Efficient Computation of the DFT: FFT Algorithms

Dr. Deepa Kundur

University of Toronto

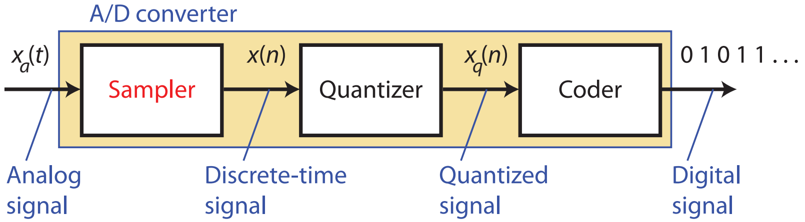
Discrete-Time Signals and Systems

Reference:

Sections 6.1, 6.2, 6.4, 6.5 of

John G. Proakis and Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, 4th edition, 2007.

Analog-to-Digital Conversion



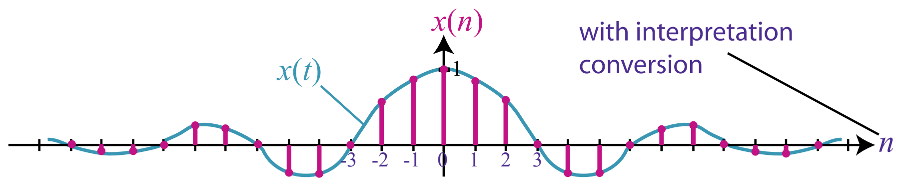
Sampling:

- conversion from cts-time to dst-time by taking "samples" at discrete time instants
- E.g., uniform sampling: $x(n) = x_a(nT)$ where T is the sampling period

Sampling Process

- To effectively reconstruct an analog signal from its samples, the sampling frequency $F_s = \frac{1}{T}$ must be selected to be "large enough".
- Sampling in the time-domain:

$$x(n) = x_a(nT), \qquad -\infty < n < \infty$$



Return

Sampling Process

- Time-Domain Sampling: frequency-domain perspective
  - The sampling frequency $F_s = \frac{1}{T}$ must be selected to be large enough such that the sampling processing does not cause any loss of spectral information (i.e., no aliasing).
- Recall CTFT and DTFT for aperiodic $x_a(t)$ and $x(n)$:

$$x_a(t) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi Ft} dF \qquad x(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f) e^{j2\pi fn} df$$

$$X_a(F) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi Ft} dt \qquad X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi fn}$$

Sampling Process

$$x(n) \equiv x_a(nT) = x_a\left(\frac{n}{F_s}\right)$$

$$\underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} X(f) e^{j2\pi fn} df}_{LHS} = \underbrace{\int_{-\infty}^{\infty} X_a(F) e^{j2\pi F \frac{n}{F_s}} dF}_{RHS}$$

- We can use the following relationship that comes about from the "reinterpretation stage".

$$t = nT \quad \text{or} \quad \frac{1}{F} = \frac{T}{f}$$

$$f = T \cdot F = \frac{F}{F_s} \Longrightarrow \boxed{f = \frac{F}{F_s}}$$

Sampling Process

$$\begin{aligned}
LHS &= \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f) e^{j2\pi fn} df \quad \text{let } f = \frac{F}{F_s} \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} X\left(\frac{F}{F_s}\right) e^{j2\pi \frac{F}{F_s} n} \frac{dF}{F_s} \\
&= \frac{1}{F_s} \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} X\left(\frac{F}{F_s}\right) e^{j2\pi \frac{F}{F_s} n} dF
\end{aligned}$$

Sampling Process

$$\begin{aligned}
RHS &= \int_{-\infty}^{\infty} X_a(F) e^{j2\pi \frac{F}{F_s} n} dF \\
&= \sum_{k=-\infty}^{\infty} \int_{-\frac{F_s}{2} - kF_s}^{+\frac{F_s}{2} - kF_s} X_a(F) e^{j2\pi \frac{F}{F_s} n} dF \quad \text{let } F' = F + kF_s \\
&= \sum_{k=-\infty}^{\infty} \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} X_a(F' - kF_s) \underbrace{e^{j2\pi \frac{F' - kF_s}{F_s} n}}_{= e^{j2\pi \frac{F'}{F_s} n}} dF' \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left[\sum_{k=-\infty}^{\infty} X_a(F - kF_s) \right] e^{j2\pi \frac{F}{F_s} n} dF
\end{aligned}$$

Sampling Process

Since,

$$\begin{aligned} LHS &= RHS \\ \frac{1}{F_s}\int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} x\left(\frac{F}{F_s}\right) e^{j2\pi \frac{F}{F_s} n} dF &= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left[\sum_{k=-\infty}^{\infty} X_a(F - kF_s)\right] e^{j2\pi \frac{F}{F_s} n} dF \end{aligned}$$

By inspection,

$$X\left(\frac{F}{F_s}\right) = F_s \sum_{k=-\infty}^{\infty} X_a(F - kF_s)$$

or letting $f = \frac{F}{F_s}$,

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a(f \cdot F_s - kF_s) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

Sampling Process

- The spectrum of the sampled signal $x(n)$ is a scaled periodic repetition of spectrum of the original analog signal $x_a(t)$.
- The form of the periodic repetition guarantees that the signal $X(f)$ is periodic with fundamental period 1 or $X(\omega)$ is periodic with fundamental period 2π.

See ▸ Figure 6.1.1 of text.

Sampling Theorem

A bandlimited continuous-time signal, with highest frequency (bandwidth) B Hz, can be uniquely recovered from its samples provided that the sampling rate $F_s \geq 2B$ samples per second.

- Perfect reconstruction is possible via the ideal interpolation formula:

$$x_a(t) = \sum_{n=-\infty}^{\infty} \underbrace{x_a(nT)}_{=x(n)} \frac{\sin(\frac{\pi}{T}(t - nT))}{\frac{\pi}{T}(t - nT)}$$

See ▸ Figure 6.1.2 of text.

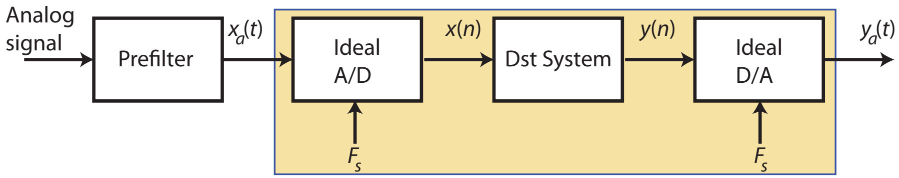
Aliasing

- Sampling and reconstruction of nonbandlimited signals results in aliasing.
- The degree of aliasing/quality of the reconstruction depends on the sampling rate in relation to the decay of the analog signal spectrum.
- Example:

$$x_a(t) = e^{-A|t|} \overset{\mathcal{F}}{\longleftrightarrow} X_a(F) = \frac{2A}{A^2 + (2\pi F)^2}, \quad A > 0$$

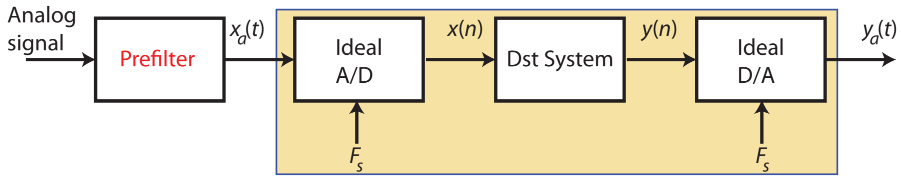
See ▸ Figure 6.1.8 of text.

Overview



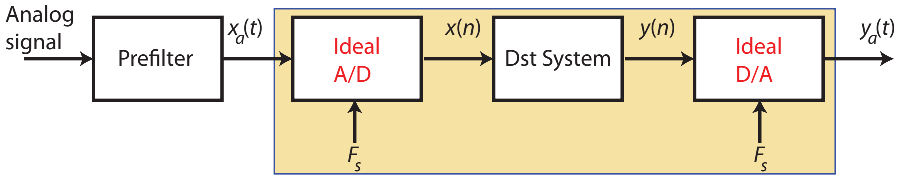
- system set up when discrete-time processing of continuous-time signals is required
- the application often defines how each block is designed

Prefilter



- ensures that bandwidth is limited to avoid aliasing or reduce subsequent computational requirements
- rejects additive noise in higher frequency ranges

A/D and D/A



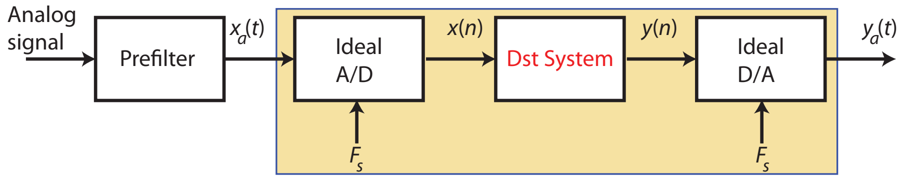
- ideal sampling and interpolation assumed:

$$x(n) = x(t)_{t=nT} = x_a(nT) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X(F) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a(F - kF_s)$$

$$y_a(t) = \sum_{n=-\infty}^{\infty} y(n)\frac{\sin(\frac{\pi}{T}(t-nT))}{\frac{\pi}{T}(t-nT)} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y_a(F) = \begin{cases} TY(F) & |F| \leq \frac{F_s}{2} \\ 0 & \text{otherwise} \end{cases}$$

See ▸ Figure 6.2.2 of text and ▸ Figure 6.2.3 of text.

Discrete-Time System Design



- Q: Is there a discrete-time system such that the overall system above is equivalent to a continuous-time LTI system?
- A: Yes if $x_a(t)$ is bandlimited and $F_s > 2B$.

Discrete-Time System Design

- Consider a desired continuous-time LTI system:

$$\begin{aligned} y_a(t) &= h_a(t) * x_a(t) = \int_{-\infty}^{\infty} h_a(\tau) x_a(t-\tau) dt \\ Y_a(F) &= H_a(F) X_a(F) \end{aligned}$$

- If $x_a(t)$ is bandlimited and $F_s > 2B$ (no overlap), then

$$X(F) = \frac{1}{T} X_a(F) \quad \text{for } |F| \leq \tfrac{F_s}{2}$$

See Figure 6.2.2 of text

Discrete-Time System Design

- Recall, from ideal interpolation and assuming dst-time system $H(F)$:

$$\begin{aligned} Y_a(F) &= \begin{cases} T\,Y(F) & |F| \leq \frac{F_s}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} T\,H(F)X(F) & |F| \leq \frac{F_s}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} TH(F)\frac{1}{T}X_a(F) & |F| \leq \frac{F_s}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} H(F)X_a(F) & |F| \leq \frac{F_s}{2} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Discrete-Time System Design

- The desired equivalent cts-time system is:

$$Y_a(F) = H_a(F) X_a(F)$$

- The actual overall response assuming a dst-time filter $H(F)$ is:

$$Y_a(F) = \begin{cases} H(F)X_a(F) & |F| \leq \frac{F_s}{2} \\ 0 & \text{otherwise} \end{cases}$$

Discrete-Time System Design

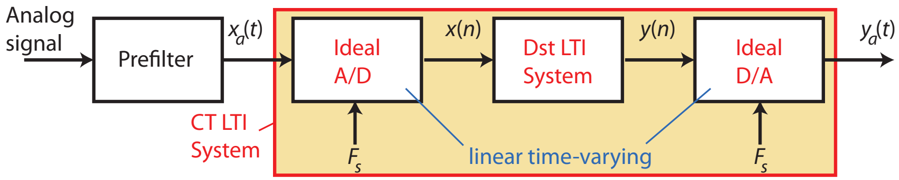
- Therefore,

$$H_a(F) = \begin{cases} H(F) & |F| \leq \frac{F_s}{2} \\ 0 & |F| > \frac{F_s}{2} \end{cases}$$

- Naturally,

$$\begin{aligned} H(F) &= H_a(F) \quad \text{for } |F| \leq \tfrac{F_s}{2} \\ H(F) &= \sum_{k=-\infty}^{\infty} H_a(F - kF_s) \\ h(n) &= T\ h_a(nT) \end{aligned}$$

Discrete-Time System Design



- Under the conditions discussed, the cascade of a linear time-varying system (A/D converter), an LTI system, and a linear time-varying system (D/A converter) is equivalent to a continuous-time LTI system.

Example: Ideal bandlimited differentiator

Goal: to design a discrete-time filter such that the overall system (including A/D and D/A conversion) is defined by

$$y_a(t) = \frac{dx_a(t)}{dt}$$

The frequency response of the overall system is given by:

$$\begin{aligned} H_a(F) &= \frac{Y_a(F)}{X_a(F)} = j2\pi F \\ &= \begin{cases} j2\pi F & |F| \leq F_c \\ 0 & |F| > F_c \end{cases} \quad \text{for bandlimited signals} \end{aligned}$$

Example: Ideal bandlimited differentiator

Choosing $F_s = 2F_c$, we define the ideal discrete-time differentiator as:

$$H(F) = H_a(F) = j2\pi F, \quad |F| \leq \frac{F_s}{2}$$

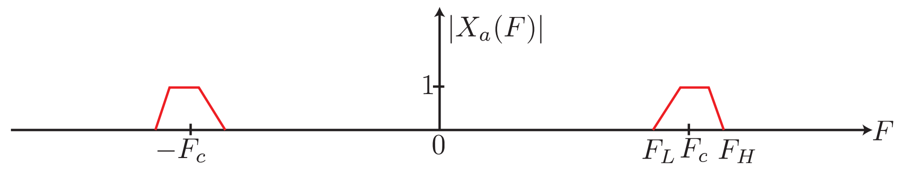
and

$$H(F) = \sum_{k=-\infty}^{\infty} H_a(F - kF_s)$$

See ‣ Figure 6.2.5 of text.

Bandlimited Signal

- A continuous-time bandpass signal with bandwidth B and center frequency F_c has its frequency content in two frequency bands defined by $0 < F_L < |F| < F_H$

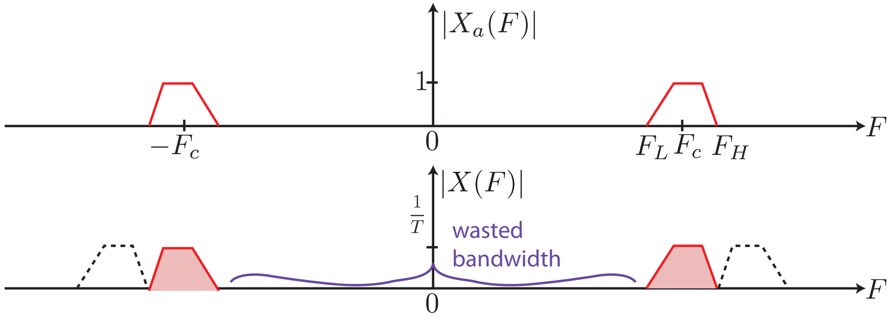


Note: $F_c = \frac{F_L+F_H}{2}$, $B = F_H - F_L$

Uniform First-Order Sampling

$$x(n) = x_a(nT) \stackrel{\mathcal{F}}{\longleftrightarrow} X(F) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a(F - kF_s)$$

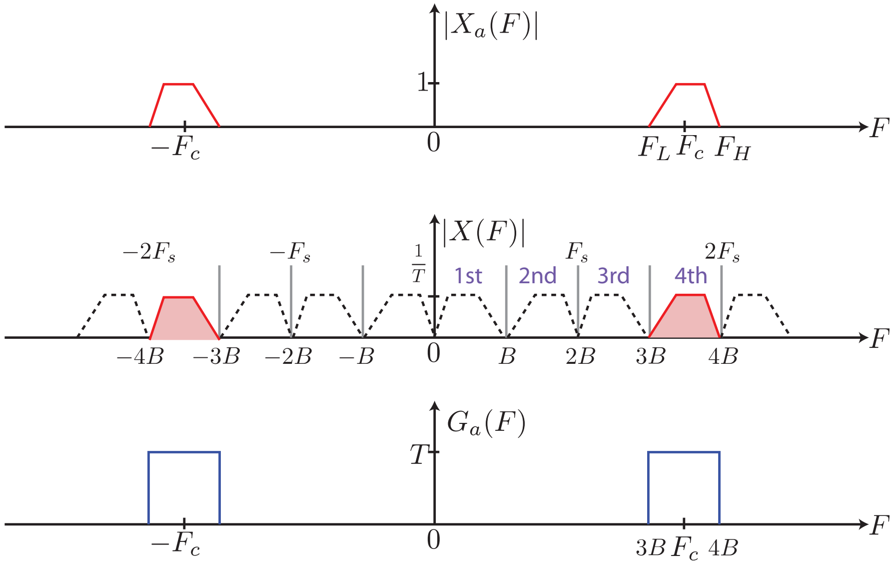
- $F_s = 2F_H$ guarantees perfect reconstruction, but wastes bandwidth for bandpass signals



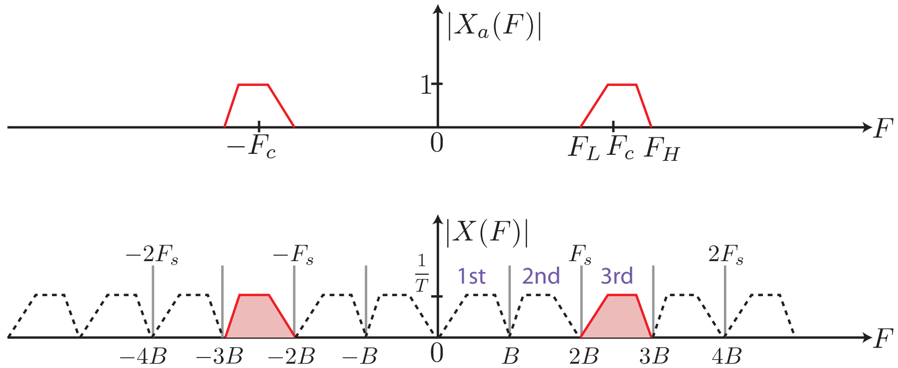
Integer Band Positioning

- Consider $F_H = mB$ where $B = F_H - F_L$
- Let $\boxed{F_s = 2B}$ results in no aliasing
- perfect reconstruction is possible with appropriate interpolation stage

Integer Band Positioning, $F_s = 2B, F_H = 4B$



Integer Band Positioning, $F_s = 2B, F_H = 3B$



Integer Band Positioning

- Perfect reconstruction via:

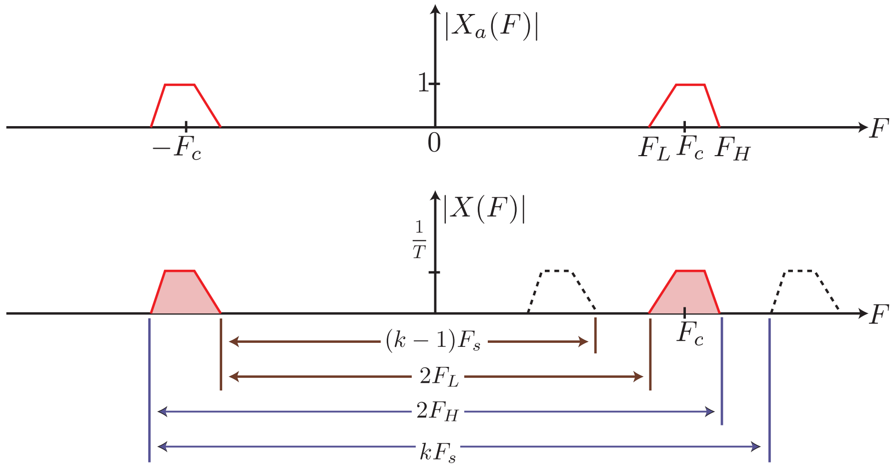
$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT) g_a(t - nT)$$

$$g_a(t) = \frac{\sin(\pi B t)}{\pi B t} \cos(2\pi F_c t)$$

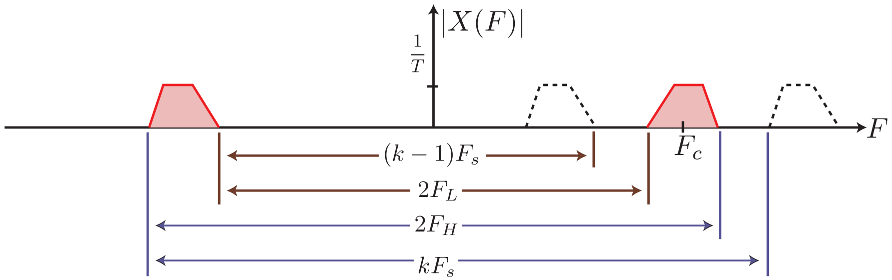
for $F_s = 2B$.

Arbitrary Band Positioning

- For $F_s \geq B$, aliasing is due to overlap of "positive" spectral band with "negative" or vice versa.



Arbitrary Band Positioning



$$(k-1)F_s \leq 2F_L$$
$$2F_H \leq kF_s$$

Arbitrary Band Positioning

$$(k-1)F_s \leq 2F_L \Longrightarrow F_s \leq \frac{2F_L}{k-1}$$
$$2F_H \leq kF_s \Longrightarrow F_s \geq \frac{2F_H}{k}$$

$$\frac{2F_H}{k} \leq F_s \leq \frac{2F_L}{k-1}$$

- If F_s obeys the above for integer $k \geq 1$, aliasing can be avoided.
- The $k = 1$ case corresponds to the Nyquist sampling criterion.
  - i.e., $F_s \geq 2F_H$
- The case $k > 1$ corresponds to sampling below Nyquist.
  - i.e., $F_s \leq 2F_L/(k-1) < 2F_H$ for all $k > 1$.

Arbitrary Band Positioning

- The maximum value of k shows the number of bands that we can fit in the range from 0 and F_H.

$$\begin{aligned} (k-1)F_s \leq 2F_L \implies \quad (k-1)F_s &\leq 2F_H - 2B \\ 2F_H \leq kF_s \implies \quad \frac{1}{F_s} &\leq \frac{k}{2F_H} \\ \text{mult. both sides} \implies \quad k-1 &\leq k - \frac{kB}{F_H} \\ k &\leq \frac{F_H}{B} \\ k_{max} &= \left\lfloor \frac{F_H}{B} \right\rfloor \end{aligned}$$

Arbitrary Band Positioning: Minimum Sampling Rate

- Recall to avoid aliasing,

$$\frac{2F_H}{k} \leq F_s \leq \frac{2F_L}{k-1}$$

- Therefore, the minimum sampling rate to avoid aliasing is given by

$$F_{s,min} = \frac{2F_H}{k_{max}}$$

Arbitrary Band Positioning

- Therefore to avoid aliasing, the range of acceptable uniform sampling rates is given by

$$\frac{2F_H}{k} \leq F_s \leq \frac{2F_L}{k-1}$$

$$\text{where } k \in \mathbb{Z}^+ \text{ and } 1 \leq k \leq \left\lfloor \frac{F_H}{B} \right\rfloor$$

See Figure 6.4.3 of text.

Sampling of Dst-Time Signals

$$x_d(n) = x(nD)$$

for all n.

Sampling of Dst-Time Signals

Consider $\boxed{x_d(n) = x(nD)}$ for all n

- $x(n)$ can be interpreted as the samples of a continuous-time signal $x_a(t)$ with rate $F_s = \frac{1}{T}$:

$$X(F) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a(F - kF_s)$$

- $x_d(n)$ can be interpreted as the samples of $x_a(t)$ with sampling rate $\frac{F_s}{D} = \frac{1}{DT}$,

$$X_d(F) = \frac{1}{DT}\sum_{k=-\infty}^{\infty} X_a(F - k\frac{F_s}{D})$$

See Figure 6.5.1 of text.

Sampling of Dst-Time Signals

- Therefore,

$$X_d(F) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(F - k\frac{F_s}{D}\right)$$

- Assuming $X_a(F) = 0, |F| > B$, to avoid aliasing for the dst-time sampling, we need:

$$\frac{F_s}{D} \geq 2B \quad \text{or} \quad B \leq \frac{F_s}{2D}$$

$$f_{max} \triangleq \frac{B}{F_s} \leq \frac{1}{2D} = \frac{f_s}{2} \quad \text{or} \quad \omega_{max} = 2\pi f_{max} \leq \frac{\pi}{D} = \frac{\omega_s}{2}$$

Sampling of Dst-Time Signals

- In cts-time sampling $x(n) = x_a(nT)$,
  - the aperiodic spectrum $X_a(F)$ is repeated an infinite number of times to create a periodic spectrum covering the infinite frequency range.
- In dst-time sampling $x_d(n) = x(nD)$,
  - the periodic spectrum $X(F)$ is repeated D times covering one period of the periodic frequency domain.

Ideal Interpolation

- Recall, the general interpolation formula is:

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n)\frac{\sin(\frac{\pi}{T}(t - nT))}{\frac{\pi}{T}(t - nT)}$$

- For dst-time sampling, it is given by:

$$x_a(t) = \sum_{m=-\infty}^{\infty} x_d(m)\frac{\sin(\frac{\pi}{DT}(t - mDT))}{\frac{\pi}{DT}(t - mDT)}$$

Ideal Interpolation

- Using

$$x_a(t) = \sum_{m=-\infty}^{\infty} x_d(m) \frac{\sin(\frac{\pi}{DT}(t - mDT))}{\frac{\pi}{DT}(t - mDT)}$$

- and the fact that $x(n) = x_a(nT)$

$$\begin{aligned} x(n) &= x_a(nT) = \sum_{m=-\infty}^{\infty} x_d(m) \frac{\sin(\frac{\pi}{DT}(nT - mDT))}{\frac{\pi}{DT}(nT - mDT)} \\ &= \sum_{m=-\infty}^{\infty} x_d(m) \frac{\sin(\frac{\pi}{D}(n - mD))}{\frac{\pi}{D}(n - mD)} \\ &= \sum_{m=-\infty}^{\infty} x_d(m) g_{BL}(n - mD) \end{aligned}$$

Ideal Interpolation

- Therefore,

$$\begin{aligned} x(n) &= \sum_{m=-\infty}^{\infty} x_d(m) g_{BL}(n - mD) \\ g_{BL}(n) &= D \frac{\sin\left(\frac{\pi}{D} n\right)}{\pi n} \end{aligned}$$

Ideal Interpolation

- where

$$g_{BL}(n) = D \frac{\sin\left(\frac{\pi}{D} n\right)}{\pi n} \stackrel{\mathcal{F}}{\longleftrightarrow} G_{BL}(\omega) = \begin{cases} D & |\omega| \le \frac{\pi}{D} \\ 0 & \frac{\pi}{D} < |\omega| \le \pi \end{cases}$$

See Figure 6.5.1 of text.

Linear Interpolation

See Figure 6.5.2 of text.

$$x_{lin}(t) = x(m-1) + \frac{x(m) - x(m-1)}{DT}(t - (m-1)DT)$$

$$\text{for } (m-1)DT \le t \le mDT$$

Linear Interpolation

- Therefore,

$$x_{lin}(n) = \sum_{m=-\infty}^{\infty} x(m) g_{lin}(n - mD)$$

$$g_{lin}(n) = \begin{cases} 1 - \frac{|n|}{D} & |n| \leq D \\ 0 & |n| > D \end{cases}$$

See ▸ Figure 6.5.2 of text.

Ideal Interpolation

- where

$$g_{lin}(n) = \begin{cases} 1 - \frac{|n|}{D} & |n| \leq D \\ 0 & |n| > D \end{cases} \stackrel{\mathcal{F}}{\longleftrightarrow} G_{lin}(\omega) = \frac{1}{D}\left[\frac{\sin(\omega \frac{D}{2})}{\sin(\frac{\omega}{2})}\right]^2$$

See ▸ Figure 6.5.3 of text.

■