# Discrete-Time Sinusoids

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## Periodicity

Recall if a signal $x(t)$ is periodic, then there exists a $T > 0$ such that

$$x(t) = x(t+T)$$

If no $T > 0$ can be found, then $x(t)$ is non-periodic.





## Continuous-time Sinusoids

To find the period $T > 0$ of a general continuous-time sinusoid $x(t) = A\cos(\omega t + \phi)$:

$$\begin{aligned} x(t) &= x(t+T) \\ A\cos(\omega t+\phi) &= A\cos(\omega(t+T)+\phi) \\ A\cos(\omega t+\phi+2\pi k) &= A\cos(\omega t+\phi+\omega T) \\ \therefore \quad 2\pi k &= \omega T \\ T &= \frac{2\pi k}{\omega} \end{aligned}$$

where $k \in \mathbb{Z}$. Note: when $k$ is the same sign as $\omega$, $T > 0$.

Therefore, there exists a $T > 0$ such that $x(t) = x(t+T)$ and therefore $x(t)$ is $\boxed{\text{periodic}}$.





## Discrete-time Sinusoids

To find the integer period $N > 0$ (i.e., ($N \in \mathbb{Z}^+$) of a general discrete-time sinusoid $x[n] = A\cos(\Omega n + \phi)$:

$$\begin{aligned} x[n] &= x[n+N] \\ A\cos(\Omega n+\phi) &= A\cos(\Omega(n+N)+\phi) \\ A\cos(\Omega n+\phi+2\pi k) &= A\cos(\Omega n+\phi+\Omega N) \\ \therefore \quad 2\pi k &= \Omega N \\ N &= \frac{2\pi k}{\Omega} \end{aligned}$$

where $k \in \mathbb{Z}$.

Note: there may not exist a $k \in \mathbb{Z}$ such that $\frac{2\pi k}{\Omega}$ is an integer.

# Discrete-time Sinusoids

Example i: $\Omega = \frac{37}{11}\pi$

$$N = \frac{2\pi k}{\Omega} = \frac{2\pi k}{\frac{37}{11}\pi} = \frac{22}{37}k$$
$$N_0 = \frac{22}{37}k = \boxed{22} \text{ for } k = 37;\ x[n] \text{ is periodic.}$$

Example ii: $\Omega = 2$

$$N = \frac{2\pi k}{\Omega} = \frac{2\pi k}{2} = \pi k$$
$$N \in \mathbb{Z}^+ \text{ does not exist for any } k \in \mathbb{Z};\ x[n] \text{ is non-periodic.}$$

Example iii: $\Omega = \sqrt{2}\pi$

$$N = \frac{2\pi k}{\Omega} = \frac{2\pi k}{\sqrt{2}\pi} = \sqrt{2}k$$
$$N \in \mathbb{Z}^+ \text{ does not exist for any } k \in \mathbb{Z};\ x[n] \text{ is not periodic.}$$





# Discrete-time Sinusoids

$$N = \frac{2\pi k}{\Omega}$$
$$\Omega = \frac{2\pi k}{N} = 2\pi\frac{k}{N} = \pi \cdot \underbrace{\boxed{\frac{2k}{N}}}_{\text{RATIONAL}}$$

Therefore, a discrete-time sinusoid is periodic if its radian frequency $\Omega$ is a rational multiple of $\pi$.

Otherwise, the discrete-time sinusoid is non-periodic.





# Example 1: $\Omega = \pi/6 = \pi \cdot \boxed{\frac{1}{6}}$

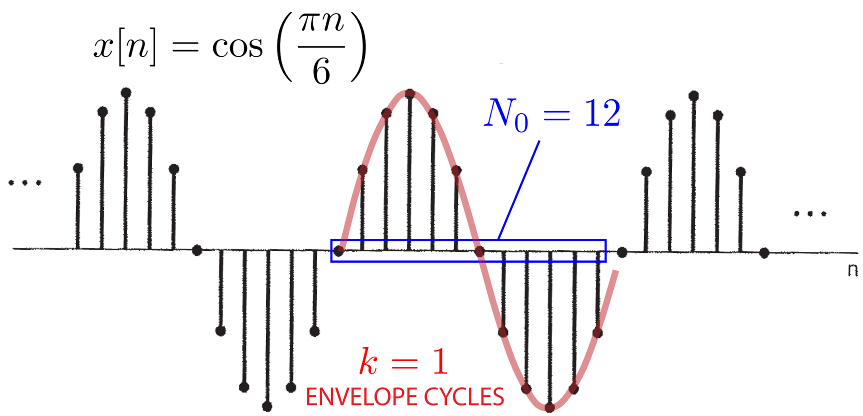
$$x[n] = \cos\left(\frac{\pi n}{6}\right)$$

$$N = \frac{2\pi k}{\Omega} = \frac{2\pi k}{\pi\frac{1}{6}} = 12k$$
$$N_0 = 12 \quad \text{for } k = 1$$

The fundamental period is 12 which corresponds to $k = 1$ envelope cycles.







## Example 2: $\Omega = 8\pi/31 = \pi \cdot \boxed{\frac{8}{31}}$

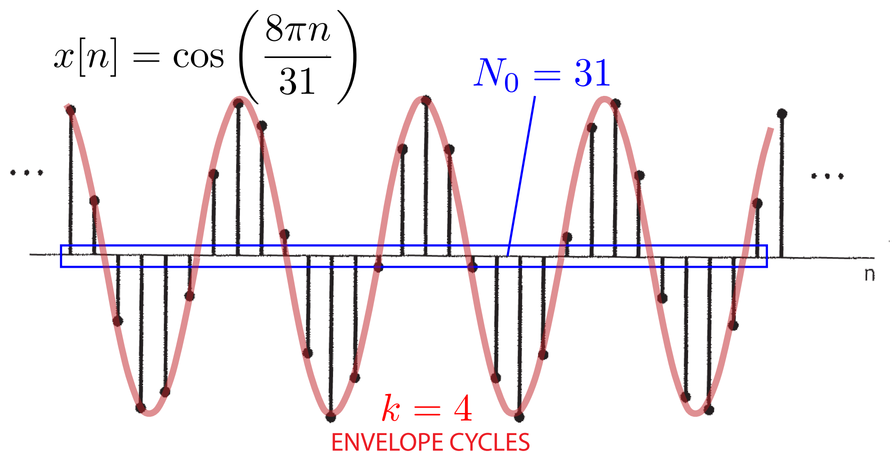
$$x[n] = \cos\left(\frac{8\pi n}{31}\right)$$

$$\begin{aligned} N &= \frac{2\pi k}{\Omega} = \frac{2\pi k}{\pi \frac{8}{31}} = \frac{31}{4}k \\ N_0 &= 31 \quad \text{for } k = 4 \end{aligned}$$

The fundamental period is 31 which corresponds to $k = 4$ envelope cycles.











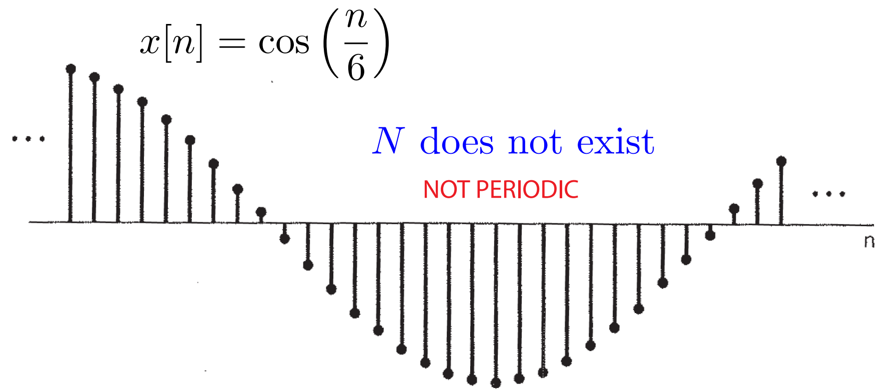

## Example 3: $\Omega = 1/6 = \pi \cdot \boxed{\frac{1}{6\pi}}$

$$x[n] = \cos\left(\frac{n}{6}\right)$$

$$\begin{aligned} N &= \frac{2\pi k}{\Omega} = \frac{2\pi k}{\frac{1}{6}} = 12\pi k \\ N &\in \mathbb{Z}^+ \quad \text{does not exist for any } k \in \mathbb{Z};\ x[n] \text{ is non-periodic.} \end{aligned}$$







## Continuous-Time Sinusoids: Frequency and Rate of Oscillation

$$x(t) = A\cos(\omega t + \phi)$$

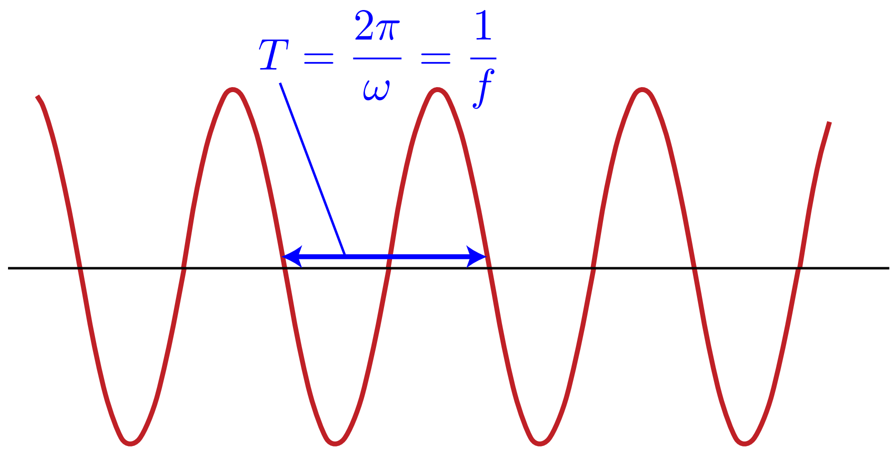
$$T \;=\; \frac{2\pi}{\omega} = \frac{1}{f}$$

Rate of oscillation increases as $\omega$ increases (or $T$ decreases).





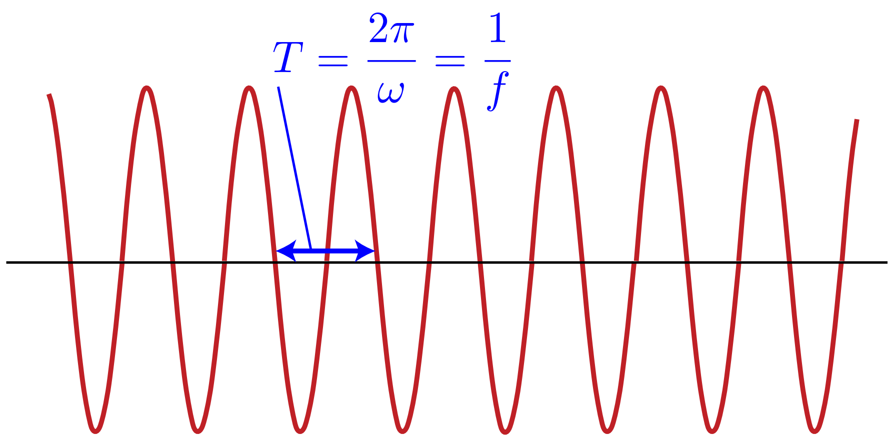
## $\omega$ smaller








## $\omega$ larger, rate of oscillation higher





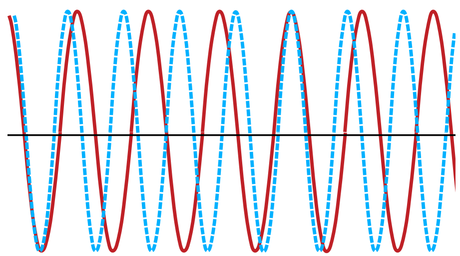



## Continuous-Time Sinusoids: Frequency and Rate of Oscillation

Also, note that $x_1(t) \neq x_2(t)$ for all $t$ for

$$x_1(t) = A\cos(\omega_1 t + \phi) \quad \text{and} \quad x_2(t) = A\cos(\omega_2 t + \phi)$$

when $\omega_1 \neq \omega_2$.



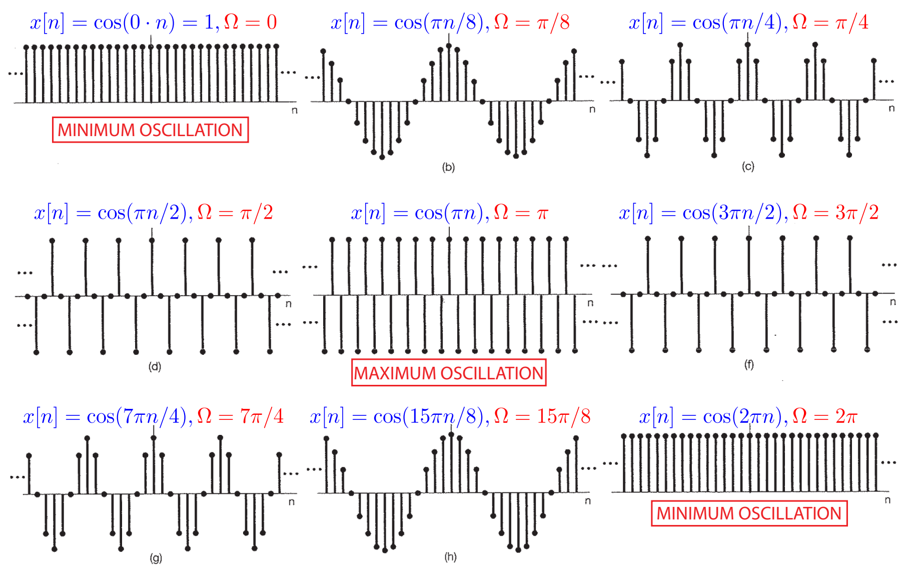
# Discrete-Time Sinusoids: Frequency and Rate of Oscillation

$$x[n] = A\cos(\Omega n + \phi)$$

Rate of oscillation increases as $\Omega$ increases UP TO A POINT then decreases again and then increases again and then decreases again ....












# Discrete-Time Sinusoids: Frequency and Rate of Oscillation

$$x[n] = A\cos(\Omega n + \phi)$$

$\boxed{\text{Discrete-time sinusoids repeat as } \Omega \text{ increases!}}$





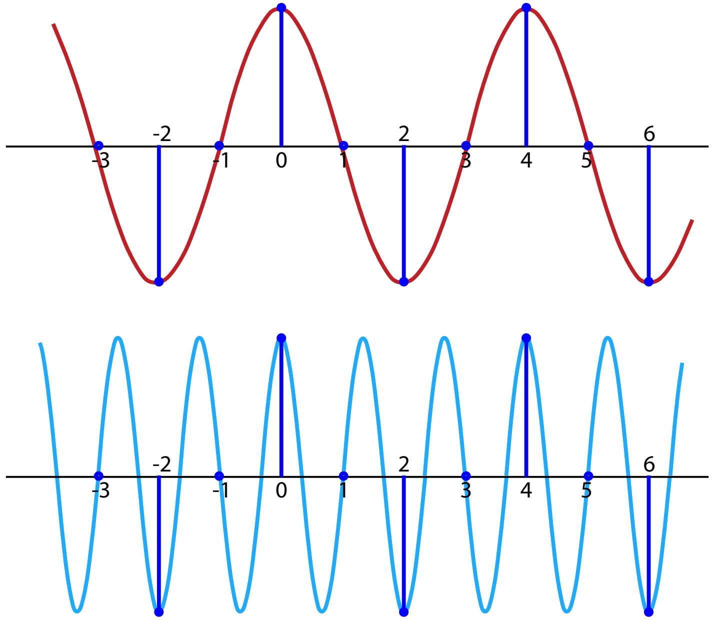
# Discrete-Time Sinusoids: Frequency and Rate of Oscillation

Let

$$x_1[n] = A\cos(\Omega_1 n + \phi) \quad \text{and} \quad x_2[n] = A\cos(\Omega_2 n + \phi)$$

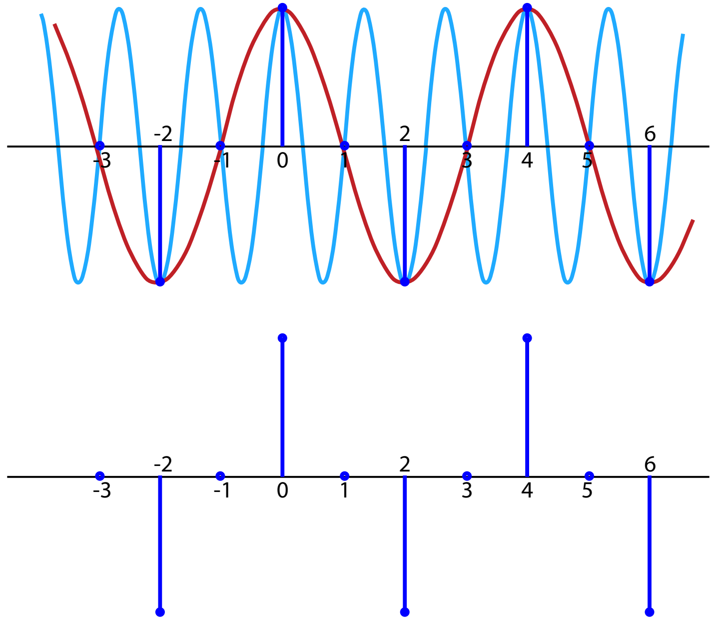
and $\Omega_2 = \Omega_1 + 2\pi k$ where $k \in \mathbb{Z}$:

$$\begin{aligned} x_2[n] &= A\cos(\Omega_2 n + \phi) \\ &= A\cos((\Omega_1 + 2\pi k)n + \phi) \\ &= A\cos(\Omega_1 n + 2\pi k n + \phi) \\ &= A\cos(\Omega_1 n + \phi) = x_1[n] \end{aligned}$$












## Discete-Time Sinusoids: Frequency and Rate of Oscillation

$$x[n] = A\cos(\Omega n + \phi)$$

can be considered a sampled version of

$$x(t) = A\cos(\Omega t + \phi)$$

at integer time instants.

As $\Omega$ increases, the samples miss the faster oscillatory behavior. ■