Discrete-Time Sinusoids

Professor Deepa Kundur

University of Toronto

Periodicity

Recall if a signal $x(t)$ is periodic, then there exists a $T > 0$ such that

$$x(t) = x(t + T)$$

If no $T > 0$ can be found, then $x(t)$ is non-periodic.

Continuous-time Sinusoids

To find the period $T > 0$ of a general continuous-time sinusoid $x(t) = A\cos(\Omega t + \phi)$:

$$\begin{aligned} x(t) &= x(t+T) \\ A\cos(\Omega t + \phi) &= A\cos(\Omega(t+T) + \phi) \\ A\cos(\Omega t + \phi + 2\pi k) &= A\cos(\Omega t + \phi + \Omega T) \\ \therefore \quad 2\pi k &= \Omega T \\ T &= \frac{2\pi k}{\Omega} \end{aligned}$$

where $k \in \mathbb{Z}$. Note: when k is the same sign as Ω, $T > 0$.

Therefore, there exists a $T > 0$ such that $x(t) = x(t + T)$ and therefore $x(t)$ is $\boxed{\text{periodic}}$.

Discrete-time Sinusoids

To find the integer period $N > 0$ (i.e., ($N \in \mathbb{Z}^+$) of a general discrete-time sinusoid $x[n] = A\cos(\omega n + \phi)$:

$$\begin{aligned} x[n] &= x[n+N] \\ A\cos(\omega n + \phi) &= A\cos(\omega(n+N) + \phi) \\ A\cos(\omega n + \phi + 2\pi k) &= A\cos(\omega n + \phi + \omega N) \\ \therefore \quad 2\pi k &= \omega N \\ N &= \frac{2\pi k}{\omega} \end{aligned}$$

where $k \in \mathbb{Z}$.

Note: there may not exist a $k \in \mathbb{Z}$ such that $\frac{2\pi k}{\omega}$ is an integer.

Discrete-time Sinusoids

Example i: $\omega = \frac{37}{11}\pi$

$$N = \frac{2\pi k}{\omega} = \frac{2\pi k}{\frac{37}{11}\pi} = \frac{22}{37}k$$

$$N_0 = \frac{22}{37}k = \boxed{22} \text{ for } k = 37;\ x[n] \text{ is periodic.}$$

Example ii: $\omega = 2$

$$N = \frac{2\pi k}{\omega} = \frac{2\pi k}{2} = \pi k$$

$$N \in \mathbb{Z}^+ \text{ does not exist for any } k \in \mathbb{Z};\ x[n] \text{ is non-periodic.}$$

Example iii: $\omega = \sqrt{2}\pi$

$$N = \frac{2\pi k}{\omega} = \frac{2\pi k}{\sqrt{2}\pi} = \sqrt{2}k$$

$$N \in \mathbb{Z}^+ \text{ does not exist for any } k \in \mathbb{Z};\ x[n] \text{ is not periodic.}$$

Discrete-time Sinusoids

$$N = \frac{2\pi k}{\omega}$$

$$\omega = \frac{2\pi k}{N} = 2\pi\frac{k}{N} = \pi \cdot \underbrace{\boxed{\frac{2k}{N}}}_{\text{RATIONAL}}$$

Therefore, a discrete-time sinusoid is periodic if its radian frequency ω is a rational multiple of π.

Otherwise, the discrete-time sinusoid is non-periodic.

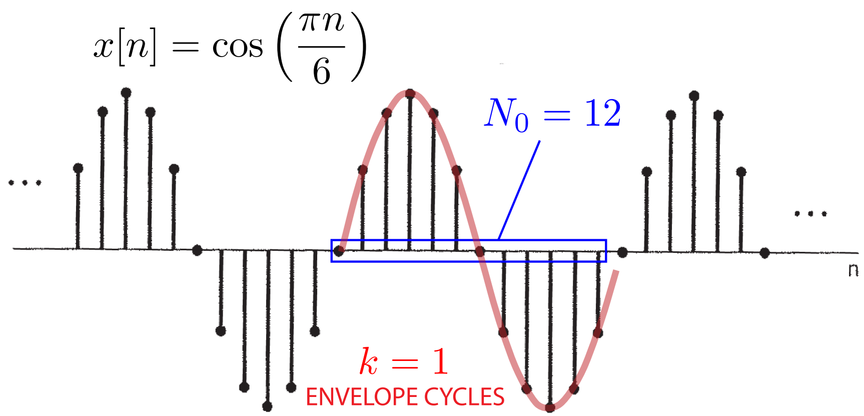
Example 1: $\omega = \pi/6 = \pi \cdot \boxed{\frac{1}{6}}$

$$x[n] = \cos\left(\frac{\pi n}{6}\right)$$

$$N = \frac{2\pi k}{\omega} = \frac{2\pi k}{\pi\frac{1}{6}} = 12k$$

$$N_0 = 12 \text{ for } k = 1$$

The fundamental period is 12 which corresponds to $k = 1$ envelope cycles.

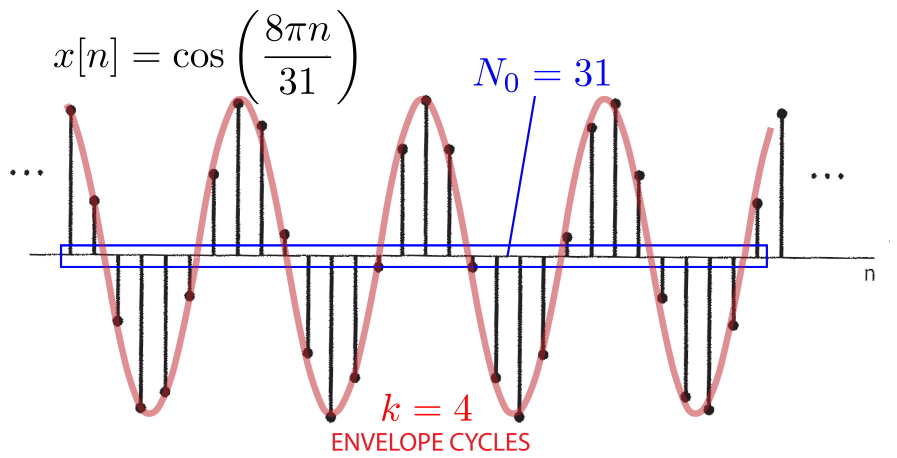


Example 2: $\omega = 8\pi/31 = \pi \cdot \boxed{\frac{8}{31}}$

$$x[n] = \cos\left(\frac{8\pi n}{31}\right)$$

$$\begin{aligned} N &= \frac{2\pi k}{\omega} = \frac{2\pi k}{\pi \frac{8}{31}} = \frac{31}{4}k \\ N_0 &= 31 \quad \text{for } k = 4 \end{aligned}$$

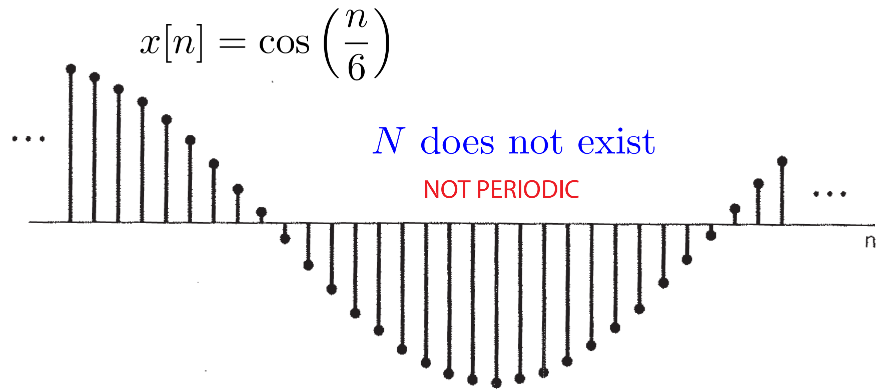
The fundamental period is 31 which corresponds to $k = 4$ envelope cycles.



Example 3: $\omega = 1/6 = \pi \cdot \boxed{\frac{1}{6\pi}}$

$$x[n] = \cos\left(\frac{n}{6}\right)$$

$$\begin{aligned} N &= \frac{2\pi k}{\omega} = \frac{2\pi k}{\frac{1}{6}} = 12\pi k \\ N &\in \mathbb{Z}^+ \quad \text{does not exist for any } k \in \mathbb{Z};\ x[n] \text{ is non-periodic.} \end{aligned}$$



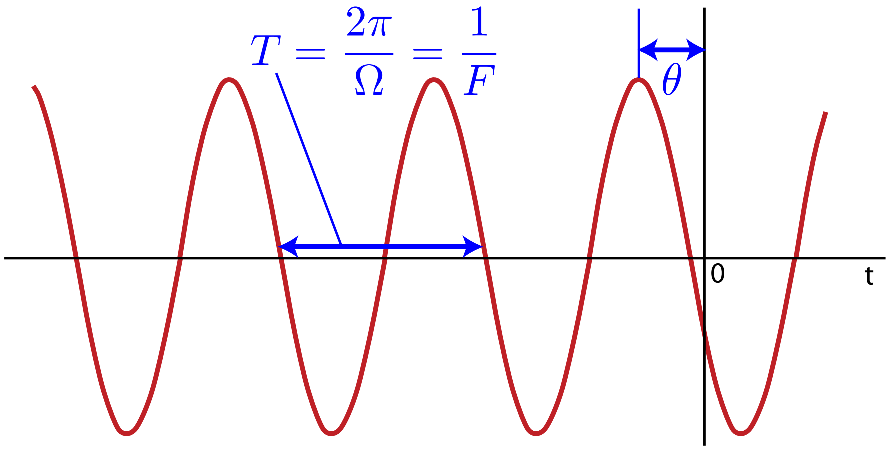
Continuous-Time Sinusoids: Frequency and Rate of Oscillation

$$x(t) = A\cos(\Omega t + \phi)$$

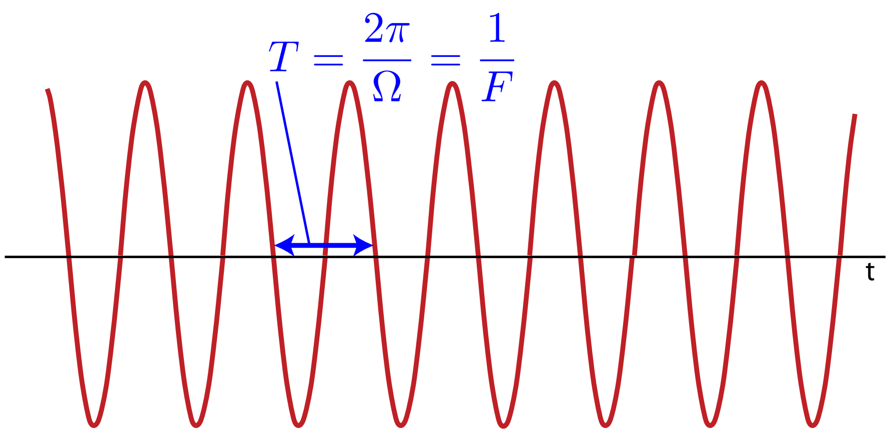
$$T = \frac{2\pi}{\Omega} = \frac{1}{f}$$

Rate of oscillation increases as Ω increases (or T decreases).

Ω smaller



Ω larger, rate of oscillation higher

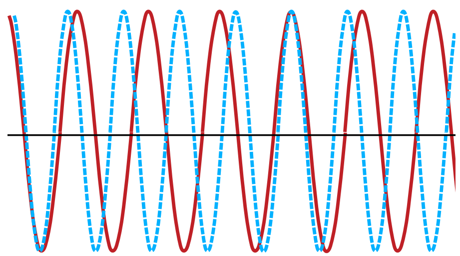


Continuous-Time Sinusoids: Frequency and Rate of Oscillation

Also, note that $x_1(t) \neq x_2(t)$ for all t for

$$x_1(t) = A\cos(\Omega_1 t + \phi) \quad \text{and} \quad x_2(t) = A\cos(\Omega_2 t + \phi)$$

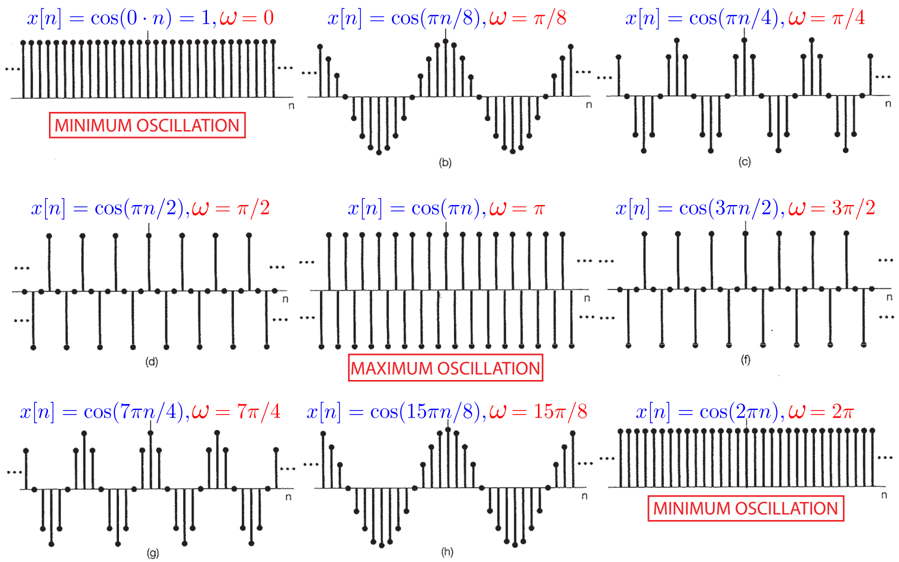
when $\Omega_1 \neq \Omega_2$.



Discrete-Time Sinusoids: Frequency and Rate of Oscillation

$$x[n] = A\cos(\omega n + \phi)$$

Rate of oscillation increases as ω increases UP TO A POINT then decreases again and then increases again and then decreases again ....



Discrete-Time Sinusoids: Frequency and Rate of Oscillation

$$x[n] = A\cos(\omega n + \phi)$$

Discrete-time sinusoids repeat as ω increases!

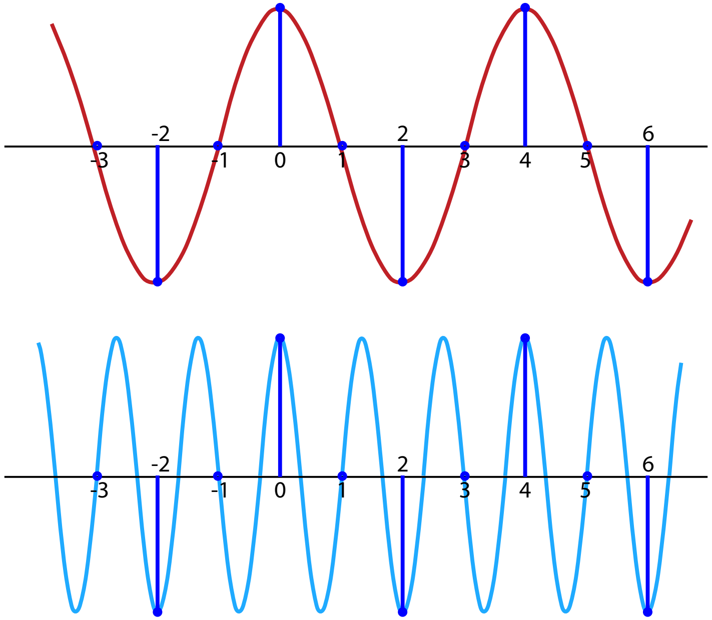
Discrete-Time Sinusoids: Frequency and Rate of Oscillation

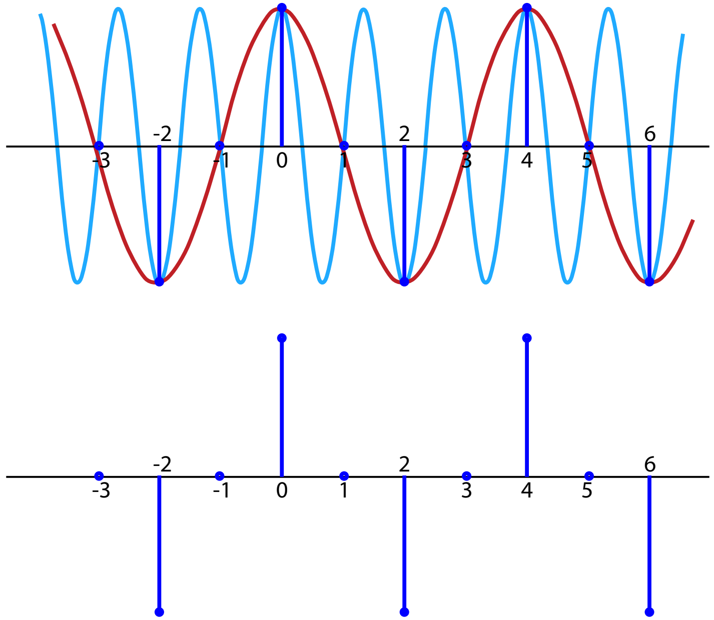
Let

$$x_1[n] = A\cos(\omega_1 n + \phi) \quad \text{and} \quad x_2[n] = A\cos(\omega_2 n + \phi)$$

and $\omega_2 = \omega_1 + 2\pi k$ where $k \in \mathbb{Z}$:

$$\begin{aligned} x_2[n] &= A\cos(\omega_2 n + \phi) \\ &= A\cos((\omega_1 + 2\pi k)n + \phi) \\ &= A\cos(\omega_1 n + 2\pi kn + \phi) \\ &= A\cos(\omega_1 n + \phi) = x_1[n] \end{aligned}$$





Discete-Time Sinusoids: Frequency and Rate of Oscillation

$$x[n] = A\cos(\omega n + \phi)$$

can be considered a sampled version of

$$x(t) = A\cos(\omega t + \phi)$$

at integer time instants.

As ω increases, the samples miss the faster oscillatory behavior. ■