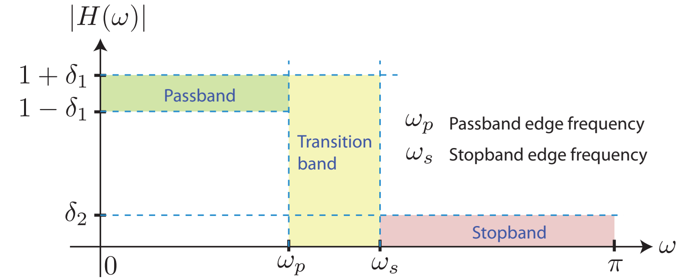
Practical Frequency-Selective Digital Filter Design

Dr. Deepa Kundur

University of Toronto

Digital Filter Design

- Desired filter characteristics are specified in the frequency domain in terms of desired magnitude and phase response of the filter; i.e., $H(\omega)$ is specified.



- Filter design involves determining the coefficients of a causal FIR or IIR filter that closely approximates the desired frequency response specifications.

FIR versus IIR Filters

- FIR filters: normally used when there is a requirement of linear phase
  - FIR filter with the following symmetry is linear phase:

$$h(n) = \pm h(M-1-n) \qquad n = 0, 1, 2, \ldots, M-1$$

- IIR filters: normally used when linear phase is not required and cost effectiveness is needed
  - IIR filter has lower sidelobes in the stopband than an FIR having the same number of parameters
  - if some phase distortion is tolerable, an IIR filter has an implementation with fewer parameters requiring less memory and lower complexity

Linear Phase

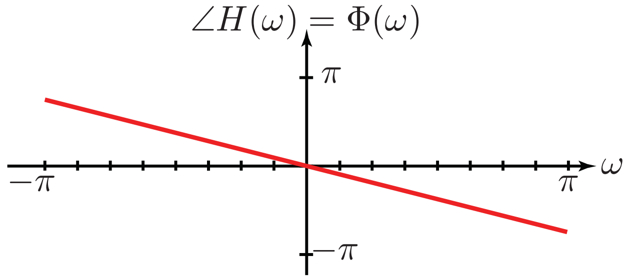
Q: What is linear phase?

A: The phase is a straight line in the passband of the system.

Linear Phase

Example: linear phase (all pass system)

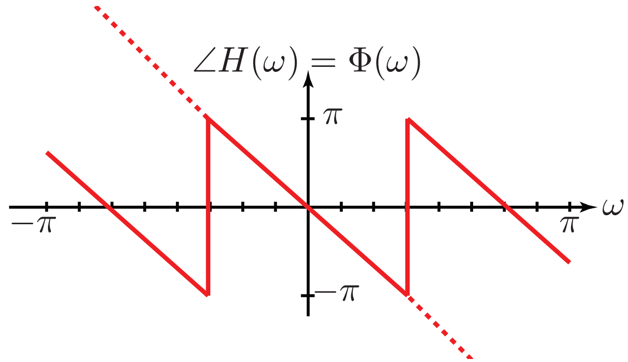
- Group delay is given by the negative of the slope of the line (more on this soon).



Linear Phase

Example: linear phase (all pass system)

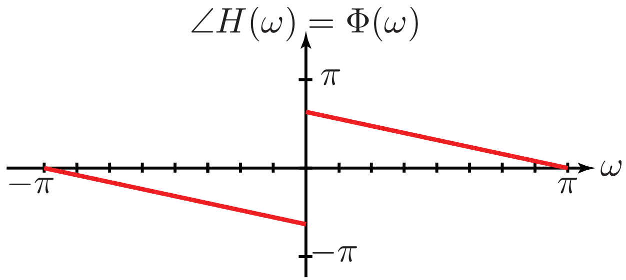
- Phase wrapping may occur, but the phase is still considered to be linear.



Linear Phase

Example: linear phase (high pass system)

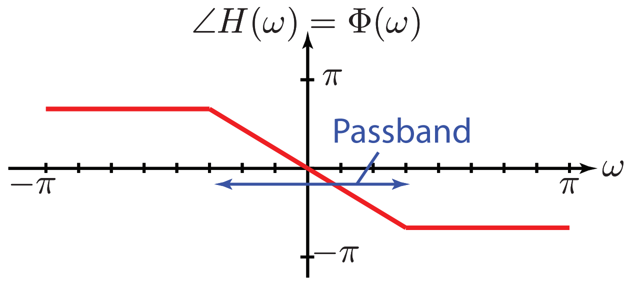
- Discontinuities at the origin still correspond to a linear phase system.



Linear Phase

Example: linear phase (low pass system)

- Linear characteristics only need to pertain to the passband frequencies only.



DTFT Theorems and Properties

Recall,

Property	Time Domain	Frequency Domain
Notation:	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_1(\omega)$
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting:	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution:	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation:	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$ $= X_1(\omega)X_2^*(\omega)$ [if $x_2(n)$ real]
Wiener-Khintchine:	$r_{xx}(l) = x(l) * x(-l)$	$S_{xx}(\omega) = \lvert X(\omega)\rvert^2$

among others ...

Group Delay

Therefore,

$$y(n) = x(n - \underbrace{n_0}_{\text{group delay}}) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(\omega) = X(\omega)e^{-j\omega n_0}$$

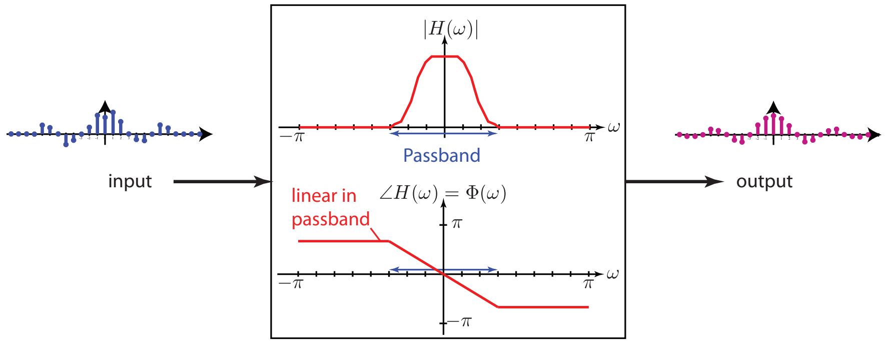
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = e^{-j\omega n_0}$$

$$\angle H(\omega) = \Phi(\omega) = -\omega n_0 = -\omega \cdot \text{group delay}$$

In general (even for nonlinear phase systems),

$$\text{group delay} \equiv -\frac{d\Phi(\omega)}{d\omega}$$

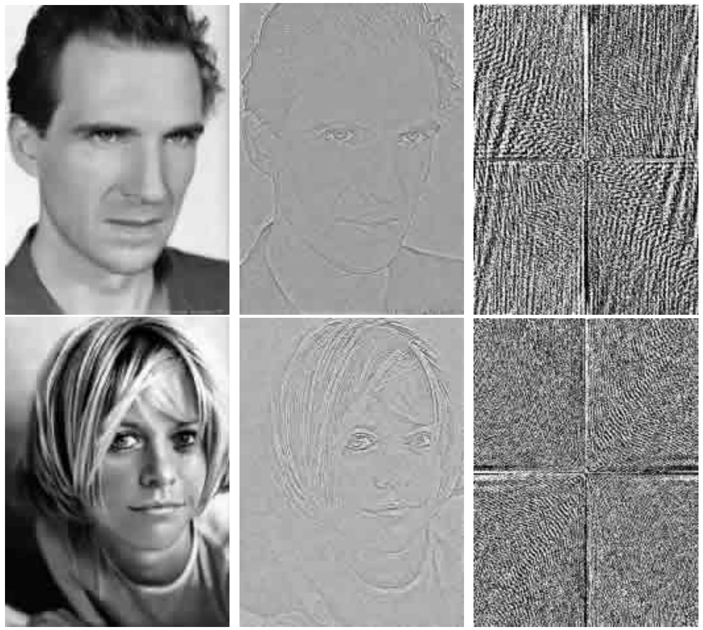
- Linear phase filters maintain the relative positioning of the sinusoids in the filter passband.
- This maintains the structure of the signal while removing unwanted frequency components.

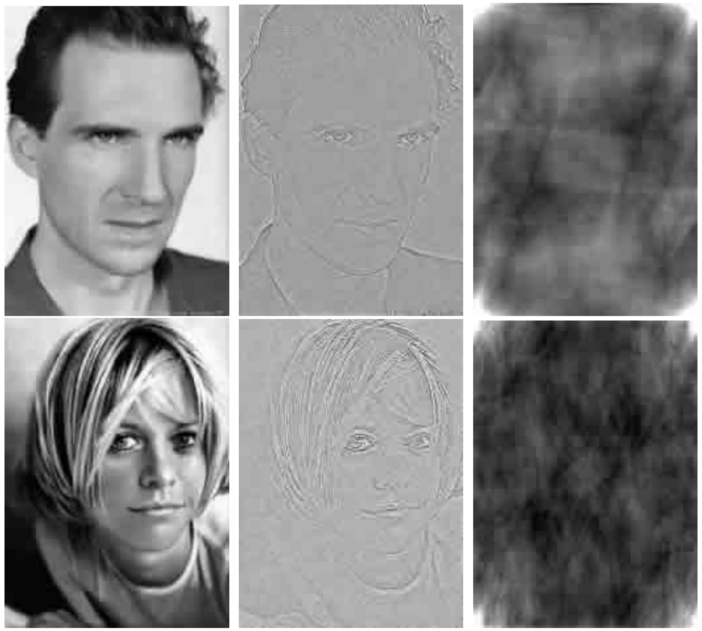


Signal Magnitude versus Signal Phase

Q: Why is linear phase important?

Q: What can happen when there is loss of phase information?





Signal Magnitude versus Signal Phase

A: To maintain the original "structure" of a signal in the passband frequency range, linear phase (or close to linear phase) is required.

Linear Phase FIR Filters

- As mentioned previously, FIR filters with the following symmetry are linear phase:

$$h(n) = \pm h(M-1-n) \qquad n = 0, 1, 2, \ldots, M-1$$

- Note that this means that

$$h(n) = +h(M-1-n)$$

for $n = 0, 1, 2, \ldots, M-1$, or

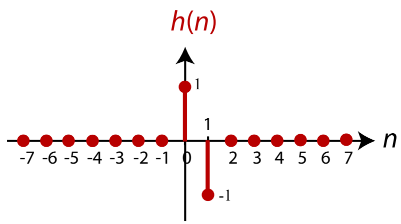
$$h(n) = -h(M-1-n)$$

for $n = 0, 1, 2, \ldots, M-1$.

Linear Phase FIR Filters: Example

Q: Show that $h(n) = \delta(n) - \delta(n-1)$ is linear phase by determining the associated phase and group delay.

Note: $M = 2$ and $h(n) = -h(1-n) = -h(M-1-n)$ for $n = 0, 1$.



For $n = 0$, $h(0) = -h(1-0) = 1$ and $n = 1$, $h(1) = -h(1-1) = -1$.

Linear Phase FIR Filters: Example

Q: Show that $h(n) = \delta(n) - \delta(n-1)$ is linear phase by determining the associated phase and group delay.

Note: This system corresponds to:

$$\begin{aligned} y(n) &= x(n) * h(n) = x(n) * [\delta(n) - \delta(n-1)] \\ &= x(n) * \delta(n) - x(n) * \delta(n-1) \\ &= x(n) - x(n-1) \quad \text{(first difference system)} \\ \text{first difference} &\Leftrightarrow \text{dst-time derivative} \Rightarrow \text{highpass filter} \end{aligned}$$

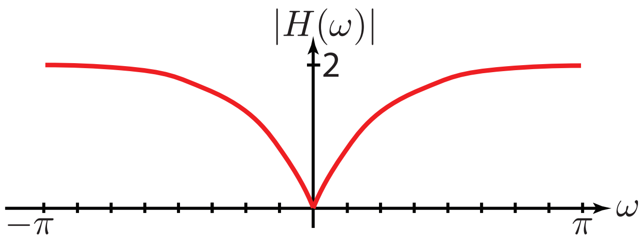
Linear Phase FIR Filters: Example

$$\begin{aligned} H(\omega) &= \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n} \\ &= 1 \cdot e^{-j\omega \cdot 0} + (-1) \cdot e^{-j\omega \cdot 1} \\ &= 1 - e^{-j\omega} = e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right) \\ &= e^{-j\omega/2} \cdot 2j \sin(\omega/2) = 2je^{-j\omega/2} \sin(\omega/2) \end{aligned}$$

Linear Phase FIR Filters: Example

Note:

$$\begin{aligned} |H(\omega)| &= |2je^{-j\omega/2} \sin(\omega/2)| \\ &= |2| \cdot |j| \cdot |e^{-j\omega/2}| \cdot |\sin(\omega/2)| \\ &= 2 \cdot 1 \cdot 1 \cdot |\sin(\omega/2)| = 2|\sin(\omega/2)| \end{aligned}$$



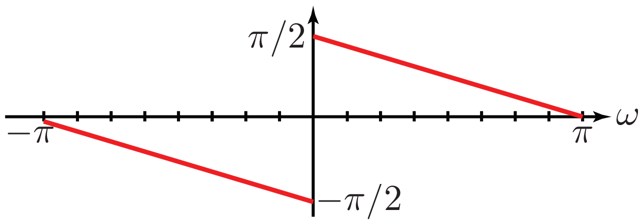
Linear Phase FIR Filters: Example

$$\begin{aligned} \Phi(\omega) &= \angle 2je^{-j\omega/2} \sin(\omega/2) \\ &= \underbrace{\angle 2}_{=0} + \underbrace{\angle j}_{=\pi/2} + \underbrace{\angle e^{-j\omega/2}}_{=-\omega/2} + \underbrace{\angle \sin(\omega/2)}_{=\begin{cases} 0 & 0 < \omega < \pi \\ \pi & -\pi < \omega < 0 \end{cases}} \\ &= \begin{cases} \frac{\pi}{2} - \frac{\omega}{2} & 0 < \omega < \pi \\ \frac{3\pi}{2} - \frac{\omega}{2} - 2\pi & -\pi < \omega < 0 \end{cases} \\ &= \begin{cases} \frac{\pi - \omega}{2} & 0 < \omega < \pi \\ \frac{-\pi - \omega}{2} & -\pi < \omega < 0 \end{cases} \end{aligned}$$

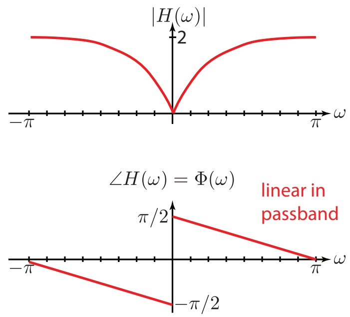
Linear Phase FIR Filters: Example

$$\Phi(\omega) = \begin{cases} \frac{\pi-\omega}{2} & 0 < \omega < \pi \\ \frac{-\pi-\omega}{2} & -\pi < \omega < 0 \end{cases}$$

$$\angle H(\omega) = \Phi(\omega)$$



Linear Phase FIR Filters: Example



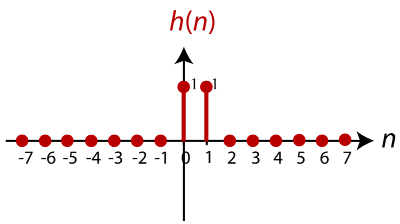
Group Delay:

$$\begin{aligned} -\frac{d\Phi(\omega)}{d\omega} &= \frac{1}{2} - \pi\delta(\omega) \\ &= \begin{cases} \frac{1}{2} & \omega \neq 0 \\ -\pi\delta(\omega) & \omega = 0 \end{cases} \\ &= \frac{1}{2} = \text{constant} \\ & \quad \text{(in passband)} \end{aligned}$$

Linear Phase FIR Filters: Example 2

Q: Show that $h(n) = \delta(n) + \delta(n-1)$ is linear phase by determining the associated phase and group delay.

Note: $M = 2$ and $h(n) = +h(1-n) = +h(M-1-n)$ for $n = 0, 1$.



For $n = 0$, $h(0) = +h(1-0) = 1$ and $n = 1$, $h(1) = +h(1-1) = 1$.

Linear Phase FIR Filters: Example 2

Q: Show that $h(n) = \delta(n) + \delta(n-1)$ is linear phase by determining the associated phase and group delay.

Note: This system corresponds to:

$$\begin{aligned} y(n) &= x(n) * h(n) = x(n) * [\delta(n) + \delta(n-1)] \\ &= x(n) * \delta(n) + x(n) * \delta(n-1) \\ &= x(n) + x(n-1) \quad \text{(scaled averaging system)} \end{aligned}$$

averager $\Rightarrow$ dst-time smoother $\Rightarrow$ lowpass filter

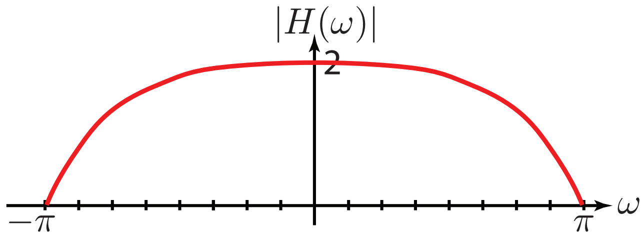
Linear Phase FIR Filters: Example 2

$$
\begin{aligned}
H(\omega) &= \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} \\
&= 1 \cdot e^{-j\omega \cdot 0} + (+1) \cdot e^{-j\omega \cdot 1} \\
&= 1 - e^{-j\omega} = e^{-j\omega/2}\left(e^{j\omega/2} + e^{-j\omega/2}\right) \\
&= e^{-j\omega/2} \cdot 2\cos(\omega/2) = 2e^{-j\omega/2}\cos(\omega/2)
\end{aligned}
$$

Linear Phase FIR Filters: Example 2

Note:

$$
\begin{aligned}
|H(\omega)| &= |2e^{-j\omega/2}\cos(\omega/2)| \\
&= |2| \cdot |e^{-j\omega/2}| \cdot |\cos(\omega/2)| \\
&= 2 \cdot 1 \cdot |\cos(\omega/2)| = 2|\cos(\omega/2)|
\end{aligned}
$$



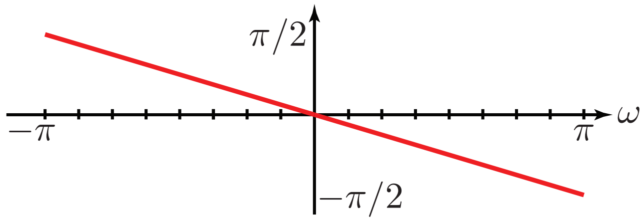
Linear Phase FIR Filters: Example 2

$$
\begin{aligned}
\Phi(\omega) &= \angle 2e^{-j\omega/2}\cos(\omega/2) \\
&= \underbrace{\angle 2}_{=0} + \underbrace{\angle e^{-j\omega/2}}_{=-\omega/2} + \underbrace{\angle \cos(\omega/2)}_{=0} \\
&= -\frac{\omega}{2}
\end{aligned}
$$

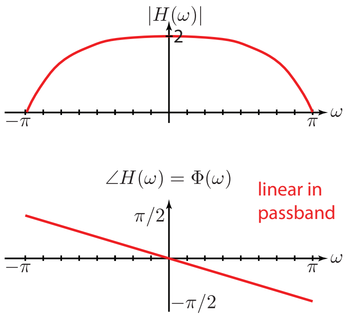
Linear Phase FIR Filters: Example 2

$$
\Phi(\omega) = -\frac{\omega}{2}
$$

$$
\angle H(\omega) = \Phi(\omega)
$$



Linear Phase FIR Filters: Example 2



Group Delay:

$$-\frac{d\Phi(\omega)}{d\omega} = \frac{1}{2} = \text{constant}$$

(in passband)

Ideal Filters

An ideal lowpass filter is given by:

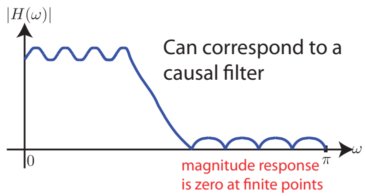
$$H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

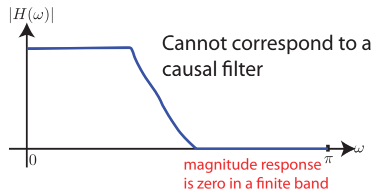
The impulse response is given by:

$$h(n) = \begin{cases} \frac{\omega_c}{\pi} & n = 0 \\ \frac{\omega_c}{\pi}\frac{\sin(\omega_c n)}{\omega_c n} & n \neq 0 \end{cases}$$

Limitations of Practical Filters

- An ideal filter is not causal since $h(n) \neq 0$ for $n < 0$.
  - From the Paley-Wiener Theorem: for causal LTI systems where necessarily $h(n) = 0$ for $n < 0$, $|H(\omega)|$ can be zero only at a finite set of points in a frequency interval, but not over a finite band of frequencies.





Limitations of Practical Filters

- Rippling occur in the passband and stopband – Why?
  - imposing causality is like truncating $h(n)$ so it has no negative part, which results in Gibbs phenomenon – i.e., ringing/rippling effect for $H(\omega)$

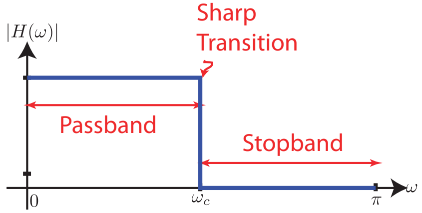
$$\text{ringing} \overset{\mathcal{F}}{\longleftrightarrow} \text{truncation} \quad \text{(Gibbs in time-domain)}$$

$$\text{truncation} \overset{\mathcal{F}}{\longleftrightarrow} \text{ringing} \quad (H(\omega) \text{ pass/stopband rippling})$$

- In addition, filters with finite parameters will demonstrate a measurable transition between passband and stopband.

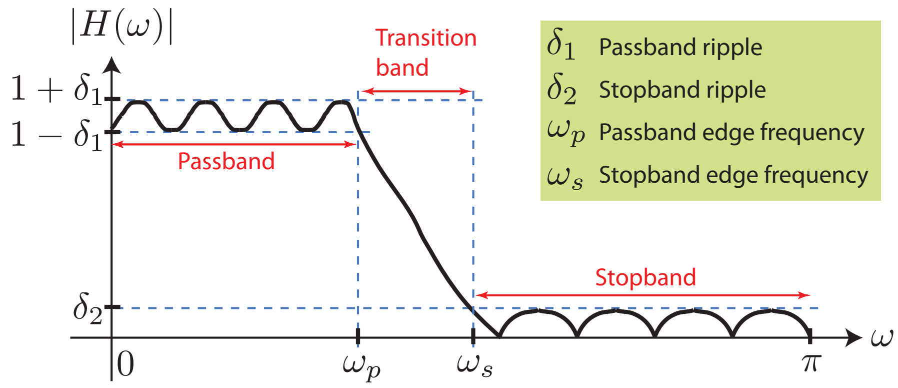
Practical Frequency Selective Filters

- Ideal filter characteristics of sharp transitions and flat gains may not be absolutely necessary for most practical applications.



- Relaxing these conditions provides an opportunity to realize causal finite parameter filters that approximate ideal filters as close as we desire.

Practical Frequency Selective Filters



Desired Frequency Response

Given: $H_d(\omega)$ (desired frequency response)

$$H_d(\omega) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n}$$

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$$

Recall for a digital FIR implementation, $h_d(n)$ needs to be finite duration; say, of length M. Therefore, it is required that $h_d(n) = 0$ for $n < 0$ and $n > M - 1$.

In general, $h_d(n)$ is infinite duration ...

Design of Linear-Phase FIR Filters using Windows

Q: How do we make $h_d(n)$ finite duration?

A: windowing ...

Consider the rectangular window

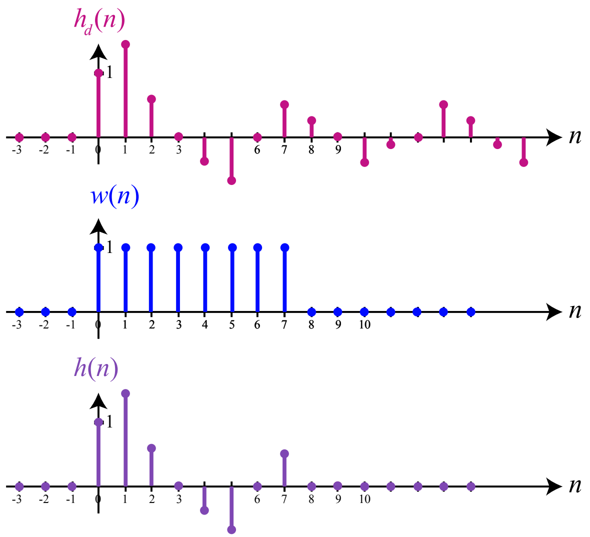
$$w(n) = \begin{cases} 1 & n = 0, 1, \ldots, M-1 \\ 0 & \text{otherwise} \end{cases}$$

$$h(n) = h_d(n)\, w(n)$$

$$= \begin{cases} h_d(n) & n = 0, 1, \ldots, M-1 \\ 0 & \text{otherwise} \end{cases}$$

Example

Rectangular window, $M = 8$



Windowing Distortion

Q: What is the distortion introduced by windowing?

A: Look in the frequency domain ...

$$\text{convolution} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \text{multiplication}$$

$$\text{multiplication} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \text{convolution}$$

$$h_d(n)w(n) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad H_d(\omega) * W(\omega)$$

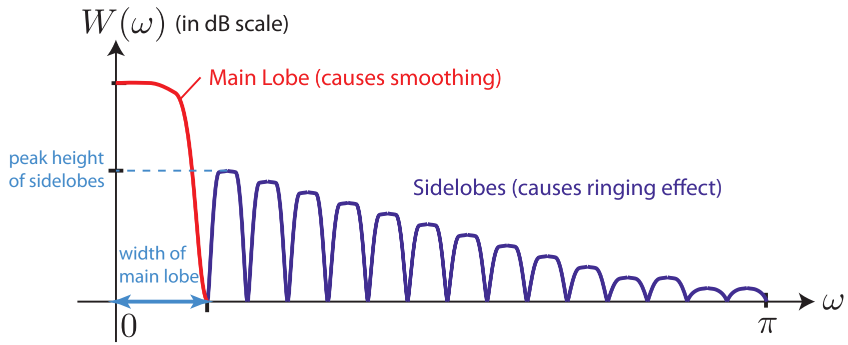
$$h(n) = h_d(n)w(n) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad H(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) \underbrace{W(\omega - \nu)}_{\text{depends on } w(n)} d\nu$$

$$W(\omega) = \sum_{n=0}^{M-1} w(n)e^{-j\omega n}$$

Windowing Distortion

- Convolving $H_d(\omega)$ with $W(\omega)$ has the effect of smoothing out the frequency response of the resulting filter.
- For no distortion from windowing, want $W(\omega)$ to be close to a delta function, $\delta(\omega)$
- $W(\omega)$ is partially characterized by:
  - main lobe width (in rad/s)
  - peak amplitude of side lobe (in dB)

Windowing Distortion



- increasing window length generally reduces the width of the main lobe
- peak of sidelobes is generally independent of M

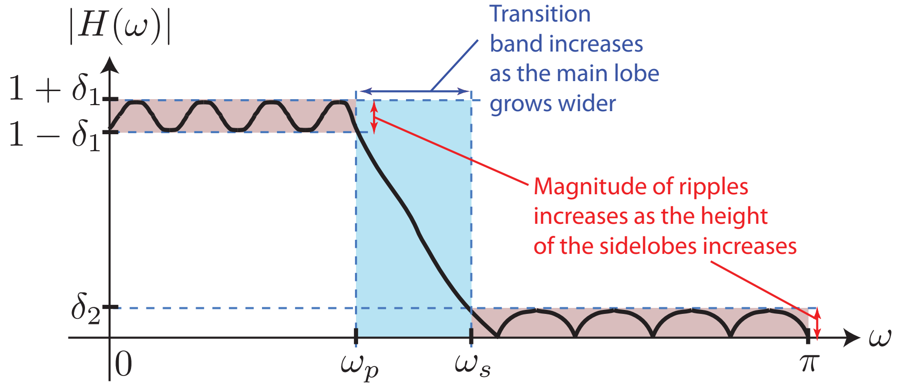
Characteristics of Different Windows

Window type	Main lobe width	Peak sidelobe (dB)
Rectangular	$4\pi/M$	-13
Bartlett	$8\pi/M$	-25
Hanning	$8\pi/M$	-31
Hamming	$8\pi/M$	-41
Blackman	$12\pi/M$	-57

Note:

- the larger the main lobe, the larger the filter transition region
- the larger the peak sidelobe, the higher the degree of ringing in the pass/stopbands

Effects of Windowing in Frequency Domain



Design of Linear-Phase FIR Filters using Windows

1. Begin with a desired frequency response $H_d(\omega)$ that is linear phase with a delay of $(M-1)/2$ units in anticipation of forcing the filter to be length M.

   Example:

$$H_d(\omega) = \begin{cases} 1 \cdot e^{-j\omega(M-1)/2} & 0 \leq |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Design of Linear-Phase FIR Filters using Windows

2. The corresponding impulse response is given by:

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} H_d(\omega)e^{j\omega n}d\omega$$

   Example:

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-(M-1)/2)}d\omega \\ &= \begin{cases} \frac{\sin \omega_c\left(n-\frac{M-1}{2}\right)}{\pi\left(n-\frac{M-1}{2}\right)} & n \neq \frac{M-1}{2} \\ \frac{\omega_c}{\pi} & n = \frac{M-1}{2} \end{cases} \quad \text{(if } M \text{ is odd)} \\ &\quad\ \ \frac{\sin \omega_c\left(n-\frac{M-1}{2}\right)}{\pi\left(n-\frac{M-1}{2}\right)} \quad \text{(if } M \text{ is even)} \end{aligned}$$

Design of Linear-Phase FIR Filters using Windows

3. Multiply $h_d(n)$ with a window of length M.

$$h(n) = h_d(n) \cdot w(n)$$

Example: rectangular window

$$w(n) = \begin{cases} 1 & n = 0, 1, \ldots, M-1 \\ 0 & \text{otherwise} \end{cases}$$

$$h(n) = h_d(n) \cdot w(n)$$

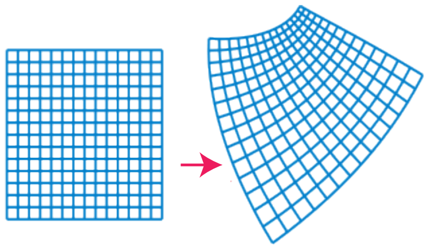
$$= \begin{cases} \frac{\sin \omega_c \left(n - \frac{M-1}{2}\right)}{\pi \left(n - \frac{M-1}{2}\right)} & 0 \leq n \leq M-1, n \neq \frac{M-1}{2} \\ \frac{\omega_c}{\pi} & n = \frac{M-1}{2} \text{ and } M \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

Can we get better filter performance?

- Yes. Use IIR filters.
- IIR digital filters can be designed by converting a well-known analog filter into a digital one.
- For the same number of parameters, better compromises between ringing and transition band width can be found.

IIR Filter Design via Bilinear Transformation

- bilinear transformation: mapping from the s-plane to the z-plane
  - conformal mapping (mapping that preserves local angles among curves) that transforms the vertical axis of the s-plane into the unit circle in the z-plane

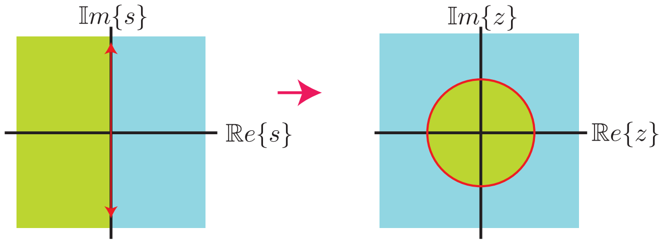


IIR Filter Design via Bilinear Transformation

- bilinear transformation: mapping from the s-plane to the z-plane
  - conformal mapping (mapping that preserves local angles among curves) that transforms the vertical axis of the s-plane into the unit circle in the z-plane
  - all points in the left half plane (LHP) of s are mapped into corresponding points inside the unit circle in the z-plane
  - all points in the right half plane (RHP) of s are mapped into corresponding points outside the unit circle in the z-plane

IIR Filter Design via Bilinear Transformation

- bilinear transformation: mapping from the s-plane to the z-plane



Bilinear Transformation: Example

$$H_a(s) = \frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

$$\begin{aligned} Y(s)(s+a) &= bX(s) \\ sY(s) + aY(s) &= bX(s) \end{aligned}$$

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

Note: we will use $\frac{dy(t)}{dt}$ and $y'(t)$ interchangeably.

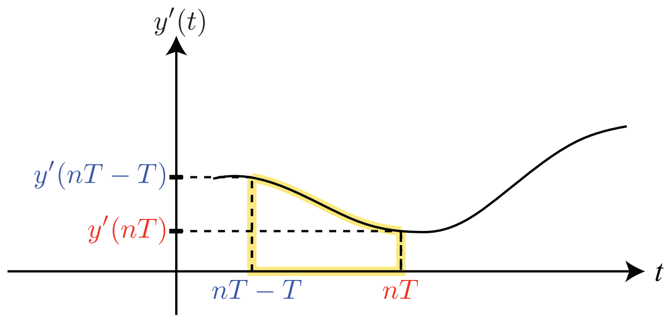
Bilinear Transformation: Example

Consider:

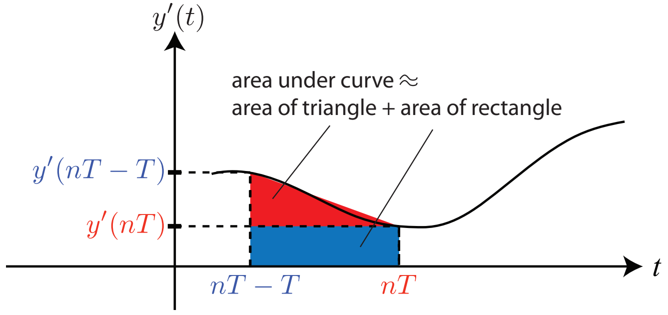
$$y(t) = \underbrace{\int_{t_0}^{t} y'(\tau)d\tau}_{\equiv I} + y(t_0)$$

Let $t = nT$ and $t_0 = nT - T$ and using the trapezoidal approximation of the integral:

$$y(nT) = \underbrace{\frac{T}{2}\left[y'(nT) + y'(nT-T)\right]}_{\text{we will show this } \approx I} + y(nT-T)$$



$$\text{Area under curve} = I = \int_{nT-T}^{nT} y'(\tau)d\tau$$



$$
\begin{aligned}
I = \int_{nT-T}^{nT} y'(\tau)d\tau &\approx b_{rect} \times h_{rect} + \frac{b_{tri} \times h_{tri}}{2} \\
&= T \cdot y'(nT) + \frac{T \cdot (y'(nT-T) - y'(nT))}{2} \\
&= \frac{T}{2}\left[y'(nT) + y'(nT-T)\right]
\end{aligned}
$$

Bilinear Transformation: Example

Therefore we indeed have:

$$y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT-T)\right] + y(nT-T).$$

Plugging $t = nT, nT - T$ into $y'(t) + ay(t) = bx(t)$ gives:

$$y(nT) = \frac{T}{2}\Bigg[\underbrace{y'(nT)}_{-ay(nT)+bx(nT)} + \underbrace{y'(nT-T)}_{-ay(nT-T)+bx(nT-T)}\Bigg] + y(nT-T)$$

and letting $x(n) \equiv x(nT)$ and $y(n) \equiv y(nT)$, we obtain ...

Bilinear Transformation: Example

$$y(n) = \frac{T}{2}\left[-ay(n) + bx(n) - ay(n-1) + b(n-1)\right] + y(n-1)$$

$$
\begin{aligned}
\left(1 + \frac{aT}{2}\right) y(n) - \left(1 - \frac{aT}{2}\right) y(n-1) &= \frac{bT}{2}\left[x(n) + x(n-1)\right] \\
\left(1 + \frac{aT}{2}\right) Y(z) - \left(1 - \frac{aT}{2}\right) z^{-1}Y(z) &= \frac{bT}{2}\left[X(z) + z^{-1}X(z)\right]
\end{aligned}
$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a}$$

Bilinear Transformation: Example

Compare:

$$H(z) = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a}$$

to:

$$H_a(s) = \frac{b}{s + a}$$

Therefore, the bilinear transformation mapping is:

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

Bilinear Transformation

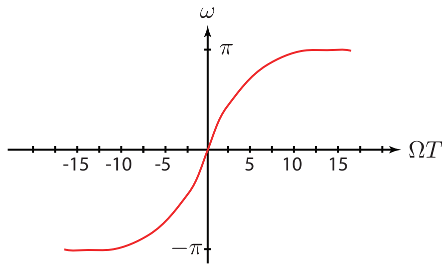
The mapping $s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ will work for any order of differential equation to convert $H_a(s)$ to $H(z)$.

General Methodology:

1. Start with $H_a(s)$ expression.
2. Determine T through the problem specifications.
3. $H(z) = H_a\left(\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)$

Bilinear Transformation

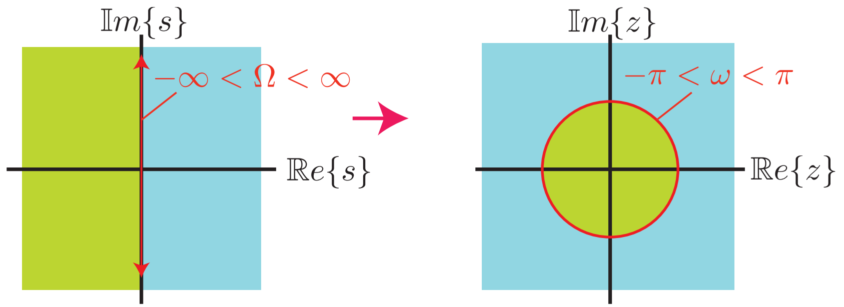
For $s = j\Omega$ and $z = e^{j\omega}$:



The entire $-\infty < \Omega < \infty$ axis is mapped to $-\pi < \omega < \pi$. There is a huge compression of the frequency response at large Ω-values.

Bilinear Transformation

For $s = j\Omega$ and $z = e^{j\omega}$:



■