The z-Transform and Its Application

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Discrete-Time Signals and Systems

Reference:

Sections 3.1 - 3.4 of

John G. Proakis and Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, 4th edition, 2007.

The Direct z-Transform

- Direct z-Transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

- Notation:

$$X(z) \equiv \mathcal{Z}\{x(n)\}$$

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

Region of Convergence

- the region of convergence (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value
- The z-Transform is, therefore, uniquely characterized by:
  1. expression for $X(z)$
  2. ROC of $X(z)$

Power Series Convergence

- For a power series,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n = a_0 + a_1(z-c) + a_2(z-c)^2 + \cdots$$

there exists a number $0 \le r \le \infty$ such that the series
  - convergences for $|z-c| < r$, and
  - diverges for $|z-c| > r$
  - may or may not converge for values on $|z-c| = r$.

Power Series Convergence

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$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^{-n} = a_0 + \frac{a_1}{(z-c)} + \frac{a_2}{(z-c)^2} + \cdots$$

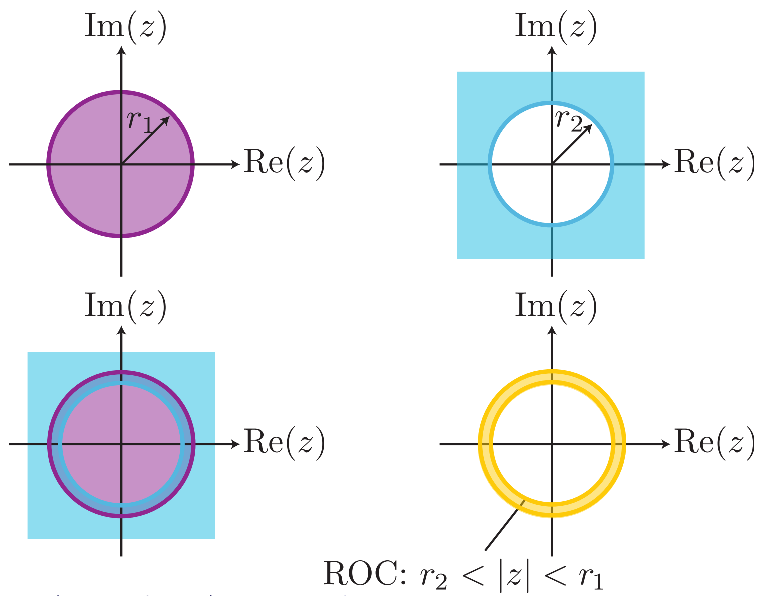
there exists a number $0 \le r \le \infty$ such that the series
  - convergences for $|z-c| > r$, and
  - diverges for $|z-c| < r$
  - may or may not converge for values on $|z-c| = r$.

Region of Convergence

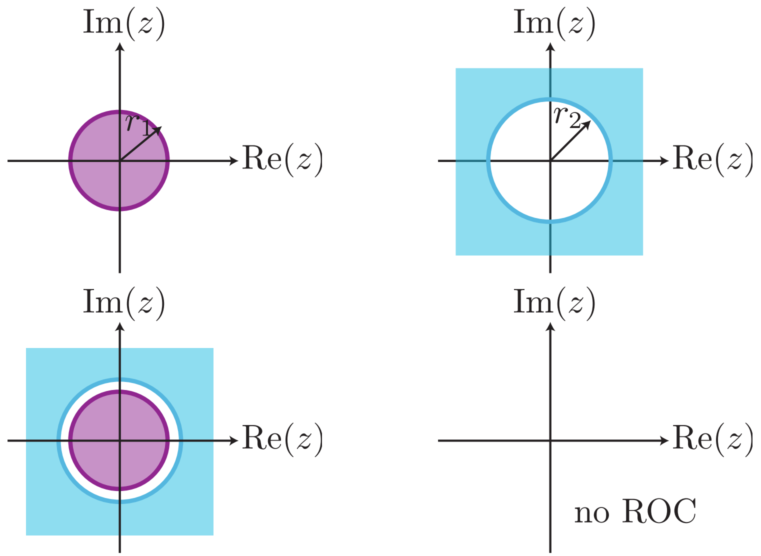
- Consider

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x(n) z^{-n} \\
&= \sum_{n=-\infty}^{-1} x(n) z^{-n} + \sum_{n=0}^{\infty} x(n) z^{-n} \\
&= \underbrace{\sum_{n'=0}^{\infty} x(-n') z^{n'}}_{\text{ROC: } |z| < r_1} + \underbrace{\sum_{n=0}^{\infty} x(n) z^{-n}}_{\text{ROC: } |z| > r_2} - \underbrace{x(0)}_{\text{ROC: all } z}
\end{aligned}$$

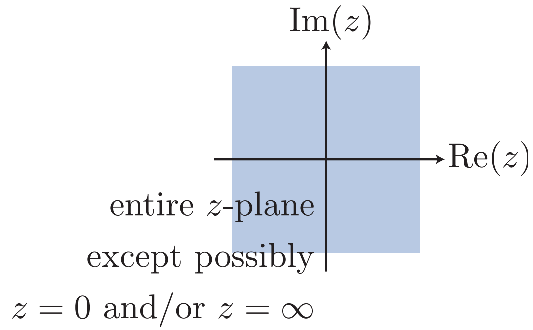
Region of Convergence: $r_1 > r_2$



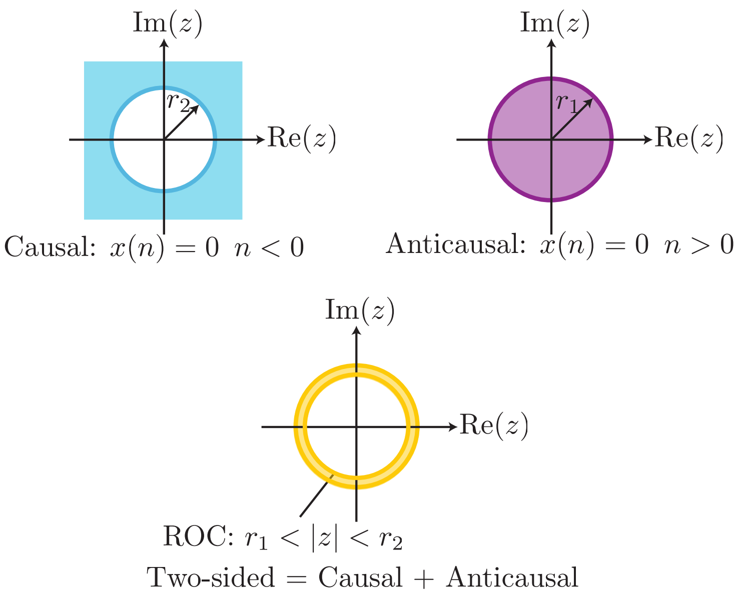
Region of Convergence: $r_1 < r_2$



ROC Families: Finite Duration Signals



ROC Families: Infinite Duration Signals



z-Transform Properties

Property	Time Domain	z-Domain	ROC
Notation:	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_1(z)$	ROC_2
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least $\mathrm{ROC}_1 \cap \mathrm{ROC}_2$
Time shifting:	$x(n-k)$	$z^{-k}X(z)$	ROC, except $z = 0$ (if $k > 0$) and $z = \infty$ (if $k < 0$)
z-Scaling:	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation:	$x^*(n)$	$X^*(z^*)$	ROC
z-Differentiation:	$n\ x(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution:	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least $\mathrm{ROC}_1 \cap \mathrm{ROC}_2$

among others ...

Convolution Property

$$x(n) = x_1(n) * x_2(n) \iff X(z) = X_1(z) \cdot X_2(z)$$

Convolution using the z-Transform

Basic Steps:

1. Compute z-Transform of each of the signals to convolve (time domain $\rightarrow$ z-domain):

$$\begin{aligned} X_1(z) &= \mathcal{Z}\{x_1(n)\} \\ X_2(z) &= \mathcal{Z}\{x_2(n)\} \end{aligned}$$

2. Multiply the two z-Transforms (in z-domain):

$$X(z) = X_1(z)X_2(z)$$

3. Find the inverse z-Transformof the product (z-domain $\rightarrow$ time domain):

$$x(n) = \mathcal{Z}^{-1}\{X(z)\}$$

Common Transform Pairs

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7	$\cos(\omega_0 n)u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
8	$\sin(\omega_0 n)u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
9	$a^n\cos(\omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
10	$a^n\sin(\omega_0 n)u(n)$	$\frac{1-az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

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10	$a^n\sin(\omega_0 n)u(n)$	$\frac{1-az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Why Rational?

- $X(z)$ is a rational function iff it can be represented as the ratio of two polynomials in z^{-1} (or z):

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

- For LTI systems that are represented by LCCDEs, the z-Transform of the unit sample response $h(n)$, denoted $H(z) = \mathcal{Z}\{h(n)\}$, is rational

Poles and Zeros

- zeros of $X(z)$: values of z for which $X(z) = 0$
- poles of $X(z)$: values of z for which $X(z) = \infty$

Poles and Zeros of the Rational z-Transform

Let $a_0, b_0 \neq 0$:

$$\begin{aligned}
X(z) = \frac{B(z)}{A(z)} &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} \\
&= \left(\frac{b_0 z^{-M}}{a_0 z^{-N}}\right) \frac{z^M + (b_1/b_0) z^{M-1} + \cdots + b_M/b_0}{z^N + (a_1/a_0) z^{N-1} + \cdots + a_N/a_0} \\
&= \frac{b_0}{a_0} z^{-M+N} \frac{(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_N)} \\
&= G z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}
\end{aligned}$$

Poles and Zeros of the Rational z-Transform

$$X(z) = G z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)} \quad \text{where } G \equiv \frac{b_0}{a_0}$$

Note: "finite" does not include zero or ∞.

- $X(z)$ has M finite zeros at $z = z_1, z_2, \ldots, z_M$
- $X(z)$ has N finite poles at $z = p_1, p_2, \ldots, p_N$
- For $N - M \neq 0$
  - if $N - M > 0$, there are $|N - M|$ zero at origin, $z = 0$
  - if $N - M < 0$, there are $|N - M|$ poles at origin, $z = 0$

$$\boxed{\text{Total number of zeros} = \text{Total number of poles}}$$

Poles and Zeros of the Rational z-Transform

Example:

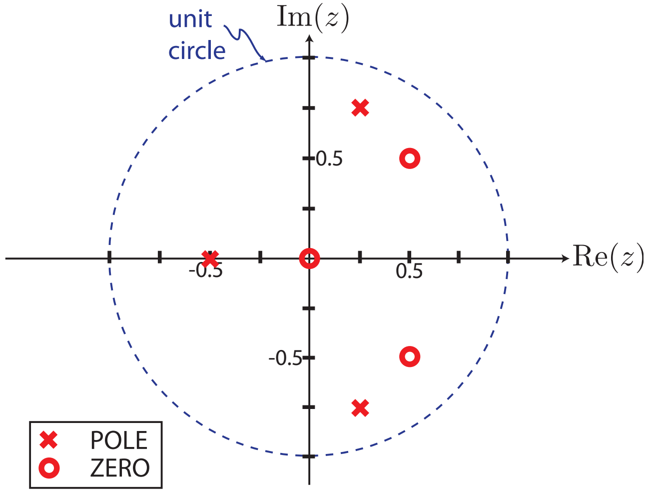
$$X(z) = z\frac{2z^2 - 2z + 1}{16z^3 + 6z + 5} = (z - 0)\frac{(z - (\frac{1}{2} + j\frac{1}{2}))(z - (\frac{1}{2} - j\frac{1}{2}))}{(z - (\frac{1}{4} + j\frac{3}{4}))(z - (\frac{1}{4} - j\frac{3}{4}))(z - (-\frac{1}{2}))}$$

$\boxed{\text{poles: } z = \frac{1}{4} \pm j\frac{3}{4}, -\frac{1}{2}}$

$\boxed{\text{zeros: } z = 0, \frac{1}{2} \pm j\frac{1}{2}}$

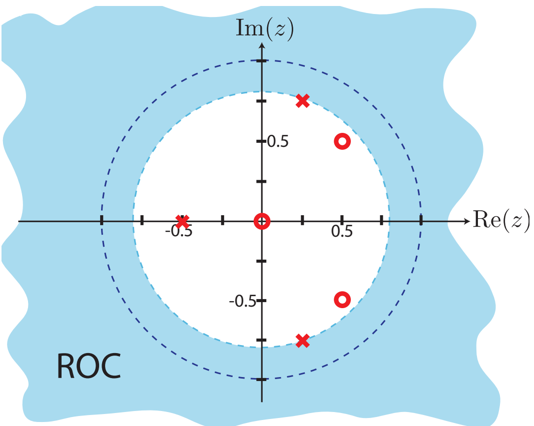
Pole-Zero Plot

Example: $\boxed{\text{poles: } z = \frac{1}{4} \pm j\frac{3}{4}, -\frac{1}{2}}$, $\boxed{\text{zeros: } z = 0, \frac{1}{2} \pm j\frac{1}{2}}$



Pole-Zero Plot

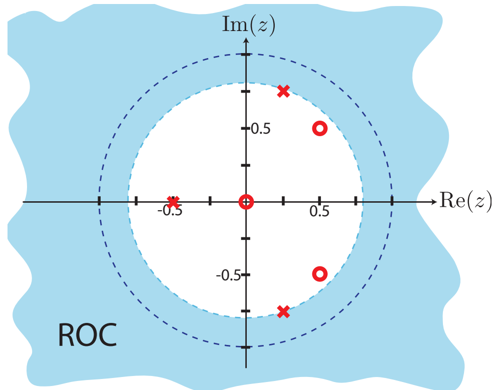
- Graphical interpretation of characteristics of $X(z)$ on the complex plane
- ROC cannot include poles; assuming causality ...



Pole-Zero Plot

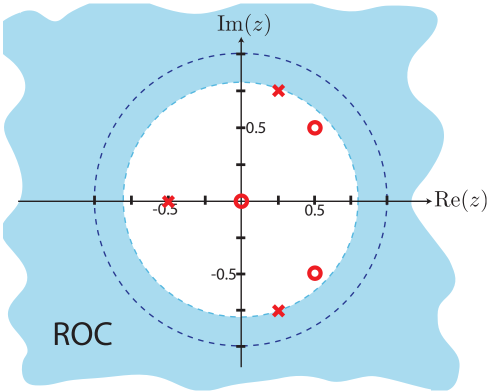
- For real time-domain signals, the coefficients of $X(z)$ are necessarily real
  - complex poles and zeros must occur in conjugate pairs
  - note: real poles and zeros do not have to be paired up

$$X(z) = z\frac{2z^2 - 2z + 1}{16z^3 + 6z + 5} \Longrightarrow$$



Pole-Zero Plot Insights

- For causal systems, the ROC will be the outer region of the smallest (origin-centered) circle encompassing all the poles.
- For stable systems, the ROC will include the unit circle.



- Causal? Yes.
- Stable? Yes.

- For stability of a causal system, the poles will lie inside the unit circle.

The System Function

$$\begin{aligned} h(n) &\overset{\mathcal{Z}}{\longleftrightarrow} H(z) \\ \text{time-domain} &\overset{\mathcal{Z}}{\longleftrightarrow} z\text{-domain} \\ \text{impulse response} &\overset{\mathcal{Z}}{\longleftrightarrow} \text{system function} \end{aligned}$$

$$y(n) = x(n) * h(n) \overset{\mathcal{Z}}{\longleftrightarrow} Y(z) = X(z) \cdot H(z)$$

Therefore,

$$H(z) = \frac{Y(z)}{X(z)}$$

The System Function of LCCDEs

$$\begin{aligned} y(n) &= -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \\ \mathcal{Z}\{y(n)\} &= \mathcal{Z}\{-\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)\} \\ \mathcal{Z}\{y(n)\} &= -\sum_{k=1}^{N} a_k \mathcal{Z}\{y(n-k)\} + \sum_{k=0}^{M} b_k \mathcal{Z}\{x(n-k)\} \\ Y(z) &= -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z) \end{aligned}$$

The System Function of LCCDEs

$$\begin{aligned} Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) &= \sum_{k=0}^{M} b_k z^{-k} X(z) \\ Y(z) \left[1 + \sum_{k=1}^{N} a_k z^{-k} \right] &= X(z) \sum_{k=0}^{M} b_k z^{-k} \\ H(z) = \frac{Y(z)}{X(z)} &= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right]} \\ \text{LCCDE} &\longleftrightarrow \text{Rational System Function} \end{aligned}$$

Many signals of practical interest have a rational z-Transform.

Inversion of the z-Transform

Three popular methods:

- Contour integration:
$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$
- Expansion into a power series in z or z^{-1}:
$$X(z) = \sum_{k=-\infty}^{\infty} x(k)z^{-k}$$
and obtaining $x(k)$ for all k by inspection
- Partial-fraction expansion and table lookup

Expansion into Power Series

Example:

$$\begin{aligned} X(z) &= \log(1+az^{-1}), \quad |z|>|a| \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}a^n z^{-n}}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}a^n}{n}z^{-n} \end{aligned}$$

By inspection:

$$x(n) = \begin{cases} \frac{(-1)^{n+1}a^n}{n} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

Partial-Fraction Expansion

1. Find the distinct poles of $X(z)$: $p_1, p_2, \ldots, p_K$ and their corresponding multiplicities $m_1, m_2, \ldots, m_K$.
2. The partial-fraction expansion is of the form:
$$X(z) = \sum_{k=1}^{K}\left(\frac{A_{1k}}{z-p_k} + \frac{A_{2k}}{(z-p_k)^2} + \cdots + \frac{A_{mk}}{(z-p_k)^{m_k}}\right)$$
where p_k is an m_kth order pole (i.e., has multiplicity m_k).
3. Use an appropriate approach to compute $\{A_{ik}\}$

Partial-Fraction Expansion

Example: Find $x(n)$ given poles of $X(z)$ at $p_1 = -2$ and a double pole at $p_2 = p_3 = 1$; specifically,

$$\begin{aligned} X(z) &= \frac{1}{(1+2z^{-1})(1-z^{-1})^2} \\ \frac{X(z)}{z} &= \frac{z^2}{(z+2)(z-1)^2} \\ \frac{z^2}{(z+2)(z-1)^2} &= \frac{A_1}{z+2} + \frac{A_2}{z-1} + \frac{A_3}{(z-1)^2} \end{aligned}$$

Note: we need a strictly proper rational function.
DO NOT FORGET TO MULTIPLY BY z IN THE END.

Partial-Fraction Expansion

$$\frac{z^2(z+2)}{(z+2)(z-1)^2} = \frac{A_1(z+2)}{z+2} + \frac{A_2(z+2)}{z-1} + \frac{A_3(z+2)}{(z-1)^2}$$
$$\frac{z^2}{(z-1)^2} = A_1 + \frac{A_2(z+2)}{z-1} + \left.\frac{A_3(z+2)}{(z-1)^2}\right|_{z=-2}$$
$$A_1 = \frac{4}{9}$$

Partial-Fraction Expansion

$$\frac{z^2(z-1)^2}{(z+2)(z-1)^2} = \frac{A_1(z-1)^2}{z+2} + \frac{A_2(z-1)^2}{z-1} + \frac{A_3(z-1)^2}{(z-1)^2}$$
$$\frac{z^2}{(z+2)} = \frac{A_1(z-1)^2}{z+2} + A_2(z-1) + \left.A_3\right|_{z=1}$$
$$A_3 = \frac{1}{3}$$

Partial-Fraction Expansion

$$\frac{z^2(z-1)^2}{(z+2)(z-1)^2} = \frac{A_1(z-1)^2}{z+2} + \frac{A_2(z-1)^2}{z-1} + \frac{A_3(z-1)^2}{(z-1)^2}$$
$$\frac{z^2}{(z+2)} = \frac{A_1(z-1)^2}{z+2} + A_2(z-1) + A_3$$
$$\frac{d}{dz}\left[\frac{z^2}{(z+2)}\right] = \left.\frac{d}{dz}\left[\frac{A_1(z-1)^2}{z+2} + A_2(z-1) + A_3\right]\right|_{z=1}$$
$$A_2 = \frac{5}{9}$$

Partial-Fraction Expansion

Therefore, assuming causality, and using the following pairs:

$$a^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}}$$
$$na^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{az^{-1}}{(1-az^{-1})^2}$$

$$X(z) = \frac{4}{9}\frac{1}{1+2z^{-1}} + \frac{5}{9}\frac{1}{1-z^{-1}} + \frac{1}{3}\frac{z^{-1}}{(1-z^{-1})^2}$$
$$x(n) = \frac{4}{9}(-2)^n u(n) + \frac{5}{9}u(n) + \frac{1}{3}nu(n)$$
$$= \left[\frac{(-2)^{n+2}}{9} + \frac{5}{9} + \frac{n}{3}\right]u(n)$$

$\blacksquare$