Magnitude and Phase

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Complex Nature of $X(j\omega)$

Recall, Fourier Transform:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \quad \in \quad \mathbb{C}$$

and Inverse Fourier Transform:

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{0} X(j\omega)e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{0}^{\infty} X(j\omega)e^{j\omega t}d\omega \end{aligned}$$

Note: If $x(t)$ is real, then the imaginary part of the negative frequency sinusoids (i.e., $e^{j\omega t}$ for $\omega<0$) cancel out the imaginary part of the positive frequency sinusoids (i.e., $e^{j\omega t}$ for $\omega>0$)

Complex Nature of $X(j\omega)$

- Rectangular coordinates: rarely used in signal processing

$$X(j\omega) = X_R(j\omega) + j\, X_I(j\omega)$$

where $X_R(j\omega), X_I(j\omega) \in \mathbb{R}$.

- Polar coordinates: more intuitive way to represent frequency content

$$X(j\omega) = |X(j\omega)|\; e^{j\angle X(j\omega)}$$

where $|X(j\omega)|, \angle X(j\omega) \in \mathbb{R}$.

Magnitude and Phase of $X(j\omega)$

- $|X(j\omega)|$: determines the relative presence of a sinusoid $e^{j\omega t}$ in $x(t)$
- $\angle X(j\omega)$: determines how the sinusoids line up relative to one another to form $x(t)$

Magnitude and Phase of $X(j\omega)$

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|\ e^{j\angle X(j\omega)}e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|\ e^{j(\omega t+\angle X(j\omega))}d\omega \end{aligned}$$

- Recall, $e^{j(\omega t+\angle X(j\omega))} = \cos(\omega t + \angle X(j\omega)) + j\sin(\omega t + \angle X(j\omega))$.
- The larger $|X(j\omega)|$ is, the more prominent $e^{j\omega t}$ is in forming $x(t)$.
- $\angle X(j\omega)$ determines the relative phases of the sinusoids (i.e. how they line up with respect to one another).

Magnitude versus Phase

Q: Which is more important for a given signal?

- Does one component (magnitude or phase) contain more information than another?
- When filtering, if we had to preserve on component (magnitude or phase) more, which one would it be?

Example: audio information signal

- An audio signal is represented by a real function $x(t)$.
- The function $x(-t)$ represents playing the audio signal backwards.
- Since $x(t)$ is real:

$$\begin{aligned} X(j\omega) &= X^*(-j\omega) \\ |X(j\omega)| &= |X^*(-j\omega)| = |X(-j\omega)| \quad \text{since } |c| = |c^*| \text{ for } c \in \mathbb{C} \end{aligned}$$

- Therefore,

$$|X(j\omega)| = |X(-j\omega)|$$

  That is, the FT magnitude is even for $x(t)$ real.

Example: audio information signal

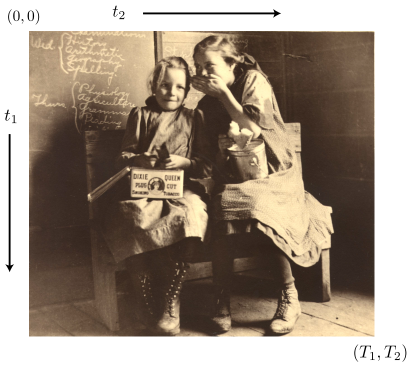
- Recall, $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$ $\quad x(-t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(-j\omega)$
- Therefore,

$$\underbrace{|X(j\omega)|}_{\text{spectrum magnitude of } x(t)} = \overbrace{|X(-j\omega)|}^{\text{spectrum magnitude of } x(-t)}$$

Therefore, the magnitude of the FT of an audio signal played forward and backward is the same!

Example: grayscale still images

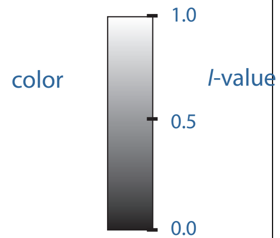
- A still image can be considered a two-dimensional signal: $x(t_1, t_2)$ where t_1 represents the horizontal dimension and t_2 represents the vertical dimension.



The image shown is "Dixie Queens" (two schoolgirls at lunch from Hadleyville, Oregon, circa 1911), Roy C. Andrews collection, PH003-P954, Special Collections and University Archives, University of Oregon, Eugene, Oregon 97403-1299.

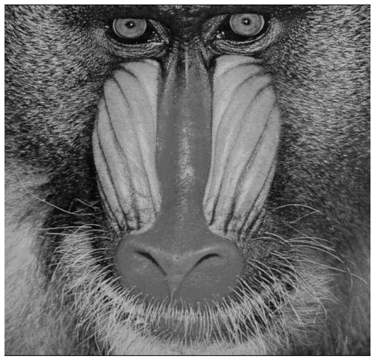
Analog Intensity Images

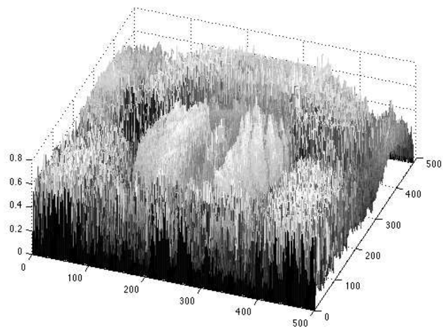
- continuous-space and continuous-amplitude image consisting of intensity (grayscale) values
- $x(t_1, t_2)$ is a two-dimensional signal representing the grayscale value at location (t_1, t_2) where:
  - $0 \leq t_1 \leq T_1$ and $0 \leq t_2 \leq T_2$
  - $x(t_1, t_2) = 0$ represents black
  - $x(t_1, t_2) = 1$ represents white
  - $0 < x(t_1, t_2) < 1$ represents proportional gray-value



Analog Intensity Images

- $x(t_1, t_2)$ can be displayed as an intensity image or as a mesh graph





Example: grayscale still images

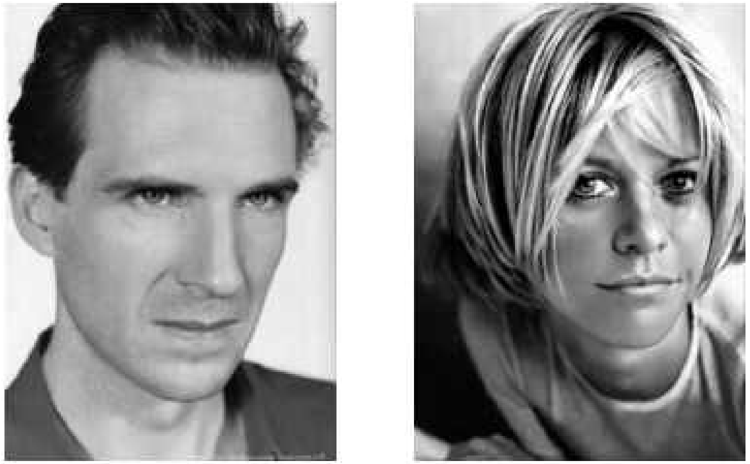
- The Fourier transform $x(t_1, t_2)$ has two frequency variables: ω_1 and ω_2 and is given by:

$$X(j\omega_1, j\omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t_1, t_2) e^{-j(\omega_1 t_1 + \omega_2 t_2)} dt_1 dt_2 \quad \in \quad \mathbb{C}$$

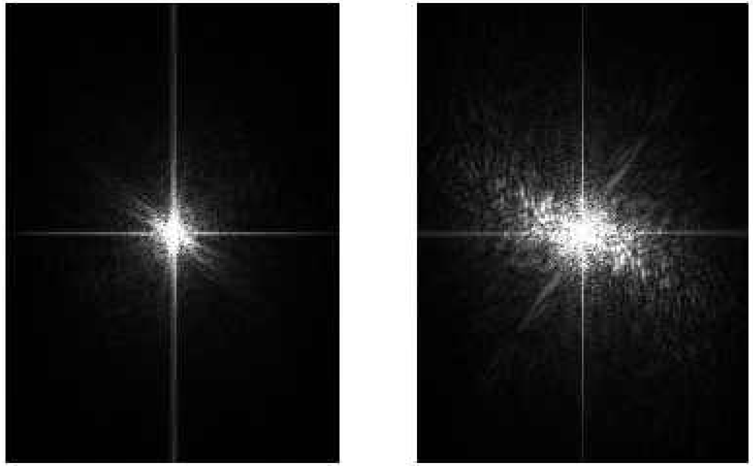
- Typically, we consider the magntiude and phase of $X(j\omega_1, j\omega_2)$:

$$|X(j\omega_1, j\omega_2)| \quad \text{and} \quad \angle X(j\omega_1, j\omega_2)$$

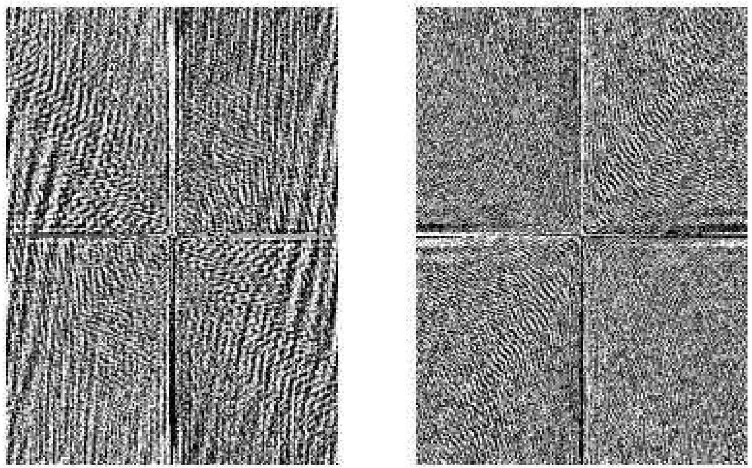
Example: $x(t_1, t_2)$

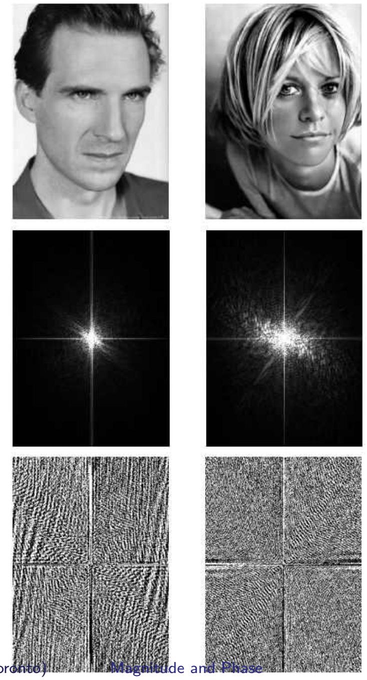


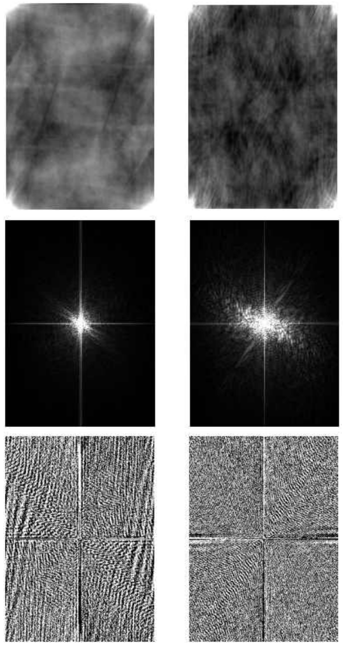
Example: $|X(j\omega_1, j\omega_2)|$



Example: $\angle X(j\omega_1, j\omega_2)$



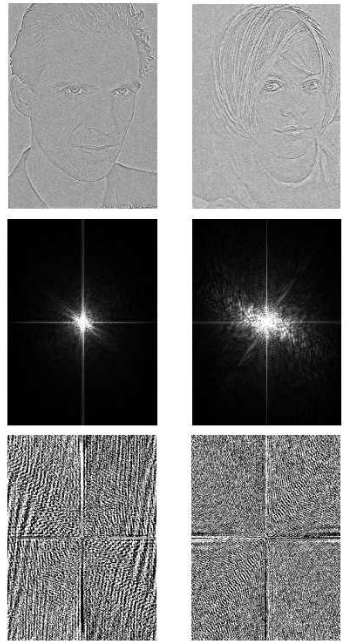




Reconstruction using magnitude only

Top Left Photo: Ralph's magnitude is the same, Phase = 0

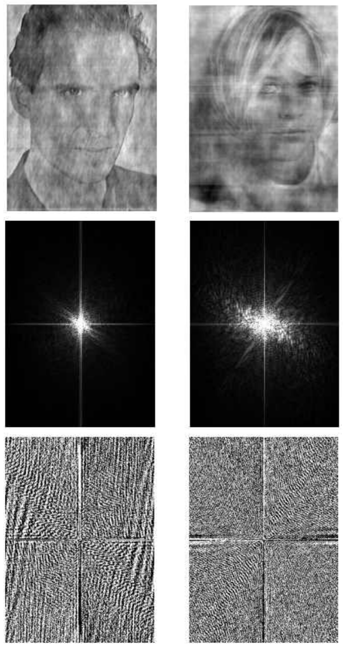
Top Right Photo: Meg's magnitude is the same, Phase = 0



Reconstruction using phase only

Top Left Photo: Ralph's magnitude normalized to one, Phase is the same

Top Right Photo: Meg's magnitude normalized to one, Phase is the same



Reconstruction swapping magnitude and phase of the images.

Top Left Photo: Ralph's phase + Meg's magnitude

Top Right Photo: Meg's phase + Ralph's magnitude

Magnitude versus Phase

Q: Which is more important for a given signal? **A:** Phase.

- Does one component (magnitude or phase) contain more information than another? **A:** Yes, typically phase.
- When filtering, if we had to preserve on component (magnitude or phase) more, which one would it be?
  A: It is important to preserve phase during filtering.

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