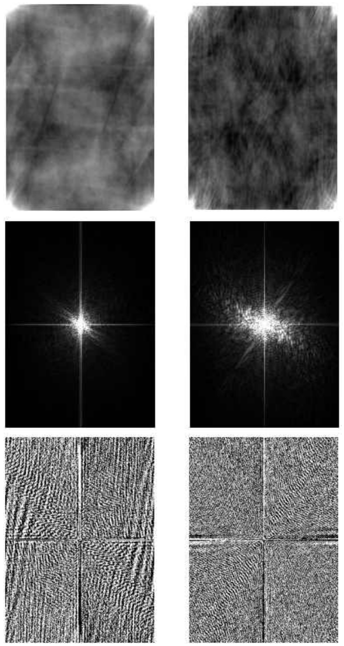
Example: $\Theta(\omega_1, \omega_2) = \angle X(\omega_1, \omega_2)$



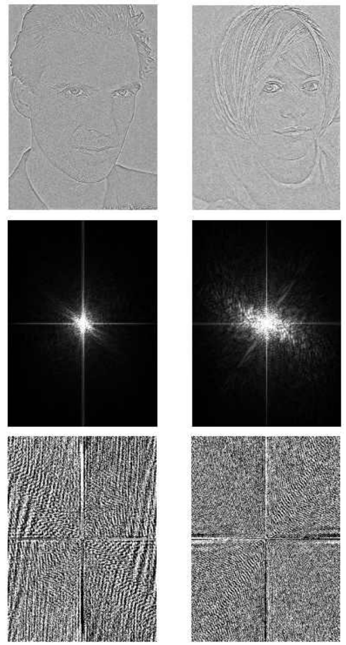




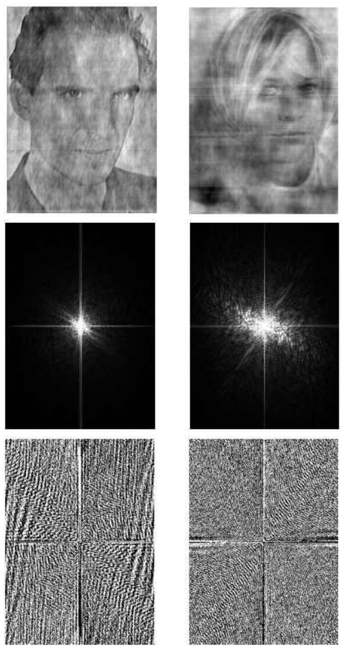
Reconstruction using
magnitude only
Top Left Photo: Ralph's magnitude is the same, Phase = 0
Top Right Photo: Meg's magnitude is the same, Phase = 0



Reconstruction using
phase only
Top Left Photo: Ralph's magnitude normalized to one, Phase is the same
Top Right Photo: Meg's magnitude normalized to one, Phase is the same



Reconstruction swapping
magnitude and phase
of the images.
Top Left Photo: Ralph's phase + Meg's magnitude
Top Right Photo: Meg's phase + Ralph's magnitude



Magnitude versus Phase

Q: Which is more important for a given signal? **A:** Phase.

- Does one component (magnitude or phase) contain more information than another? **A:** Yes, typically phase.
- When filtering, if we had to preserve on component (magnitude or phase) more, which one would it be?
  A: It is important to preserve phase during filtering.

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