Multirate Digital Signal Processing: Part I

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Discrete-Time Signals and Systems

Reference:

Sections 11.1-11.3 of

John G. Proakis and Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, 4th edition, 2007.

Multirate DSP

- sampling rate conversion: process of converting a given discrete-time signal at a given rate to a different rate
- multirate digital signal processing systems: systems that employ multiple sampling rates

Sampling vs. Sampling Rate Conversion

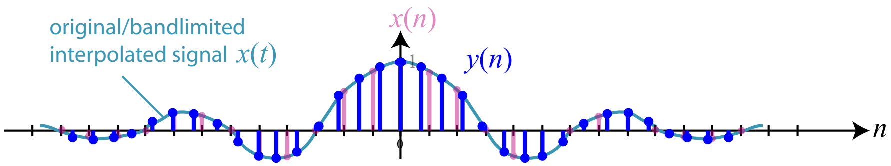
Sampling:

- conversion from cts-time to dst-time by taking "samples" at discrete time instants
- E.g., uniform sampling: $x(n) = x_a(nT)$ where T is the sampling period

Sampling rate conversion approaches:

- convert original samples to analog domain and then resample to generate new samples
- filter original samples with a discrete-time linear time-varying system to generate new samples

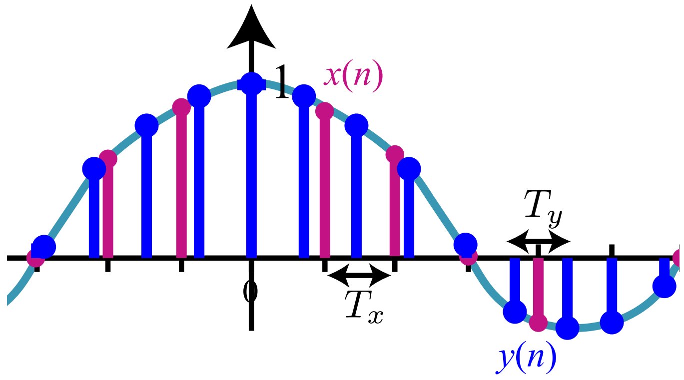
Sampling Rate Conversion



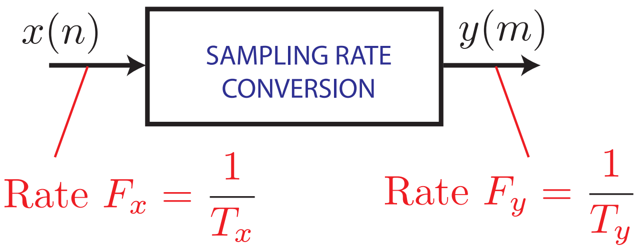
- $x(n)$: original samples at sampling rate F_x
- $y(n)$: new samples at sampling rate F_y

Ideal Sampling Rate Conversion

- $x(n)$: original samples at sampling rate $F_x = \frac{1}{T_x}$
- $y(n)$: new samples at sampling rate $F_y = \frac{1}{T_y}$



Parameter Relationships



Parameter/Variable	$x(n) \equiv x(nT_x)$	$y(m) \equiv y(mT_y)$
Rate	F_x	F_y
Period	T_x	T_y
Dst-time Frequency	ω_x	ω_y
Cts-time Frequency	F	F

Bridging the Parameter Relationships

- related to the ratio: $\boxed{\frac{T_y}{T_x}}$

$$
\begin{aligned}
F_y &= \frac{T_x}{T_y} \cdot F_x \\
\omega_x &= 2\pi f_x = \frac{2\pi F}{F_x} \\
\omega_y &= 2\pi f_y = \frac{2\pi F}{F_y} \\
\omega_x &= \frac{F_y}{F_x} \cdot \omega_y = \frac{T_x}{T_y} \cdot \omega_y
\end{aligned}
$$

Implementation of Sampling Rate Conversion

We relate the original samples $x(nT_x)$ to the new samples $y(mT_y)$ by assuming we convert the signal to analog and resample. Using the interpolation formula

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT_x)g(t - nT_x)$$

where

$$g(t) = \frac{\sin(\pi t/T_x)}{\pi t/T_x} \stackrel{\mathcal{F}}{\longleftrightarrow} G(F) = \begin{cases} T_x & |F| \leq F_x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Note: $y(t) = x(t)$ if $x(t)$ is sampled above Nyquist.

Implementation of Sampling Rate Conversion

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT_x)g(t - nT_x)$$

$$\underbrace{y(mT_y)}_{\text{desired samples}} = \sum_{n=-\infty}^{\infty} \underbrace{x(nT_x)}_{\text{original samples}} \underbrace{g(mT_y - nT_x)}_{\text{samples of } g(t)}$$

Implementation of Sampling Rate Conversion

$$\begin{aligned} y(t) &= \sum_{n=-\infty}^{\infty} x(nT_x)g(t - nT_x) \\ y(mT_y) &= \sum_{n=-\infty}^{\infty} x(nT_x)g(mT_y - nT_x) \\ &= \sum_{n=-\infty}^{\infty} x(nT_x)g\left(T_x\left(\frac{mT_y}{T_x} - n\right)\right) \end{aligned}$$

$$\frac{mT_y}{T_x} = k_m + \Delta_m$$

Implementation of Sampling Rate Conversion

$$\begin{aligned} \frac{mT_y}{T_x} &= \underbrace{k_m}_{\text{integer}} + \underbrace{\Delta_m}_{\text{remainder}} \\ k_m &= \left\lfloor \frac{mT_y}{T_x} \right\rfloor \\ \Delta_m &= \frac{mT_y}{T_x} - \left\lfloor \frac{mT_y}{T_x} \right\rfloor \in [0, 1) \end{aligned}$$

Implementation of Sampling Rate Conversion

$$
\begin{aligned}
y(mT_y) &= \sum_{n=-\infty}^{\infty} x(nT_x) g\left(T_x\left(\frac{mT_y}{T_x} - n\right)\right) \\
&= \sum_{n=-\infty}^{\infty} x(nT_x) g\left(T_x\left(k_m + \Delta_m - n\right)\right) \\
&\qquad \text{let } k = k_m - n \\
&= \sum_{k=-\infty}^{\infty} x((k_m - k)T_x) g\left(T_x\left(k + \Delta_m\right)\right) \\
&= \underbrace{\sum_{k=-\infty}^{\infty} g((k+\Delta_m)T_x) x((k_m - k)T_x)}_{\text{weighted linear combination of orig samples}}
\end{aligned}
$$

Implementation of Sampling Rate Conversion

$$
y(mT_y) = \sum_{k=-\infty}^{\infty} g((k+\Delta_m)T_x) x((k_m - k)T_x)
$$

- Δ_m: determines the set of weights
- k_m: specifies the set of input samples
- represents a discrete-time linear time-varying system
  - every output sample m requires use of a different impulse response/coefficient set:

$$
\begin{aligned}
g_m(nT_x) &= g((n+\Delta_m)T_x) \\
\Delta_m &= \frac{mT_y}{T_x} - \left\lfloor \frac{mT_y}{T_x} \right\rfloor \in [0,1)
\end{aligned}
$$

Implementation of Sampling Rate Conversion

$$
\begin{aligned}
y(mT_y) &= \sum_{k=-\infty}^{\infty} g((k+\Delta_m)T_x) x((k_m - k)T_x) \\
&= \sum_{k=-\infty}^{\infty} g((k+\Delta_m)T_x) x((k_m - k)T_x)
\end{aligned}
$$

- $g_m(nT_x)$ may have to be retrieved or computed
- in general, there are as many weights/coefficients required as input samples $\times$ output values to compute
- in general, no simplification is possible making computation of $y(mT_y)$ from $x(nT_x)$ impractical

Linear Periodically Time-Varying Implementation

- significant simplification possible for $\frac{T_y}{T_x} = \frac{F_x}{F_y} = \frac{D}{I}$ where $D, I \in \mathbb{Z}^+$ and $\text{GCD}(D, I) = 1$

$$
\begin{aligned}
\Delta_m &= \frac{mT_y}{T_x} - \left\lfloor \frac{mT_y}{T_x} \right\rfloor = \frac{mD}{I} - \left\lfloor \frac{mD}{I} \right\rfloor \\
&= \frac{1}{I}\left(mD - \left\lfloor \frac{mD}{I} \right\rfloor I\right) = \frac{1}{I}(mD) \mod I \\
&= \frac{1}{I}(mD)_I
\end{aligned}
$$

Linear Periodically Time-Varying Implementation

Note:

$$\begin{aligned}(mD)_I &\in \{0, 1, 2, \ldots, I-1\}\\ \Delta_m &= \frac{1}{I}(mD)_I \in \{0, 1/I, 2/I, \ldots, (I-1)/I\}\end{aligned}$$

- $g_m(nT_x) = g((n + \Delta_m)T_x)$ consists only of I distinct sets of coefficients!

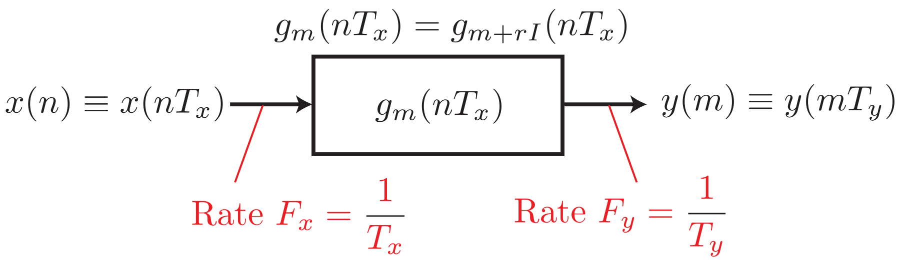
Linear Periodically Time-Varying Implementation

Furthermore, for $r \in \mathbb{Z}$:

$$\begin{aligned}\Delta_{m+rI} &= \frac{1}{I}((m+rI)D)_I = \frac{1}{I}(mD + rID)_I\\ &= \frac{1}{I}(mD)_I = \Delta_m\\ \therefore g_{m+rI}(nT_x) &= g_m(nT_x), \quad r \in \mathbb{Z}\end{aligned}$$

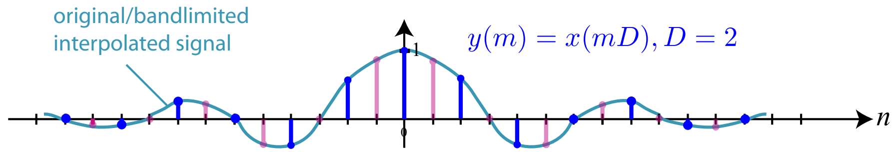
- $g_m(nT_x) = g_{m+rI}(nT_x)$ represents a discrete-time periodically time-varying system!

Linear Periodically Time-Varying Implementation

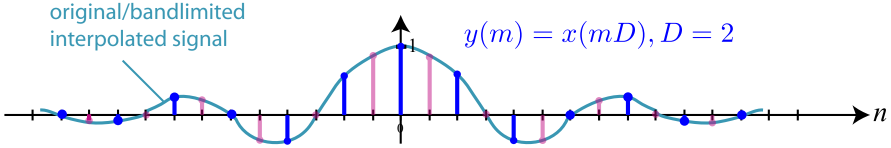


$$\frac{T_y}{T_x} = \frac{D}{I}, \quad D, I \in \mathbb{Z}, \mathrm{GCD}(D, I) = 1$$

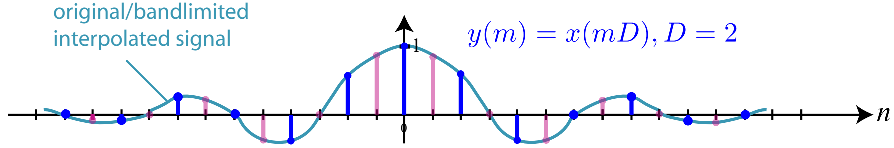
Decimation/Downsampling



$$\begin{aligned}T_y &= DT_x \Longrightarrow \frac{T_y}{T_x} = D, \quad D \in \mathbb{Z}^+\\ k_m &= \left\lfloor \frac{mT_y}{T_x} \right\rfloor = \lfloor mD \rfloor = mD \quad \because mD \in \mathbb{Z}\\ \Delta_m &= \frac{mT_y}{T_x} - \left\lfloor \frac{mT_y}{T_x} \right\rfloor = mD - \lfloor mD \rfloor = mD - mD = 0\end{aligned}$$



$$
\begin{aligned}
y(mT_y) &= \sum_{n=-\infty}^{\infty} g((n+\Delta_m)T_x)x((k_m - n)T_x) \\
&= \sum_{n=-\infty}^{\infty} g((n+0)T_x)x((mD - n)T_x) \\
&= \underbrace{\sum_{n=-\infty}^{\infty} g(nT_x)x((mD - n)T_x)}_{\text{dst-time convolution}}
\end{aligned}
$$

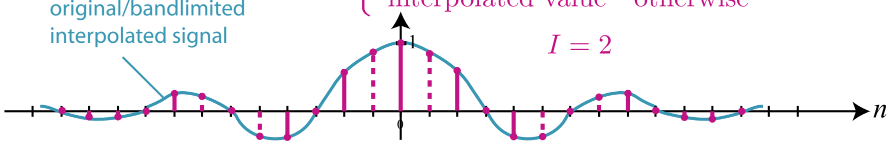


$$
\begin{aligned}
y(mT_y) &= \sum_{n=-\infty}^{\infty} g(nT_x)x((mD - n)T_x) \\
&= \sum_{n=-\infty}^{\infty} \frac{\sin(\pi n)}{\pi n}x((mD - n)T_x) \\
&= \sum_{n=-\infty}^{\infty} \delta(n)x((mD - n)T_x) = x(mDT_x)
\end{aligned}
$$

See ▸ Figure 11.1.3 of text.

Interpolation/Upsampling

$$
y(m) = \begin{cases} x(m/I) & m = 0 \pm I, \pm 2I, \ldots \\ \text{interpolated value} & \text{otherwise} \end{cases}
$$



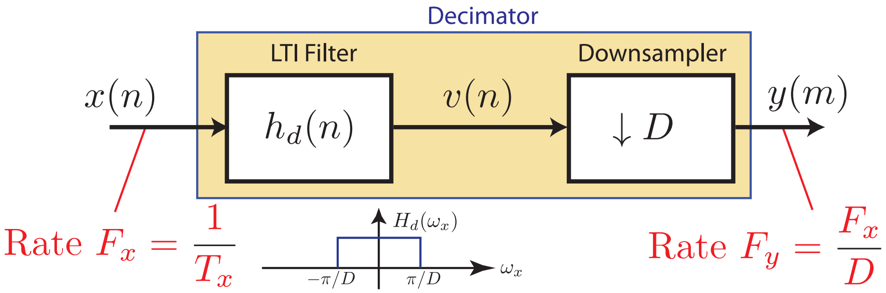
$$
\begin{aligned}
T_x &= IT_y \Longrightarrow \frac{T_y}{T_x} = \frac{1}{I}, \quad I \in \mathbb{Z}^+ \\
k_m &= \left\lfloor \frac{mT_y}{T_x} \right\rfloor = \left\lfloor \frac{m}{I} \right\rfloor \\
\Delta_m &= \frac{mT_y}{T_x} - \left\lfloor \frac{mT_y}{T_x} \right\rfloor = \frac{m}{I} - \left\lfloor \frac{m}{I} \right\rfloor \\
&\qquad \in \{0, 1/I, 2/I, \ldots, (I-1)/I\}
\end{aligned}
$$

Interpolation/Upsampling

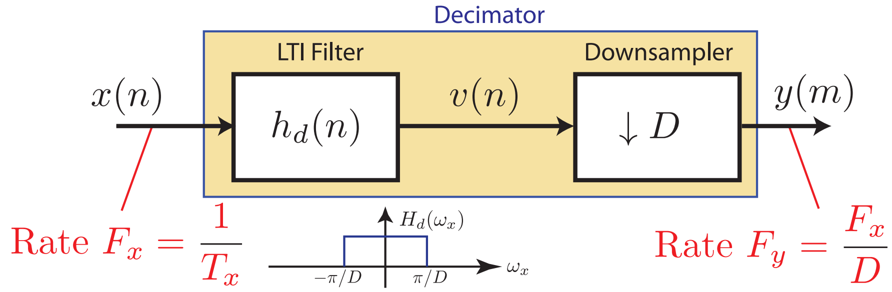
$$
\begin{aligned}
y(mT_y) &= \sum_{n=-\infty}^{\infty} x(nT_x)g(mT_y - nT_x) \\
&= \sum_{n=-\infty}^{\infty} x(nT_x)g(m\frac{T_x}{I} - nT_x) \\
&= \sum_{n=-\infty}^{\infty} x(nT_x)\frac{\sin(\pi\frac{m}{I} - \pi n)}{\pi\frac{m}{I} - \pi n} \\
&= \sum_{n=-\infty}^{\infty} x(nT_x) \underbrace{\frac{\sin(\frac{\pi}{I}(m - nI))}{\frac{\pi}{I}(m - nI)}}_{\text{sinc centered at } n = m/I}
\end{aligned}
$$

See ▸ Figure 11.1.4 of text.

Downsampling with Anti-Alaising Filter



- Downsampling alone may cause aliasing, therefore, it is desirable to introduce an anti-aliasing filter $H_d(\omega_x)$



$$v(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$y(m) = v(mD) = \underbrace{\sum_{k=-\infty}^{\infty} h(k)x(mD-k)}_{\text{linear time-varying system}}$$

Downsampling: Frequency Domain Perspective

Goal: determine relationship between input-output spectra

Create an intermediate signal $\tilde{v}(n)$ at rate F_x but with the equivalent information as $y(m)$.

$$\tilde{v}(n) = \begin{cases} v(n) & n = 0, \pm D, \pm 2D, \dots \\ 0 & \text{otherwise} \end{cases}$$

$$= v(n) \cdot p(n) \quad \text{where } p(n) = \sum_{k=-\infty}^{\infty} \delta(n-kD)$$

Note: $y(m) = v(mD) \cdot 1 = v(mD) \cdot p(mD) = \tilde{v}(mD)$

$y(m) \longleftrightarrow \tilde{v}(n)$ (equiv info)

See Figure 11.2.2 of text.

Aside: Impulse Train $p(n)$

$$p(n) = \sum_{l=-\infty}^{\infty} \delta(n-lD) \quad \text{(periodic with period } D\text{)}$$

$$c_k = \frac{1}{D}\sum_{n=0}^{D-1} p(n) e^{j2\pi kn/D}$$

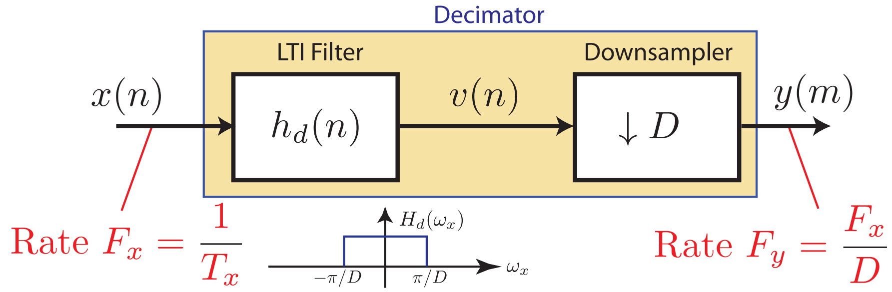
$$= \sum_{n=0}^{D-1} \underbrace{\sum_{l=-\infty}^{\infty} \delta(n-lD)}_{\text{zero for } l \neq 0} e^{j2\pi kn/D}$$

$$= \frac{1}{D}\sum_{n=0}^{D-1} \delta(n) e^{j2\pi kn/D} = \frac{1}{D}$$

$$p(n) = \sum_{k=0}^{D-1} c_k e^{j2\pi km/D} = \frac{1}{D}\sum_{k=0}^{D-1} e^{j2\pi km/D}$$

Downsampling: Frequency Domain Perspective

$$
\begin{aligned}
Y(z) &= \sum_{m=-\infty}^{\infty} y(m) z^{-m} = \sum_{m=-\infty}^{\infty} \tilde{v}(mD) z^{-m} \\
&= \cdots + \tilde{v}(-D) z^{1} + \tilde{v}(0) z^{0} + \tilde{v}(D) z^{-1} + \cdots \\
&= \sum_{m'=-\infty}^{\infty} \tilde{v}(m') z^{-m'/D} \text{ since } \tilde{v}(m) = 0 \text{ for } m \notin \{0, \pm D, \pm 2D, \ldots\} \\
&= \sum_{m'=-\infty}^{\infty} v(m) p(m) z^{-m/D} = \sum_{m=-\infty}^{\infty} v(m) \frac{1}{D} \sum_{k=0}^{D-1} e^{j2\pi km/D} z^{-m'/D} \\
&= \frac{1}{D} \sum_{k=0}^{D-1} \underbrace{\sum_{m=-\infty}^{\infty} v(m) (e^{-j2\pi k/D} z^{1/D})^{-m}}_{\equiv V(e^{-j2\pi k/D} z^{1/D}) = H_d(e^{-j2\pi k/D} z^{1/D}) X(e^{-j2\pi k/D})} \\
&= \frac{1}{D} \sum_{k=0}^{D-1} H_d(e^{-j2\pi k/D} z^{1/D}) X(e^{-j2\pi k/D} z^{1/D})
\end{aligned}
$$

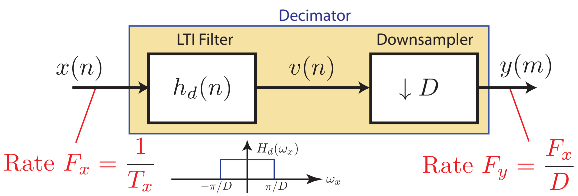


In the preceding analysis, we employed:

$$
\begin{aligned}
v(n) &\overset{Z}{\longleftrightarrow} V(z) \\
h_d(n) &\overset{Z}{\longleftrightarrow} H_d(z) \\
x(n) &\overset{Z}{\longleftrightarrow} X(z) \\
V(z) &= \sum_{m=-\infty}^{\infty} v(m) z^{-m} \\
V(z) &= H_d(z) \cdot X(z)
\end{aligned}
$$

Let $z = e^{j\omega_y}$:

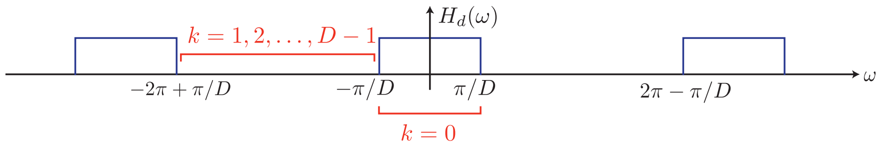
$$
\begin{aligned}
Y(z) &= \frac{1}{D} \sum_{k=0}^{D-1} H_d(e^{-j2\pi k/D} z^{1/D}) X(e^{-j2\pi k/D} z^{1/D}) \\
Y(e^{j\omega_y}) &= \frac{1}{D} \sum_{k=0}^{D-1} H_d(e^{-j2\pi k/D} e^{j\omega_y 1/D}) X(e^{-j2\pi k/D} e^{j\omega_y 1/D}) \\
&= \frac{1}{D} \sum_{k=0}^{D-1} H_d(e^{j(\omega_y - 2\pi k)/D}) X(e^{j(\omega_y - 2\pi k)/D}) \\
Y(\omega_y) &= \frac{1}{D} \sum_{k=0}^{D-1} H_d\left(\frac{\omega_y - 2\pi k}{D}\right) X\left(\frac{\omega_y - 2\pi k}{D}\right)
\end{aligned}
$$



$$
Y(\omega_y) = \frac{1}{D} \sum_{k=0}^{D-1} H_d\left(\frac{\omega_y - 2\pi k}{D}\right) X\left(\frac{\omega_y - 2\pi k}{D}\right)
$$

For $-\pi \le \omega_y \le \pi$,

$$
\begin{array}{rcccl}
-\frac{\pi}{D} \le & \frac{\omega_y - 2\pi k}{D} & \le \frac{\pi}{D} & & \text{for } k = 0 \\
-\frac{3\pi}{D} \le & \frac{\omega_y - 2\pi k}{D} & \le -\frac{\pi}{D} & & \text{for } k = 1 \\
 & \vdots & \vdots & & \\
-2\pi + \frac{\pi}{D} \le & \frac{\omega_y - 2\pi k}{D} & \le -2\pi + \frac{3\pi}{D} & & \text{for } k = D-1
\end{array}
$$



Note: For $-\pi \le \omega_y \le \pi$,

$$
H_d\left(\frac{\omega_y - 2\pi k}{D}\right) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k = 1, 2, \ldots, D-1 \end{cases}
$$

Therefore, for $-\pi \le \omega_y \le \pi$,

$$\begin{aligned} Y(\omega_y) &= \frac{1}{D}\sum_{k=0}^{D-1} H_d\left(\frac{\omega_y - 2\pi k}{D}\right) X\left(\frac{\omega_y - 2\pi k}{D}\right) \\ &= \frac{1}{D}\underbrace{H_d\left(\frac{\omega_y}{D}\right)}_{=1} X\left(\frac{\omega_y}{D}\right) + \frac{1}{D}\sum_{k=1}^{D-1}\underbrace{H_d\left(\frac{\omega_y - 2\pi k}{D}\right)}_{=0} X\left(\frac{\omega_y - 2\pi k}{D}\right) \end{aligned}$$

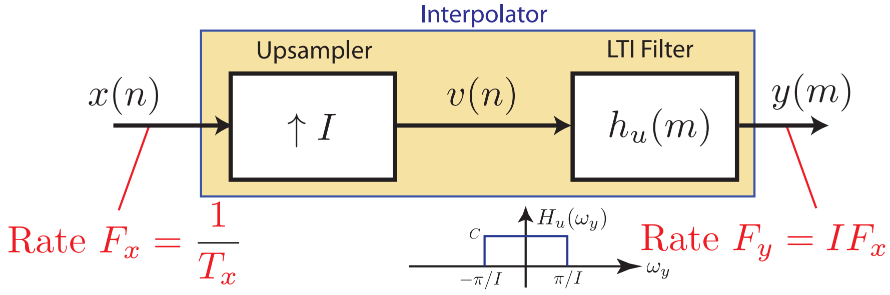
$$\boxed{Y(\omega_y) = \frac{1}{D}X\left(\frac{\omega_y}{D}\right)}$$

Note: $\omega_y = \frac{T_y}{T_x}\omega_x = D\omega_x$.

$-\frac{\pi}{D} \le \omega_x \le \frac{\pi}{D}$ of $X(\omega_x)$ is stretched into $-\pi \le \omega_y \le \pi$ for $Y(\omega_y)$.

See Figure 11.2.3 of text.

Interpolation by a Factor I

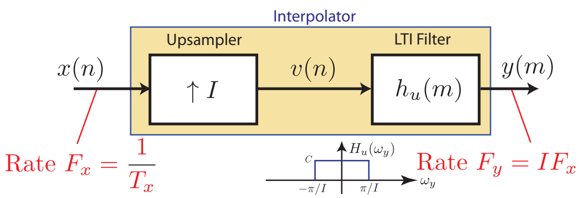


- Interpolation only increases the visible resolution of the signal.
- No information gain is achieved. At best $H_u(\omega_y)$ maintains the same information in $y(n)$ as exists in $x(n)$.

Goal: determine relationship between input-output spectra

Consider an intermediate signal $v(n)$ at rate F_y but with the equivalent information as $x(m)$.

$$\begin{aligned} v(m) &= \begin{cases} x(m/I) & m = 0, \pm I, \pm 2I, \ldots \\ 0 & \text{otherwise} \end{cases} \\ V(z) &= \sum_{m=-\infty}^{\infty} v(m) z^{-m} = \cdots + v(-I)z^{I} + v(0)z^{0} + v(I)z^{-I} + \cdots \\ &= \sum_{m=-\infty}^{\infty} x(m) z^{-mI} = \sum_{m=-\infty}^{\infty} x(m)(z^{I})^{-m} = X(z^{I}) \\ V(e^{j\omega_y}) &= X(e^{j\omega_y I}) \quad \Longrightarrow V(\omega_y) = X(\omega_y I) \end{aligned}$$



$$V(\omega_y) = X(\omega_y I)$$

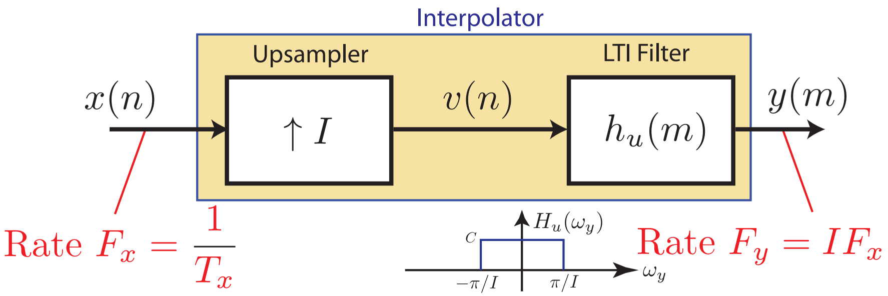
See Figure 11.3.1 of text.

$$\begin{aligned} H_u(\omega_y) &= \begin{cases} I & 0 \le |\omega_y| \le \pi/I \\ 0 & \text{otherwise} \end{cases} \\ Y(\omega_y) &= H_u(\omega_y)V(\omega_y) = \begin{cases} IX(\omega_y I) & 0 \le |\omega_y| \le \pi/I \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$\boxed{Y(\omega_y) = \begin{cases} IX(\omega_y I) & 0 \le |\omega_y| \le \pi/I \\ 0 & \text{otherwise} \end{cases}}$$

Note: $\omega_y = \frac{T_y}{T_x} = \frac{\omega_x}{I}$.

$-\pi \le \omega_x \le \pi$ is compressed into $-\pi/I \le \omega_y \le \pi/I$



$$y(m) = \sum_{k=-\infty}^{\infty} h_u(m-k)v(k)$$

$$\because v(m) = \begin{cases} x(m/I) & m = 0, \pm I, \pm 2I, \ldots \\ 0 & \text{otherwise} \end{cases} \Rightarrow v(k) = 0 \text{ for } (k)_I \neq 0$$

$$\therefore y(m) = \sum_{k=-\infty}^{\infty} h_u(m-kI)\underbrace{v(kI)}_{=x(k)} = \underbrace{\sum_{k=-\infty}^{\infty} h_u(m-kI)x(k)}_{\text{linear time-varying system}}$$

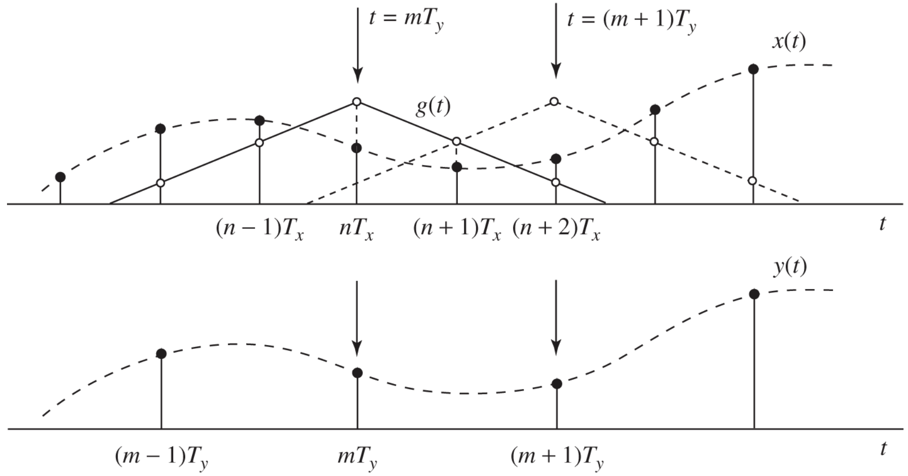


Figure 11.1.3 Illustration of timing relations for sampling rate decrease by an integer factor $D = 2$. A single impulse response, sampled with period T_x, is shifted at steps equal to $T_y = DT_x$ to generate the output samples.

Return

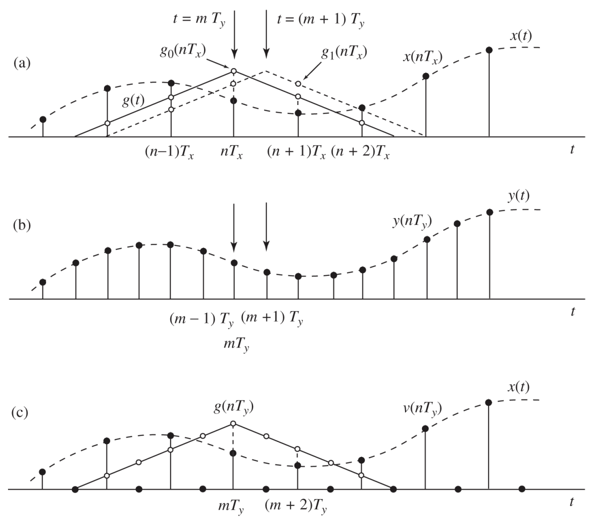


Figure 11.1.4 Illustration of timing relations for sampling rate increase by an integer factor $I = 2$. The approach in (a) requires one impulse response for the even-numbered and one for the odd-numbered output samples. The approach in (c) requires only one impulse response, obtained by interleaving the impulse responses in (a).

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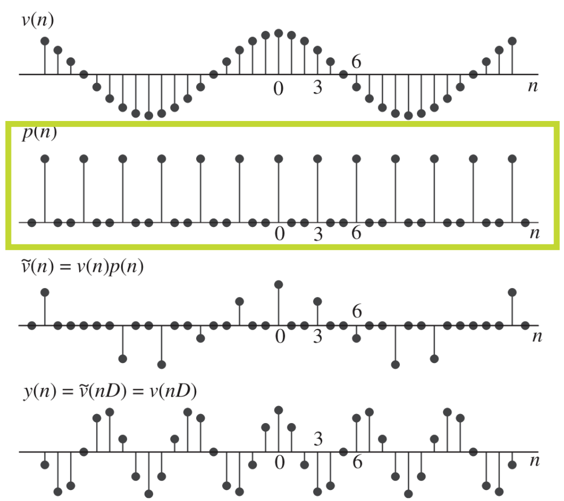

Figure 11.2.2 Steps required to facilitate the mathematical description of downsampling by a factor D, using a sinusoidal sequence for illustration.

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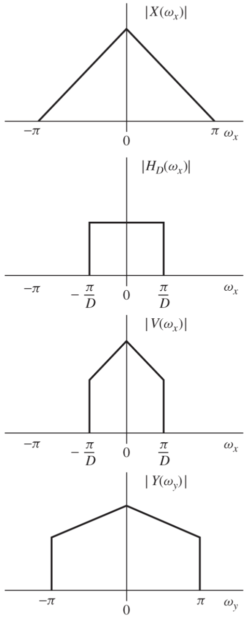

Figure 11.2.3 Spectra of signals in the decimation of $x(n)$ by a factor D.

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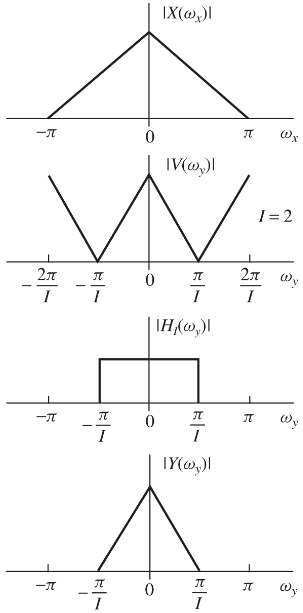

Figure 11.3.1 Spectra of $x(n)$ and $v(n)$ where $V(\omega_y) = X(\omega_y l)$.

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