Introduction to Discrete-Time Systems

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Discrete-Time Signals and Systems

Reference:

Sections 1.1 - 1.4 of

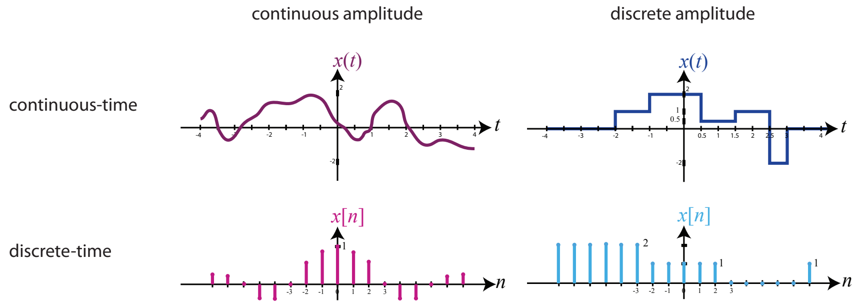
John G. Proakis and Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, 4th edition, 2007.

What is a Signal? What is a System?

- Signal:
  - any physical quantity that varies with time, space, or any other independent variable or variables
  - Examples: pressure as a function of altitude, sound as a function of time, color as a function of space, ...
  - $x(t) = \cos(2\pi t)$, $x(t) = 4\sqrt{t} + t^3$, $x(m,n) = (m+n)^2$
- System:
  - a physical device that performs an operation on a signal
  - Examples: analog amplifier, noise canceler, communication channel, transistor, ...
  - $y(t) = -4x(t)$, $\frac{dy(t)}{dt} + 3y(t) = -\frac{dx(t)}{dt} + 6x(t)$, $y(n) - \frac{1}{2}y(n-2) = 3x(n) + x(n-2)$

Analog and Digital Signals

- analog signal = continuous-time + continuous amplitude
- digital signal = discrete-time + discrete amplitude



Analog and Digital Signals

- Analog signals are fundamentally significant because we must interface with the real world which is analog by nature.
- Digital signals are important because they facilitate the use of digital signal processing (DSP) systems, which have practical and performance advantages for several applications.

Analog and Digital Systems

- analog system = analog signal input + analog signal output
  - advantages: easy to interface to real world, do not need A/D or D/A converters, speed not dependent on clock rate
- digital system = digital signal input + digital signal output
  - advantages: re-configurability using software, greater control over accuracy/resolution, predictable and reproducible behavior

Deterministic vs. Random Signals

- Deterministic signal:
  - any signal that can be uniquely described by an explicit mathematical expression, a table of data, or a well-defined rule
  - past, present and future values of the signal are known precisely without any uncertainty
- Random signal:
  - any signal that lacks a unique and explicit mathematical expression and thus evolves in time in an unpredictable manner
  - it may not be possible to accurately describe the signal
  - the deterministic model of the signal may be too complicated to be of use.

What is a “pure frequency” signal?

$$x_a(t) = A\cos(\Omega t + \theta) = A\cos(2\pi F t + \theta), \quad t \in \mathbb{R}$$

- analog signal, $\because -A \leq x_a(t) \leq A$ and $-\infty < t < \infty$
- A = amplitude
- Ω = frequency in rad/s
- F = frequency in Hz (or cycles/s); note: $\Omega = 2\pi F$
- θ = phase in rad

Continuous-time Sinusoids

$$x_a(t) = A\cos(\Omega t + \theta) = A\cos(2\pi F t + \theta), \quad t \in \mathbb{R}$$

1. For $F \in \mathbb{R}$, $x_a(t)$ is periodic
   - i.e., there exists $T_p \in \mathbb{R}^+$ such that $x_a(t) = x_a(t + T_p)$
2. distinct frequencies result in distinct sinusoids
   - i.e., for $F_1 \neq F_2$, $A\cos(2\pi F_1 t + \theta) \neq A\cos(2\pi F_2 t + \theta)$
3. increasing frequency results in an increase in the rate of oscillation of the sinusoid
   - i.e., for $|F_1| < |F_2|$, $A\cos(2\pi F_1 t + \theta)$ has a lower rate of oscillation than $A\cos(2\pi F_2 t + \theta)$

Discrete-time Sinusoids

$$x(n) = A\cos(\omega n + \theta) = A\cos(2\pi f n + \theta), \quad n \in \mathbb{Z}$$

- discrete-time signal (not digital), $\because -A \leq x_a(t) \leq A$ and $n \in \mathbb{Z}$
- A = amplitude
- ω = frequency in rad/sample
- f = frequency in cycles/sample; note: $\omega = 2\pi f$
- θ = phase in rad

Discrete-time Sinusoids

$$x(n) = A\cos(\omega n + \theta) = A\cos(2\pi f n + \theta), \quad n \in \mathbb{Z}$$

1. $x(n)$ is periodic only if its frequency f is a rational number
   - Note: rational number is of the form $\frac{k_1}{k_2}$ for $k_1, k_2 \in \mathbb{Z}$
   - periodic discrete-time sinusoids: $x(n) = 2\cos(\frac{4}{7}\pi n), x(n) = \sin(-\frac{\pi}{5}n + \sqrt{3})$
   - aperiodic discrete-time sinusoids: $x(n) = 2\cos(\frac{4}{7}n), x(n) = \sin(\sqrt{2}\pi n + \sqrt{3})$

Discrete-time Sinusoids

$$x(n) = A\cos(\omega n + \theta) = A\cos(2\pi f n + \theta), \quad n \in \mathbb{Z}$$

2. radian frequencies separated by an integer multiple of 2π are identical
   - or cyclic frequencies separated by an integer multiple are identical
3. lowest rate of oscillation is achieved for $\omega = 2k\pi$ and highest rate of oscillation is achieved for $\omega = (2k+1)\pi$, for $k \in \mathbb{Z}$
   - subsequently, this corresponds to lowest rate for $f = k$ (integer) and highest rate for $f = \frac{2k+1}{2}$ (half integer), for $k \in \mathbb{Z}$; see Figure 1.3.4 of text.

Complex Exponentials

$$e^{j\phi} = \cos(\phi) + j\sin(\phi) \quad \text{Euler's relation}$$

$$\cos(\phi) = \frac{e^{j\phi}+e^{-j\phi}}{2}$$

$$\sin(\phi) = \frac{e^{j\phi}-e^{-j\phi}}{2j}$$

where $j \triangleq \sqrt{-1}$

Complex Exponentials

Continuous-time: $A\, e^{j(\Omega t+\theta)} = A\, e^{j(2\pi F t+\theta)}$

Discrete-time: $A\, e^{j(\omega n+\theta)} = A\, e^{j(2\pi f n+\theta)}$

Periodicity: Continuous-time

$$\begin{aligned}
x(t) &= x(t+T),\ T \in \mathbb{R}^+ \\
A\, e^{j(2\pi F t+\theta)} &= A\, e^{j(2\pi F(t+T)+\theta)} \\
e^{j2\pi F t}\cdot e^{j\theta} &= e^{j2\pi F t}\cdot e^{j2\pi F T}\cdot e^{j\theta} \\
1 &= e^{j2\pi F T} \\
e^{j2\pi k} = 1 &= e^{j2\pi F T},\ \ k \in \mathbb{Z} \\
T &= \frac{k}{F}\ \ k \in \mathbb{Z} \\
T_0 &= \frac{1}{|F|},\ \ k = \text{sgn}(F)
\end{aligned}$$

Periodicity: Discrete-time

$$\begin{aligned}
x(n) &= x(n+N),\ N \in \mathbb{Z}^+ \\
A\, e^{j(2\pi f n+\theta)} &= A\, e^{j(2\pi f(n+N)+\theta)} \\
e^{j2\pi f n}\cdot e^{j\theta} &= e^{j2\pi f n}\cdot e^{j2\pi f N}\cdot e^{j\theta} \\
1 &= e^{j2\pi f N} \\
e^{j2\pi k} = 1 &= e^{j2\pi f N},\ \ k \in \mathbb{Z} \\
f &= \frac{k}{N}\ \ k \in \mathbb{Z} \\
N_0 &= \frac{k'}{f}, \min |k'| \in \mathbb{Z} \text{ such that } \frac{k'}{f} \in \mathbb{Z}^+
\end{aligned}$$

Uniqueness: Continuous-time

For $F_1 \neq F_2$,

$$A\cos(2\pi F_1 t + \theta) \neq A\cos(2\pi F_2 t + \theta)$$

except at discrete points in time.

Uniqueness: Discrete-time

Let $f_1 = f_0 + k$ where $k \in \mathbb{Z}$,

$$\begin{aligned} x_1(n) &= A\, e^{j(2\pi f_1 n + \theta)} \\ &= A\, e^{j(2\pi(f_0+k)n+\theta)} \\ &= A\, e^{j(2\pi f_0 n+\theta)} \cdot e^{j(2\pi k n)} \\ &= x_0(n) \cdot 1 = x_0(n) \end{aligned}$$

Uniqueness: Discrete-time

- Therefore, dst-time sinusoids are unique for $f \in [0, 1)$.
- For any sinusoid with $f_1 \notin [0, 1)$, $\exists f_0 \in [0, 1)$ such that

$$x_1(n) = A\, e^{j(2\pi f_1 n + \theta)} = A\, e^{j(2\pi f_0 n + \theta)} = x_0(n).$$

- Example: A dst-time sinusoid with frequency $f_1 = 4.56$ is the same as a dst-time sinusoid with frequency $f_0 = 4.56 - 4 = 0.56$.
- Example: A dst-time sinusoid with frequency $f_1 = -\frac{7}{8}$ is the same as a dst-time sinusoid with frequency $f_0 = -\frac{7}{8} + 1 = \frac{1}{8}$.

Figure 1.4.5 of text

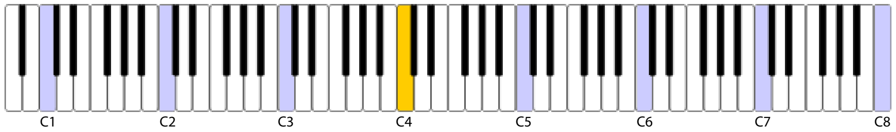
Harmonically Related Complex Exponentials

Harmonically related (cts-time) $s_k(t) = e^{jk\Omega_0 t} = e^{j2\pi k F_0 t}$, $k = 0, \pm 1, \pm 2, \ldots$

Scientific Designation	Frequency (Hz)	k for $F_0 = 8.176$
C-1	8.176	1
C0	16.352	2
C1	32.703	4
C2	65.406	8
C3	130.813	16
C4	261.626	32
⋮	⋮	
C9	8372.018	1024

Harmonically Related Complex Exponentials

Scientific Designation	Frequency (Hz)	k for $F_0 = 8.176$
C1	32.703	4
C2	65.406	8
C3	130.813	16
C4 (middle C)	261.626	32
C5	523.251	64
C6	1046.502	128
C7	2093.005	256
C8	4186.009	512



Harmonically Related Complex Exponentials

What does the family of harmonically related sinusoids $s_k(t)$ have in common?

Harmonically related (cts-time) $s_k(t) = e^{jk\Omega_0 t} = e^{j2\pi(kF_0)t}$, $k = 0, \pm1, \pm2, \ldots$

$$\begin{aligned}
\text{fund. period: } T_{0,k} &= \frac{1}{\text{cyclic frequency}} = \frac{1}{kF_0} \\
\text{period: } T_k &= \text{any integer multiple of } T_0 \\
\text{common period: } T &= k \cdot T_{0,k} = \frac{1}{F_0}
\end{aligned}$$

Harmonically Related Complex Exponentials

Discrete-time Case:

For periodicity, select $f_0 = \frac{1}{N}$ where $N \in \mathbb{Z}$:

Harmonically related (dts-time) $s_k(n) = e^{j2\pi k f_0 n} = e^{j2\pi kn/N}$, $k = 0, \pm1, \pm2, \ldots$

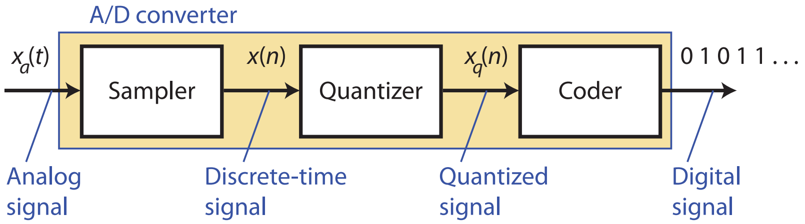
- There are only N distinct dst-time harmonics: $s_k(n)$, $k = 0, 1, 2, \ldots, N-1$.

Harmonically Related Complex Exponentials

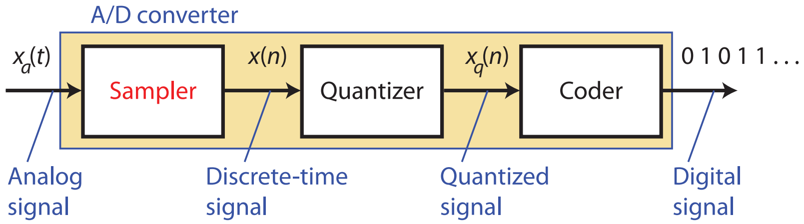
$$\begin{aligned}
s_{k+N}(n) &= e^{j2\pi(k+N)n/N} \\
&= e^{j2\pi kn/N} \cdot e^{j2\pi Nn/N} \\
&= e^{j2\pi kn/N} \cdot 1 \\
&= e^{j2\pi kn/N} = s_k(n)
\end{aligned}$$

Therefore, there are only N distinct dst-time harmonics: $s_k(n)$, $k = 0, 1, 2, \ldots, N-1$.

Analog-to-Digital Conversion



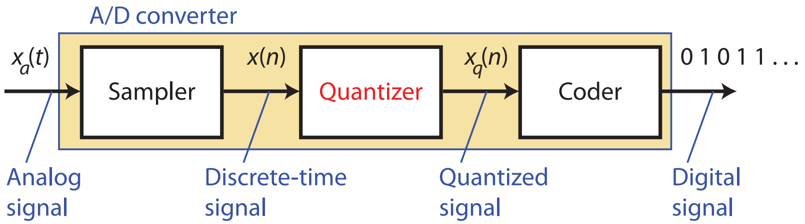
Analog-to-Digital Conversion



Sampling:

- conversion from cts-time to dst-time by taking "samples" at discrete time instants
- E.g., uniform sampling: $x(n) = x_a(nT)$ where T is the sampling period and $n \in \mathbb{Z}$

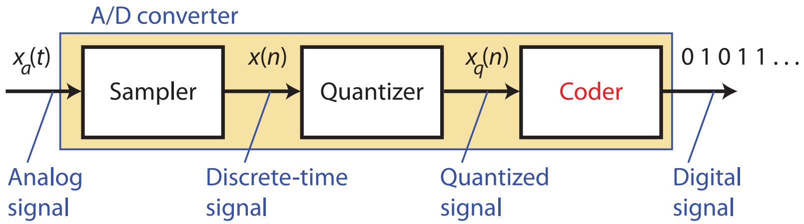
Analog-to-Digital Conversion



Quantization:

- conversion from dst-time cts-valued signal to a dst-time dst-valued signal
- quantization error: $e_q(n) = x_q(n) - x(n)$ for all $n \in \mathbb{Z}$

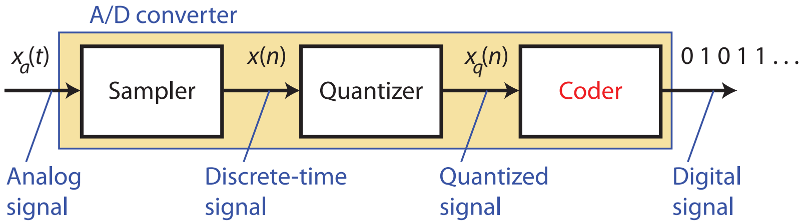
Analog-to-Digital Conversion



Coding:

- representation of each dst-value $x_q(n)$ by a b-bit binary sequence
- e.g., if for any n, $x_q(n) \in \{0, 1, \ldots, 6, 7\}$, then the coder may use the following mapping to code the quantized amplitude:

Analog-to-Digital Conversion



Example coder:

0	000	4	100
1	001	5	101
2	010	6	110
3	011	7	111

Sampling Theorem

If the highest frequency contained in an analog signal $x_a(t)$ is $F_{max} = B$ and the signal is sampled at a rate

$$F_s > 2F_{max} = 2B$$

then $x_a(t)$ can be exactly recovered from its sample values using the interpolation function

$$g(t) = \frac{\sin(2\pi Bt)}{2\pi Bt}$$

Note: $F_N = 2B = 2F_{max}$ is called the Nyquist rate.

Sampling Theorem

$$\text{Sampling Period} = T = \frac{1}{F_s} = \frac{1}{\text{Sampling Frequency}}$$

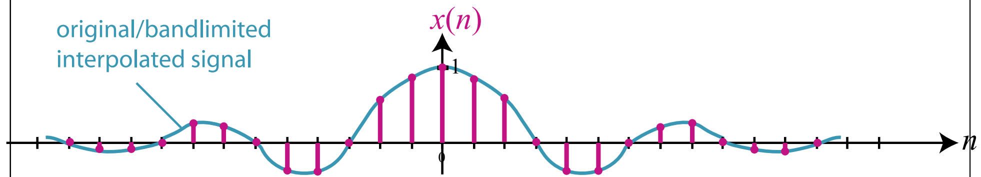
Therefore, given the interpolation relation, $x_a(t)$ can be written as

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT)g(t - nT)$$

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n) \; g(t - nT)$$

where $x_a(nT) = x(n)$; called bandlimited interpolation.
See ▸ Figure 1.4.6 of text.

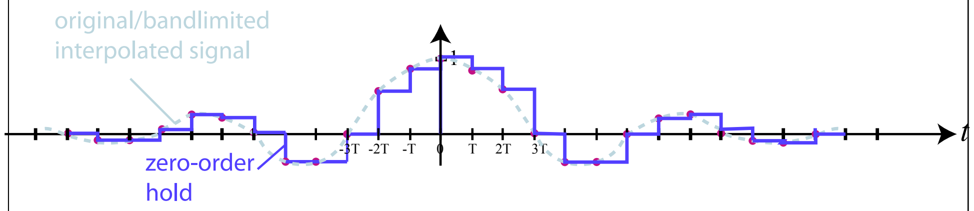
Digital-to-Analog Conversion



- Common interpolation approaches: bandlimited interpolation, zero-order hold, linear interpolation, higher-order interpolation techniques, e.g., using splines
- In practice, "cheap" interpolation along with a smoothing filter is employed.

□

Digital-to-Analog Conversion



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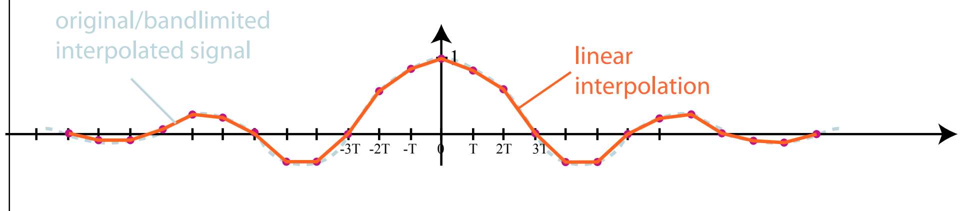
Chapter 1: Introduction 1.4 Analog-to-Digital and Digital-to-Analog Conversion

Digital-to-Analog Conversion



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