http://www.comm.utoronto.ca/~dkundur/course/discrete-time-systems/

**11.1**

(a)

Let's assume that the original spectrum $X_a(F)$ has a triangular shape with amplitude L (see figure). $X(F)$, being a sampled version of $X_a(F)$ for $F_S = 2500Hz$, it becomes a repetition of that triangular signal every Fs Hz and its amplitude is changed to $A = L \cdot F_S$ (see figure).

$$\begin{aligned} X(F) &= F_s \cdot \sum_k X_a(F - k \cdot F_s) \\ &= 2500 \cdot \sum_k X_a(F - k \cdot 2500) \end{aligned}$$

We change the spectrum from $X(F)$ to $X(\omega)$ with the relation $\omega = \frac{2\pi F}{F_S} = \frac{2\pi F}{2500}$ (see figure).

The convolution with the signal $\cos(0.8\pi)$ gives us the spectrum for $W(\omega)$:

$$W(\omega) = \frac{1}{2}\left[X(\omega + 0.8\pi) + X(\omega - 0.8\pi)\right]$$

The ideal lowpass filter spectrum is given by:

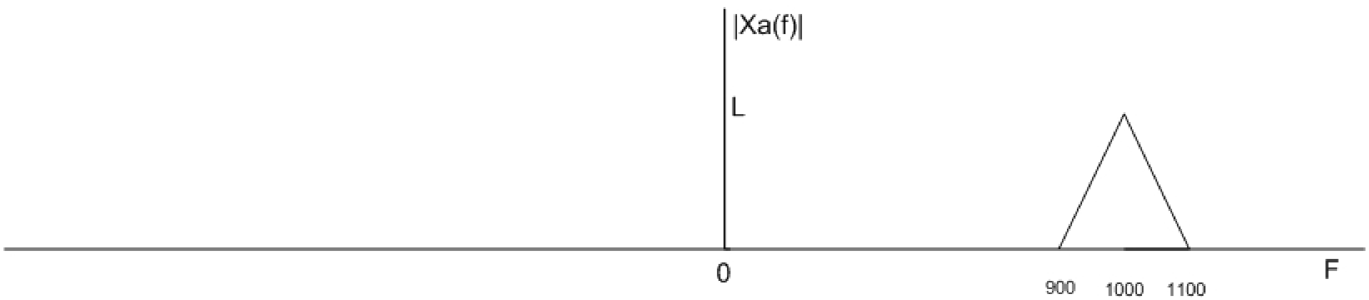
$$H(F) = \begin{cases} 1, & for \quad 0 \le |F| < 125Hz \\ 0, & otherwise \end{cases} \quad \text{or} \quad H(\omega) = \begin{cases} 1, & for \quad 0 \le |\omega| < 0.1\pi \\ 0, & otherwise \end{cases}$$

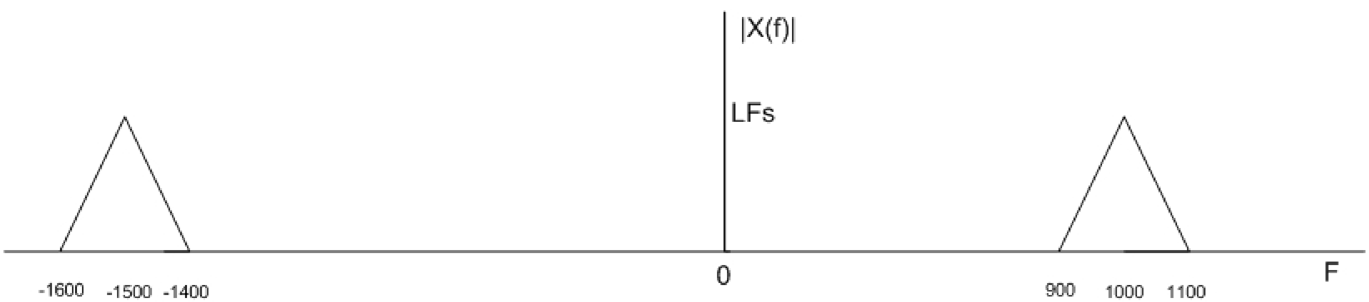
Therefore $V(\omega)$ consists of only one triangular iteration (the one centered at $\omega = 0$) after filtering $W(\omega)$ (see figure).

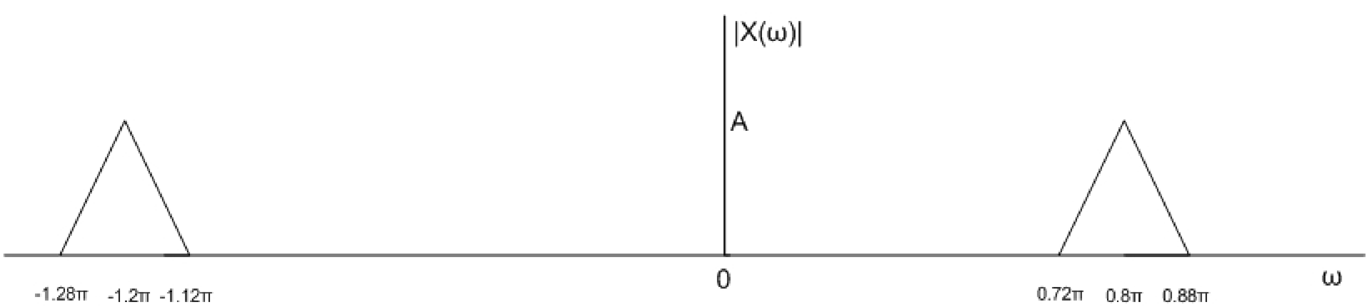
Finally, $V(\omega)$ is decimated by a factor D=10 to obtain $Y(\omega')$. The spectrum of $Y(\omega')$ is still centered at $\omega = 0$ but the decimation makes its bandwidth 10 times larger and its amplitude 10 times smaller than that of $V(\omega)$ (see figure).

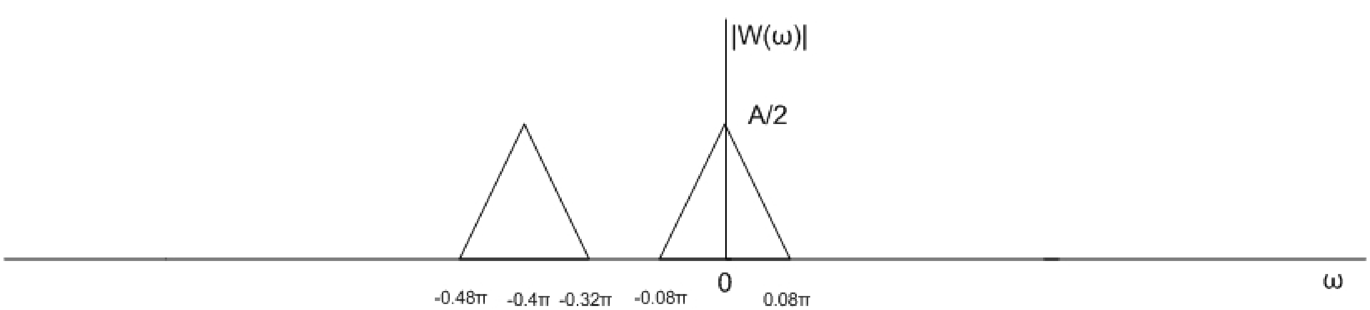
$$\begin{aligned} Y(\omega') &= \frac{1}{D} V\left(\frac{\omega'}{D}\right) \quad with \quad \omega' = D\omega \\ &= \frac{1}{10} V\left(\frac{\omega'}{10}\right) \quad with \quad \omega' = 10\omega \end{aligned}$$

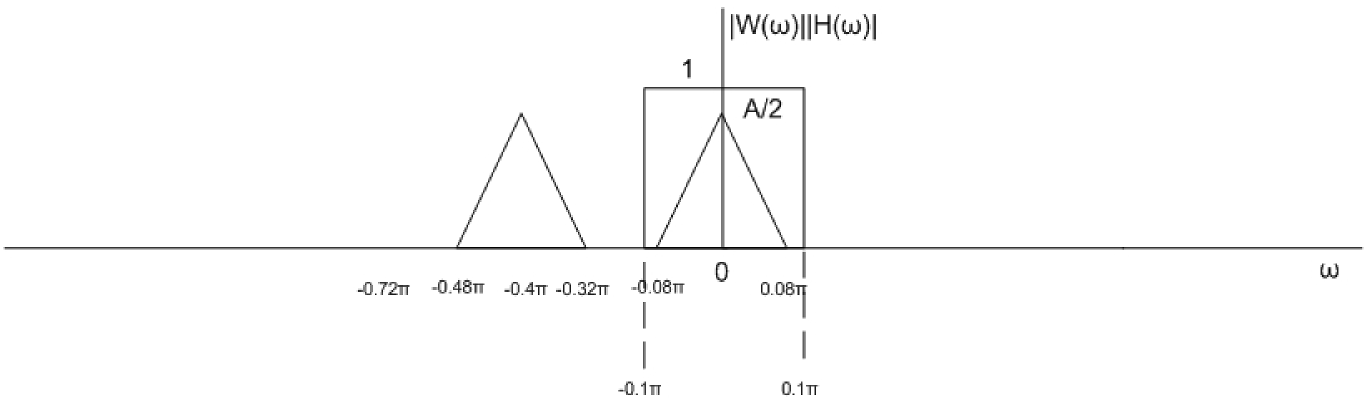
The figures below show each spectrum. It is to be noted that $X(\omega)$ and $W(\omega)$ are both 2π-periodic.

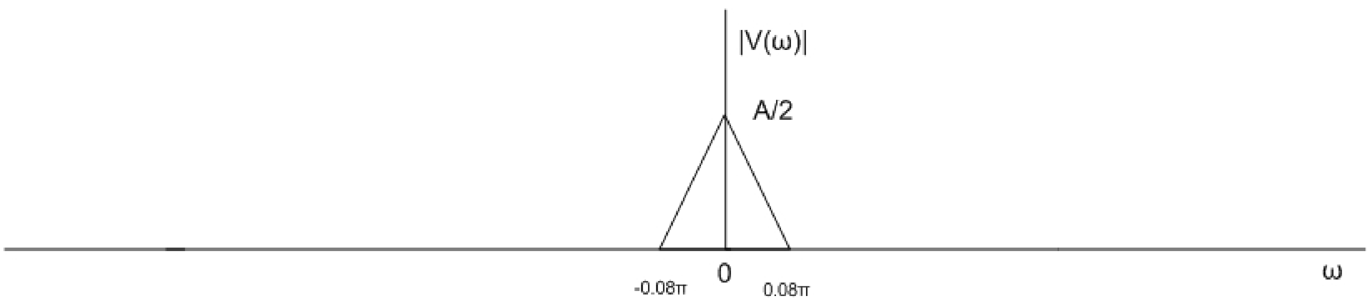

|Xa(f)|
L
0
900
1000
1100
F

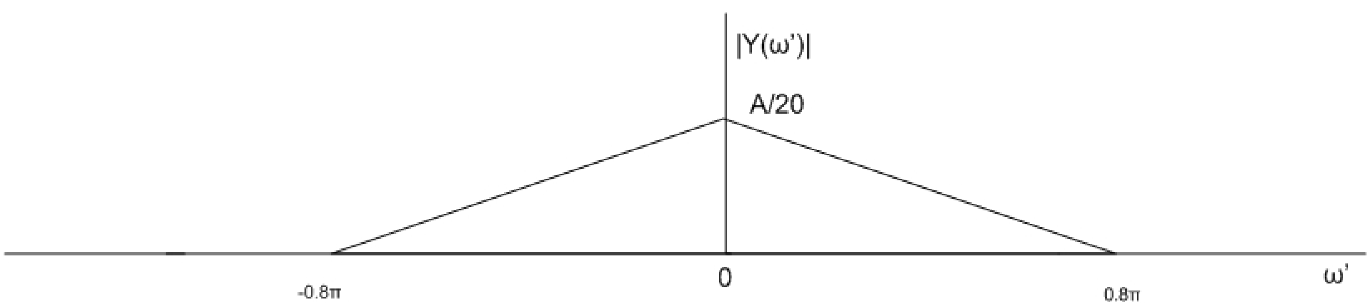

|X(f)|
LFs
-1600
-1500
-1400
0
900
1000
1100
F


|X(ω)|
A
-1.28π
-1.2π
-1.12π
0
0.72π
0.8π
0.88π
ω


|W(ω)|
A/2
-0.48π
-0.4π
-0.32π
-0.08π
0
0.08π
ω




(b) If we sample $x_a(t)$ with period $T = 4ms$ which correspond to sampling at the rate $F_S = \frac{1}{0.004} = 250Hz$, we obtain the spectrum of the sampled signal $X(F)$:

$$X(F) = F_s \cdot \sum_k X_a(F - k \cdot F_s)$$
$$= 250 \cdot \sum_k X_a(F - k \cdot 250)$$

But we have to consider the scaling of the frequency axis defined by $\omega' = \frac{2\pi F}{F_s}$ and therefore

$$X(\omega') = 250 \cdot \sum_k X_a(\omega' - k \cdot \omega'_s) \quad with \quad \omega'_s = \frac{2\pi F_s}{F_s} = 2\pi$$
$$= 250 \cdot \sum_k X_a(\omega' - k \cdot 2\pi)$$

After comparison with the result obtain for $Y(\omega')$ in question (a), we can see that except for a scaling factor of 2, $Y(\omega')$ and $X(\omega')$, obtained after sampling $x_a(t)$ with period $T = 4ms$, are the same:

$$Y(\omega') = 2 \cdot X(\omega')$$

**11.2**

We are given the signal $x(n) = a^n u(n), \quad |a| < 1$.

(a) The spectrum of the given signal is by definition $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cdot \exp(-j\omega n)$. This can we further develop as follows:

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x(n) \cdot \exp(-j\omega n) \\ &= \sum_{n=0}^{\infty} a^n \cdot \exp(-j\omega n) \\ &= \sum_{n=0}^{\infty} \left[a \cdot \exp(-j\omega)\right]^n \\ &= \frac{1}{1 - a \cdot \exp(-j\omega)}, \quad |a| < 1 \end{aligned}$$

Therefore the answer :

$$\boxed{X(\omega) = \frac{1}{1 - a \cdot \exp(-j\omega)}}$$

(b) After using a decimator that reduces the rate by a factor 2 on $x(n)$, we generate $y(n)$. $y(n)$'s spectrum, $Y(\omega)$, can be obtained using the relation 11.2.15 in the textbook which states that after a decimation of factor D we would have:

$$Y(\omega') = \frac{1}{D} X\left(\frac{\omega'}{D}\right) \quad with \quad \omega' = D\omega$$

Applied to our case for D=2, this equation gives:

$$Y(\omega') = \frac{1}{2} X\left(\frac{\omega'}{2}\right)$$

And therefore:

$$\boxed{Y(\omega') = \frac{1}{2} \cdot \frac{1}{1 - a \cdot \exp\left(-j\frac{\omega'}{2}\right)}, \quad |a| < 1}$$

(c) Lets define the signal $v(n)$ as $v(n) = x(2n)$. The Fourier transform of $v(n)$ is given by:

$$
\begin{aligned}
F\{v(n)\} &= \sum_n x(2n) \cdot \exp(-j\omega n) \\
&= \sum_m x(m) \cdot \exp\left(-j\omega \frac{m}{2}\right), \quad \mathit{for} \quad m = 2n \\
&= X\left(\frac{\omega}{2}\right) \\
&= 2 \cdot Y(\omega)
\end{aligned}
$$

NB. The book solution manual has the following solution (which somehow seems wrong to me):

$$
\begin{aligned}
F\{x(2n)\} &= \sum_n x(2n) \cdot \exp(-j\omega 2n) \\
&= \sum_n x(2n) \cdot \exp(-j\omega' n), \quad \mathit{for} \quad m = 2n \quad \mathit{and} \quad \omega' = 2\omega \\
&= Y(\omega')
\end{aligned}
$$

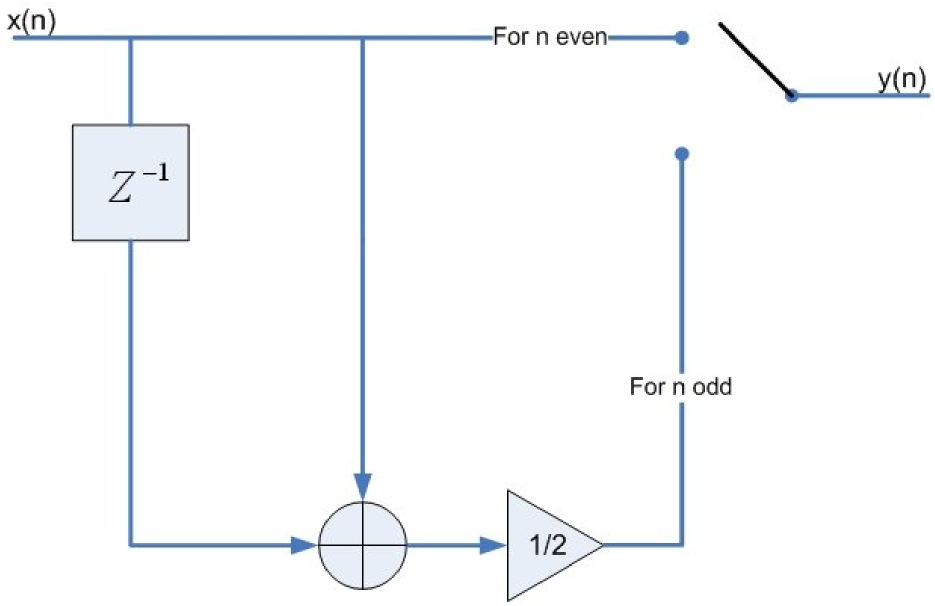
## 11.3

a) From the expression given for $y(n)$ we can write the even and odd samples of $y(n)$ as:

$$y(2m) = x(m) \quad for \quad 2m = n$$

$$y(2m-1) = \frac{1}{2} \cdot \left[ x(m-1) + x(m) \right] \quad for \quad 2m-1 = n$$

A suitable DSP implementation would therefore be the following:




b) Lets compute the spectrum of $y(n)$, $Y(\omega_y)$, from $X(\omega_x)$ where $\omega_y = 2\omega_x$:

$$Y(\omega_y) = \sum_{n=-\infty}^{\infty} y(n) \cdot \exp(-j\omega_y n)$$

$$= \sum_{n\ even} x\left(\frac{n}{2}\right) \cdot \exp(-j\omega_y n) + \frac{1}{2} \cdot \sum_{n\ odd} \left[ x\left(\frac{n-1}{2}\right) + x\left(\frac{n+1}{2}\right) \right] \cdot \exp(-j\omega_y n)$$

$$= \sum_p x(p) \cdot \exp(-j\omega_y (2p)) + \frac{1}{2} \cdot \sum_q \left[ x(q) + x(q+1) \right] \cdot \exp(-j\omega_y (2q+1)) \quad with \quad p = \frac{n}{2}, q = \frac{n-1}{2}$$

$$= X(2\omega_y) + \frac{1}{2} \cdot \left[ \exp(-j\omega_y) + \exp(j\omega_y) \right] \cdot X(2\omega_y)$$

$$= X(2\omega_y) \cdot \left[ 1 + \cos(\omega_y) \right]$$

We are given the signal with spectrum $X(\omega_x) = \begin{cases} 1, & 0 \le |\omega_x| \le 0.2\pi \\ 0, & otherwise \end{cases}$. In order to get the corresponding output spectrum $Y(\omega_y)$, lets first change the variable $\omega_x$ into $2\omega_y$ to get $X(2\omega_y)$:

$$X(2\omega_y) = \begin{cases} 1, & 0 \le |2\omega_y| \le 0.2\pi \\ 0, & otherwise \end{cases}$$

$$\Rightarrow X(2\omega_y) = \begin{cases} 1, & 0 \le |\omega_y| \le 0.1\pi \\ 0, & otherwise \end{cases}$$

And finally we obtain the output spectrum by multiply the obtained spectrum above by $1+\cos(\omega_y)$:

$$Y(\omega_y) = \begin{cases} 1+\cos(\omega_y), & 0 \le |\omega_y| \le 0.1\pi \\ 0, & otherwise \end{cases}$$

c) Applying the same method as before to the signal with spectrum $X(\omega_x) = \begin{cases} 1, & 0.7\pi \le |\omega_x| \le 0.9\pi \\ 0, & otherwise \end{cases}$, we get:

$$X(2\omega_y) = \begin{cases} 1, & 0.7\pi \le |2\omega_y| \le 0.9\pi \\ 0, & otherwise \end{cases}$$

$$\Rightarrow X(2\omega_y) = \begin{cases} 1, & 0.35\pi \le |\omega_y| \le 0.45\pi \\ 0, & otherwise \end{cases}$$

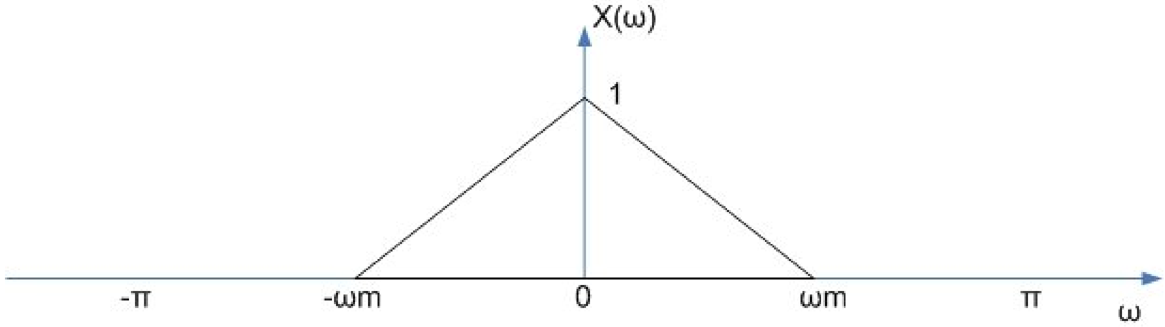
$$\text{i.e., } Y(\omega_y) = \begin{cases} 1+\cos(\omega_y), & 0.35\pi \le |\omega_y| \le 0.45\pi \\ 0, & otherwise \end{cases}$$

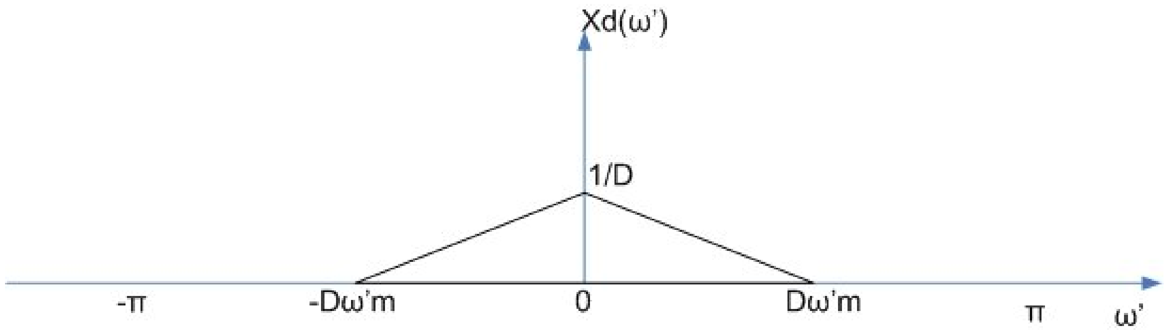
**11.4**

We are given the signal $x(n)$ with Fourier transform $X(\omega)=0,\ \ \omega_m<|\omega|\le\pi\ \ \left(f_m<|f|\le\frac{1}{2}\right)$

a) Let $x'(n)$ be the downsampled sequence, then we have: (see figure)

$$x'(n)=x(nD)\ \text{ and }\ X'(\omega')=\frac{1}{D}X\left(\frac{\omega'}{D}\right)$$




As long as $D\omega_m^{'}\le\pi$, $X(\omega)$ (hence $x(n)$) can be recovered from $X'(\omega')$ using an interpolator with a factor $D$:

$$X(\omega)=DX'(D\omega)$$

The given sampling frequency is $\omega_s=\frac{2\pi}{D}$, hence the condition $D\omega_m^{'}\le\pi$ becomes $2\omega_m^{'}\le\frac{2\pi}{D}=\omega_s$.

b) Let $x(a)$ be the real analog signal from which samples $x(n)$ were taken at rate $F_x$. There exists a signal, say $x_a^{'}(t')$, such that $x_a^{'}(t') = X_a\left(\frac{t}{T_x}\right)$. $x(n)$ may be considered to be the samples of $x'(t')$ taken at rate $f_x = 1$. Likewise $x"(n) = x(nD)$ are samples of $x'(t')$ taken at rate $f_x" = \frac{f_x}{D} = \frac{1}{D}$.

From sampling theory, we know that $x'(t')$ can be reconstructed from its samples $x"(n)$ as long as it is bandlimited to $f_m \le \frac{1}{2D}$, or $w_m \le \frac{\pi}{D}$, which is the case here. The reconstruction formula is:

$$x'(t') = \sum_k x"(k) h_r(t' - kD) \quad \text{where } h_r(t') = \frac{\sin\left(\frac{\pi}{D}t'\right)}{\left(\frac{\pi}{D}t'\right)}$$

Actually the bandwidth of the reconstruction filter may be made as small as $\omega_m^{'}$, or as large as $\frac{2\pi}{D} - \omega_m^{'}$, so $h_r$ may be:

$$h_r(t') = \frac{\sin(\omega_c^{'} t')}{(\omega_c^{'} t')} \quad \text{where } \omega_m^{'} \le \omega_c^{'} \le \frac{2\pi}{D} - \omega_m^{'}.$$

In particular, $x(n) = x'(t' = n)$, so $x(n) = \sum_k x(kD) h_r(n - kD)$

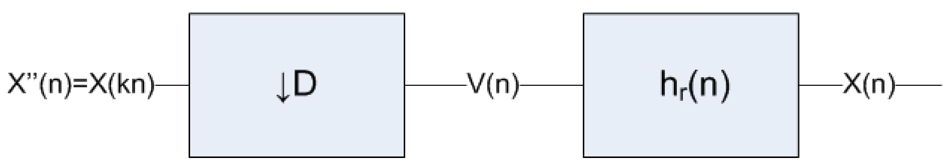
c) Clearly if we define

$$v(p) = \begin{cases} x(p), & \text{if p is an integer multiple of D} \\ 0, & \text{other p} \end{cases}$$

then, we may write:

$$x(n) = \sum_p v(p) h_r(n - p)$$

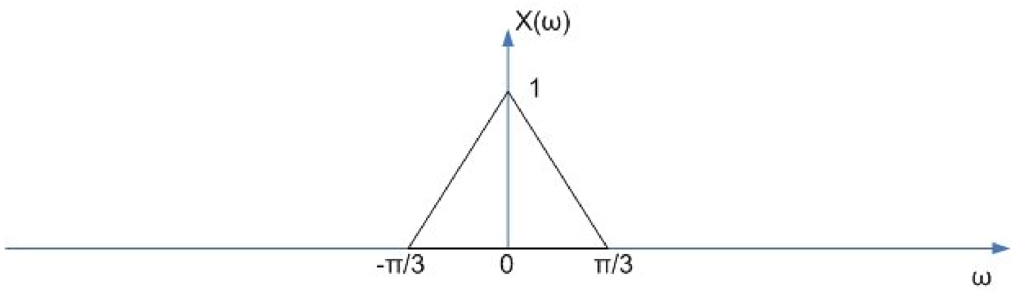
So $x(n)$ can be reconstructed using a 2-step process made of a decimator followed by a lowpass filter:

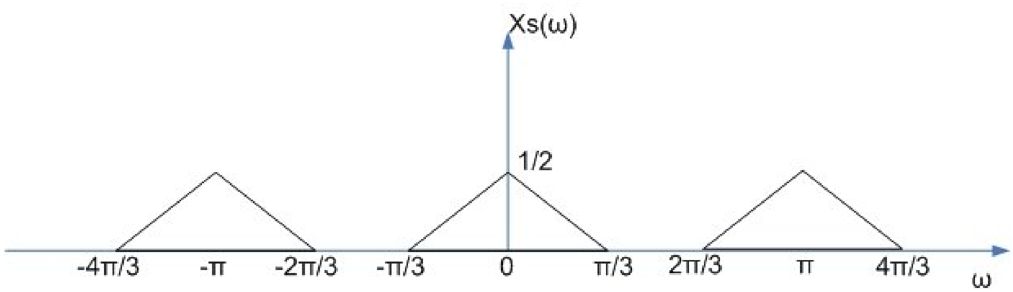


**11.5**

From relation 11.4.11 in the book we get for D=2 and $\omega = \dfrac{2\pi F}{F_x}$ :

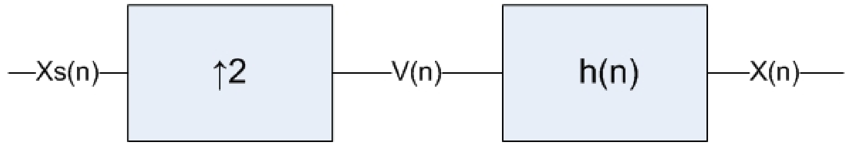
$$X_S(\omega) = \frac{1}{2}\sum_k X\left(\frac{\omega' - 2\pi k}{2}\right) \quad with \quad \omega' = 2\omega$$
$$= \frac{1}{2}\sum_k X(\omega - \pi k)$$



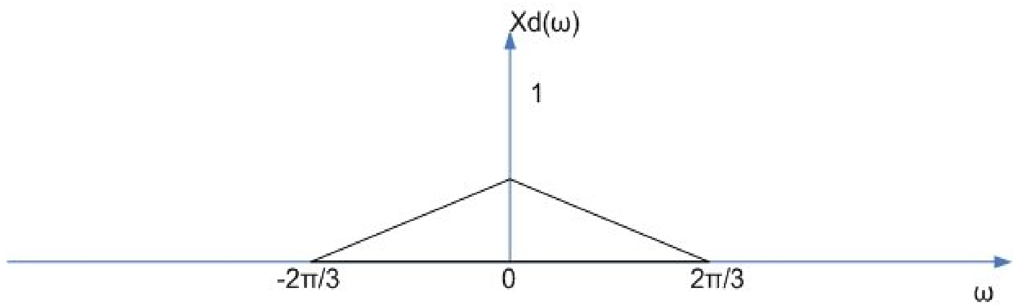

Yes, we can retrieve the original signal x(n) from its sampled version simply by using first an upsampler with factor I=2 followed by an ideal lowpass filter defined as:

$$H(\omega) = \begin{cases} 1 & for \quad 0 \le |\omega| \le \dfrac{\pi}{2} \\ 0 & eslsewhere \end{cases}$$



(b) By definition we have: $X_d(\omega) = \sum_n x_d(n) \cdot \exp(-j\omega n)$ that we can rewrite as

$$\begin{aligned} X_d(\omega) &= \sum_n x_d(n) \cdot \exp(-j\omega n) \\ &= \sum_n x(2n) \cdot \exp(-j\omega n) \\ &= \sum_m x(m) \cdot \exp\left(-j\omega \frac{m}{2}\right) \quad for \quad m \quad even \\ &= \sum_l x_s(l) \cdot \exp\left(-j\frac{\omega}{2} l\right) \\ &= X_s\left(\frac{\omega}{2}\right) \end{aligned}$$




There is no information loss in that case because the decimated sample rate is more than twice the bandlimit of the original signal.