

Chapter 4: Frequency Analysis of Signals 4.1 Frequency Analysis of Continuous-Time Signals

- ► For continuous-time periodic signals
- Synthesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

► Analysis equation:

$$c_k = \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi k F_0 t}$$

Convergence?

Discrete-Time Signals and Systems

Reference:

Sections 4.1 - 4.3 of

John G. Proakis and Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, 4th edition, 2007.

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Chapter 4: Frequency Analysis of Signals 4.1 Frequency Analysis of Continuous-Time Signals

Continuous-Time Fourier Series (CTFS)

- **Q:** For what conditions is $\sum_{\forall k} c_k e^{j2\pi kF_0 t}$ equal to x(t)?
- A: Sufficient conditions are given by Dirichlet conditions:
 - 1. x(t) has a finite number of discontinuities in any period.
 - 2. x(t) contains a finite number of maxima and minima during any period.
 - 3. x(t) is absolutely integrable in any period:

$$\int_{\mathcal{T}_p} |x(t)| dt < \infty$$

- Note: the Dirichlet conditions guarantee equality except at values of t for which x(t) is discontinuous.
 - At discontinuities, $\sum_{\forall k} c_k e^{j2\pi kF_0 t}$ convergences to the midpoint of the discontinuity.

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Chapter 4: Frequency Analysis of Signals 4.1 Frequency Analysis of Continuous-Time Signals

Continuous-Time Fourier Transform (CTFT)

- ► For continuous-time aperiodic signals
- CTFT pair using cyclic frequency:

$$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF$$
$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt$$

CTFT pair using radian frequency:

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega \\ X(\Omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \end{aligned}$$

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Chapter 4: Frequency Analysis of Signals 4.2 Frequency Analysis of Discrete-Time Signals

Discrete-Time Fourier Series (DTFS)

- ► For discrete-time periodic signals
- ► DTFS pair:

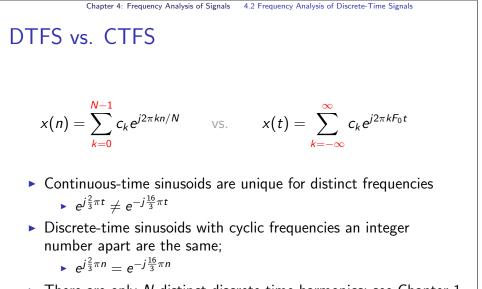
$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$
$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

Chapter 4: Frequency Analysis of Signals 4.1 Frequency Analysis of Continuous-Time Signals Continuous-Time Fourier Transform (CTFT) • CTFT convergence is guaranteed for Dirichlet conditions outlined previously allowing $T_p \to \infty$ 1. x(t) has a finite number of finite discontinuities. 2. x(t) has a finite number of maxima and minima. 3. x(t) is absolutely integrable: $\int_{-\infty}^{\infty} |x(t)| dt < \infty$

Frequency Analysis of Signals

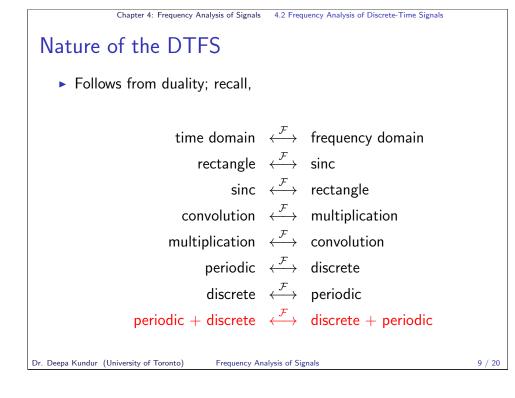
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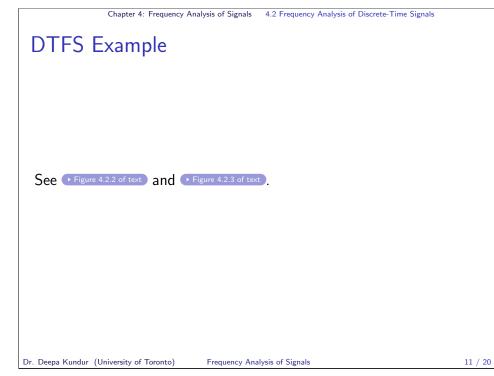
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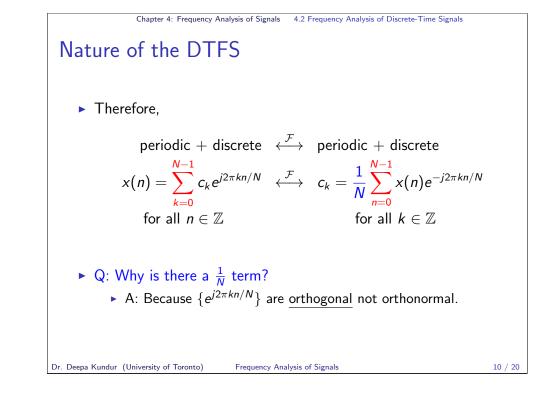


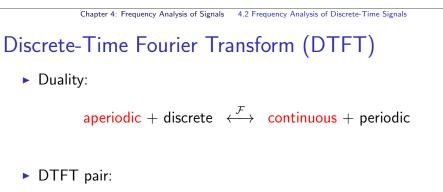
 There are only N distinct discrete-time harmonics; see Chapter 1 notes.

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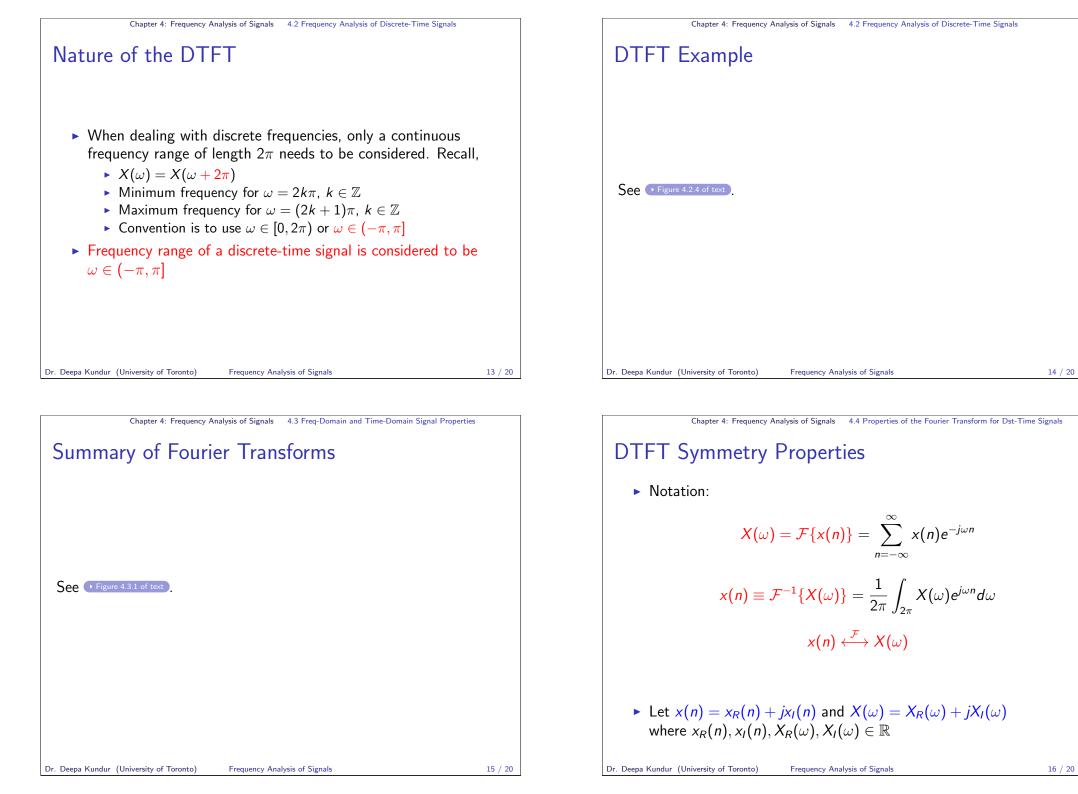




$$x(n) = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

- X(ω) is the decomposition of x(n) into its frequency components.
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	Chapter 4: Frequency Analysis of Signals 4	4 Properties of the Fourier Transform for	Dst-Time Signals
DTFT S	Symmetry Propertie	2S	
	Time Sequence	DTFT	
	x(n)	$X(\omega)$	
	$x^*(n)$	$X^*(-\omega)$	
	$x^{*}(-n)$	$X^*(\omega)$	
	x(-n)	$X(-\omega)$	
	$x_R(n)$ $jx_I(n)$	$rac{1}{2}[X(\omega) + X^*(-\omega)]$ $rac{1}{2}[X(\omega) - X^*(-\omega)]$	
		$\frac{2[X(\omega) - X^*(-\omega)]}{X(\omega) = X^*(-\omega)}$	
		$X_R(\omega) = X_R(-\omega)$	
	x(n) real	$X_{I}(\omega) = -X_{I}(-\omega)$	
		$ X(\omega) = X(-\omega) $	
		$\angle X(\omega) = -\angle X(-\omega)$	
	$x'_{e}(n) = \frac{1}{2}[x(n) + x^{*}(-n)]$	$X_R(\omega)$	
	$x'_o(n) = \frac{1}{2}[x(n) - x^*(-n)]$	$jX_{I}(\omega)$	
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Chapter 4: Frequency Analysis of Signals 4.4 Properties of the Fourier Transform for Dst-Time Signals

DTFT Theorems and Properties

Property	Time Domain	Frequency Domain
Notation:	<i>x</i> (<i>n</i>)	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_1(\omega)$
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting:	x(n-k)	$e^{-j\omega k}X(\omega)$
Time reversal	x(-n)	$X(-\omega)$
Convolution:		$X_1(\omega)X_2(\omega)$
Correlation:	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$
		$= X_1(\omega) X_2^*(\omega) \text{ [if } x_2(n) \text{ real]}$
Wiener-Khintchine:	$r_{xx}(l) = x(l) * x(-l)$	
		[if $x(n)$ real]
		among others
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DTFT Example
Find the DTFT of

$$x(n) = \begin{cases} A & -M \le n \le M \\ 0 & \text{elsewhere} \end{cases}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=-M}^{M} Ae^{-j\omega n} = A \sum_{n=-M}^{M} e^{-j\omega n}$$

$$= A \sum_{n=-M}^{M} [e^{-j\omega}]^n = A \frac{\sin((M + \frac{1}{2})\omega)}{\sin(\omega/2)}$$
See Figure 4.4.5 of text
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Chapter 4: Frequency Analysis of Signals 4.4 Properties of the Fourier Transform for Dst-Time Signals

Wiener-Khintchine Theorem

Consider

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$$\begin{array}{rcl} x(n) & \stackrel{\mathcal{F}}{\longleftrightarrow} & X(\omega) \\ r_{xx}(l) = x(l) * x(-l) & \stackrel{\mathcal{F}}{\longleftrightarrow} & S_{xx}(\omega) = |X(\omega)|^2 \end{array}$$

 Lack of phase information in S_{xx}(ω) suggests that x(n) cannot uniquely be reconstructed from r_{xx}(l).

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