Frequency Analysis of Signals

Dr. Deepa Kundur

University of Toronto

Discrete-Time Signals and Systems

Reference:

Sections 4.1 - 4.3 of

John G. Proakis and Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, 4th edition, 2007.

Continuous-Time Fourier Series (CTFS)

- For continuous-time periodic signals
- Synthesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

- Analysis equation:

$$c_k = \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi k F_0 t}$$

- Convergence?

Continuous-Time Fourier Series (CTFS)

- **Q:** For what conditions is $\sum_{\forall k} c_k e^{j2\pi k F_0 t}$ equal to $x(t)$?
- **A:** Sufficient conditions are given by Dirichlet conditions:
    1. $x(t)$ has a finite number of discontinuities in any period.
    2. $x(t)$ contains a finite number of maxima and minima during any period.
    3. $x(t)$ is absolutely integrable in any period:

$$\int_{T_p} |x(t)| dt < \infty$$

- Note: the Dirichlet conditions guarantee equality except at values of t for which $x(t)$ is discontinuous.
    - At discontinuities, $\sum_{\forall k} c_k e^{j2\pi k F_0 t}$ convergences to the midpoint of the discontinuity.

Continuous-Time Fourier Transform (CTFT)

- For continuous-time aperiodic signals
- CTFT pair using cyclic frequency:

$$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi F t} dF$$
$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi F t} dt$$

- CTFT pair using radian frequency:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$$
$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

Continuous-Time Fourier Transform (CTFT)

- CTFT convergence is guaranteed for Dirichlet conditions outlined previously allowing $T_p \to \infty$
  1. $x(t)$ has a finite number of finite discontinuities.
  2. $x(t)$ has a finite number of maxima and minima.
  3. $x(t)$ is absolutely integrable:

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

Discrete-Time Fourier Series (DTFS)

- For discrete-time periodic signals
- DTFS pair:

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$
$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

DTFS vs. CTFS

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \qquad \text{vs.} \qquad x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

- Continuous-time sinusoids are unique for distinct frequencies
  - $e^{j\frac{2}{3}\pi t} \neq e^{-j\frac{16}{3}\pi t}$
- Discrete-time sinusoids with cyclic frequencies an integer number apart are the same;
  - $e^{j\frac{2}{3}\pi n} = e^{-j\frac{16}{3}\pi n}$
- There are only N distinct discrete-time harmonics; see Chapter 1 notes.

Nature of the DTFS

- Follows from duality; recall,

$$\begin{aligned}
\text{time domain} &\overset{\mathcal{F}}{\longleftrightarrow} \text{frequency domain}\\
\text{rectangle} &\overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}\\
\text{sinc} &\overset{\mathcal{F}}{\longleftrightarrow} \text{rectangle}\\
\text{convolution} &\overset{\mathcal{F}}{\longleftrightarrow} \text{multiplication}\\
\text{multiplication} &\overset{\mathcal{F}}{\longleftrightarrow} \text{convolution}\\
\text{periodic} &\overset{\mathcal{F}}{\longleftrightarrow} \text{discrete}\\
\text{discrete} &\overset{\mathcal{F}}{\longleftrightarrow} \text{periodic}\\
\text{periodic} + \text{discrete} &\overset{\mathcal{F}}{\longleftrightarrow} \text{discrete} + \text{periodic}
\end{aligned}$$

Nature of the DTFS

- Therefore,

$$\text{periodic} + \text{discrete} \overset{\mathcal{F}}{\longleftrightarrow} \text{periodic} + \text{discrete}$$

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \overset{\mathcal{F}}{\longleftrightarrow} c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

$$\text{for all } n \in \mathbb{Z} \qquad\qquad \text{for all } k \in \mathbb{Z}$$

- Q: Why is there a $\frac{1}{N}$ term?
  - A: Because $\{e^{j2\pi kn/N}\}$ are orthogonal not orthonormal.

DTFS Example

See Figure 4.2.2 of text and Figure 4.2.3 of text.

Discrete-Time Fourier Transform (DTFT)

- Duality:

$$\text{aperiodic} + \text{discrete} \overset{\mathcal{F}}{\longleftrightarrow} \text{continuous} + \text{periodic}$$

- DTFT pair:

$$\begin{aligned}
x(n) &= \frac{1}{2\pi}\int_{2\pi} X(\omega) e^{j\omega n} d\omega\\
X(\omega) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}
\end{aligned}$$

- $X(\omega)$ is the decomposition of $x(n)$ into its frequency components.

Nature of the DTFT

- When dealing with discrete frequencies, only a continuous frequency range of length 2π needs to be considered. Recall,
  - $X(\omega) = X(\omega + 2\pi)$
  - Minimum frequency for $\omega = 2k\pi$, $k \in \mathbb{Z}$
  - Maximum frequency for $\omega = (2k+1)\pi$, $k \in \mathbb{Z}$
  - Convention is to use $\omega \in [0, 2\pi)$ or $\omega \in (-\pi, \pi]$
- Frequency range of a discrete-time signal is considered to be $\omega \in (-\pi, \pi]$

DTFT Example

See Figure 4.2.4 of text.

Summary of Fourier Transforms

See Figure 4.3.1 of text.

DTFT Symmetry Properties

- Notation:

$$X(\omega) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$x(n) \equiv \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega$$

$$x(n) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

- Let $x(n) = x_R(n) + jx_I(n)$ and $X(\omega) = X_R(\omega) + jX_I(\omega)$ where $x_R(n), x_I(n), X_R(\omega), X_I(\omega) \in \mathbb{R}$

DTFT Symmetry Properties

Time Sequence	DTFT
$x(n)$	$X(\omega)$
$x^*(n)$	$X^*(-\omega)$
$x^*(-n)$	$X^*(\omega)$
$x(-n)$	$X(-\omega)$
$x_R(n)$	$\frac{1}{2}[X(\omega) + X^*(-\omega)]$
$jx_I(n)$	$\frac{1}{2}[X(\omega) - X^*(-\omega)]$
$x(n)$ real	$X(\omega) = X^*(-\omega)$
	$X_R(\omega) = X_R(-\omega)$
	$X_I(\omega) = -X_I(-\omega)$
	$\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$
	$\angle X(\omega) = -\angle X(-\omega)$
$x_e'(n) = \frac{1}{2}[x(n) + x^*(-n)]$	$X_R(\omega)$
$x_o'(n) = \frac{1}{2}[x(n) - x^*(-n)]$	$jX_I(\omega)$

DTFT Example

Find the DTFT of

$$x(n) = \begin{cases} A & -M \le n \le M \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=-M}^{M} Ae^{-j\omega n} = A \sum_{n=-M}^{M} e^{-j\omega n} \\ &= A \sum_{n=-M}^{M} [e^{-j\omega}]^n = A\frac{\sin((M+\frac{1}{2})\omega)}{\sin(\omega/2)} \end{aligned}$$

See Figure 4.4.5 of text

DTFT Theorems and Properties

Property	Time Domain	Frequency Domain
Notation:	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_1(\omega)$
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting:	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution:	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation:	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$ $= X_1(\omega)X_2^*(\omega)$ [if $x_2(n)$ real]
Wiener-Khintchine:	$r_{xx}(l) = x(l) * x(-l)$	$S_{xx}(\omega) = \lvert X(\omega)\rvert^2$ [if $x(n)$ real]

among others . . .

Wiener-Khintchine Theorem

- Consider

$$x(n) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

$$r_{xx}(l) = x(l) * x(-l) \stackrel{\mathcal{F}}{\longleftrightarrow} S_{xx}(\omega) = \lvert X(\omega)\rvert^2$$

- Lack of phase information in $S_{xx}(\omega)$ suggests that $x(n)$ cannot uniquely be reconstructed from $r_{xx}(l)$.

■