# System Properties

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## Classification of Discrete-Time Systems

Why is this so important?

- mathematical techniques developed to analyze systems are often contingent upon the general characteristics of the systems being considered
- for a system to possess a given property, the property must hold for every possible input to the system
  - to disprove a property, need a single counterexample
  - to prove a property, need to prove for the general case

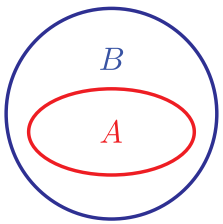




## Terminology: Implication

If "$A$" then "$B$" Shorthand: $A \Longrightarrow B$

Example 1:
it is snowing $\Longrightarrow$ it is at or below freezing temperature
Example 2:
$\alpha \geq 5.2 \Longrightarrow \alpha$ is positive
Note: For both examples above, $B \not\Longrightarrow A$








## Terminology: Equivalence

If "$A$" then "$B$" Shorthand: $A \Longrightarrow B$

$\boxed{\text{and}}$

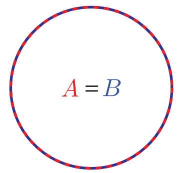
If "$B$" then "$A$" Shorthand: $B \Longrightarrow A$

can be rewritten as

"$A$" if and only if "$B$" Shorthand: $A \Longleftrightarrow B$

We can also say:

- $A$ is EQUIVALENT to $B$
- $A = B$



## Terminology: Systems

- A cts-time system processes a cts-time input signal to produce a cts-time output signal.

$$y(t) = H\{x(t)\}$$

- A dst-time system processes a dst-time input signal to produce a dst-time output signal.

$$y[n] = H\{x[n]\}$$

$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t) \qquad x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

Note: iff = "if and only if"





## Stability

- Bounded Input-Bounded output (BIBO) stable system: every bounded input produces a bounded output

$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t) \qquad x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

- a cts-time system is BIBO stable iff

$$|x(t)| \leq M_x < \infty \implies |y(t)| \leq M_y < \infty$$

for all $t$.

- a dst-time system is BIBO stable iff

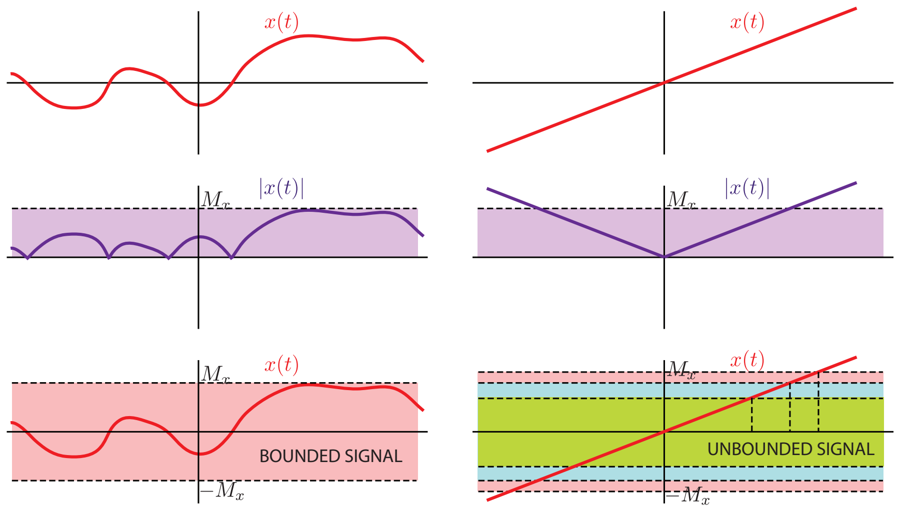
$$|x[n]| \leq M_x < \infty \implies |y[n]| \leq M_y < \infty$$

for all $n$.





## Bounded Signals








## Stability

Examples: Are each of the following systems BIBO stable?

1. $y(t) = A\, x(t)$, note: $|A| < \infty$
2. $y(t) = A\, x(t) + B$, , note: $|A|, |B| < \infty$, $B \neq 0$
3. $y[n] = n\, x[n]$
4. $y(t) = x(t) \cos(\omega_c t)$
5. $y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$
6. $y[n] = r^n x[n]$, note: $|r| > 1$
7. $y[n] = \frac{1}{1 - x[n+2]}$
8. $y(t) = e^{3x(t)}$

Ans: Y, Y, N, Y, Y, N, N, Y

# Memory

- Memoryless system: output signal depends only on the present value of the input signal
  - cts-time: $y(t)$ only depends on $x(t)$ for all $t$
  - dst-time: $y[n]$ only depends on $x[n]$ for all $n$

- Note: a system that is not memoryless has memory
- System with Memory: output signal depends on past or future values of the input signal





# Memory

Examples: Do each of the following systems have memory?

1. $y(t) = A\, x(t)$
2. $y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$
3. $y[n] = n\, x[n]$
4. $y(t) = x(t)\cos(\omega_c(t-1))$
5. $y[n] = x[-n]$
6. $y(t) = a_0 + a_1 x(t) + a_2 x^2(t) + a_3 x^3(t) \cdots$
7. $y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$
8. $y(t) = e^{3x(t)}$

Ans: N, Y, N, N, Y, N, Y, N





# Causality

- Causal system: present value of the output signal depends only on the present or past values of the input signal
- a cts-time system is causal iff

$$y(t) = F\left[x(\tau)|\tau \leq t\right]$$

  for all $t$
- a dst-time system is causal iff

$$y[n] = F\left[x[n], x[n-1], x[n-2], \ldots\right]$$

  for all $n$





# Causality

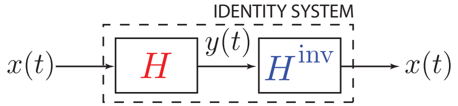
Examples: Are each of the following systems causal?

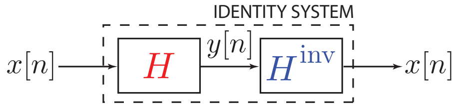
1. $y(t) = A\, x(t)$
2. $y(t) = A\, x(t) + B,\ B \neq 0$
3. $y[n] = (n+1)\, x[n]$
4. $y(t) = x(t)\cos(\omega_c(t+1))$
5. $y[n] = x[-n]$
6. $y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$
7. $y[n] = \frac{1}{1-x[n+2]}$
8. $y(t) = e^{3x(t)}$

Ans: Y, Y, Y, Y, N, N, N, Y

# Invertibility

- Invertible system: input of the system can always be recovered from the output
- a system is invertible iff there exists an inverse system as follows




- Consider

$$
\begin{aligned}
x(t) &= H^{\text{inv}}\{y(t)\} = H^{\text{inv}}\{H\{x(t)\}\} \\
x(t) &= H^{\text{inv}}\{H\{x(t)\}\} \qquad \text{IDENTITY SYSTEM}
\end{aligned}
$$





# Invertibility

- A system that is invertible has a one-to-one mapping between input and output. That is, a given output can be mapped to a single possible input that generated it.
- A system that is not invertible can be shown to have two or more input signals that produce the same output signal.





# Invertibility

Examples: Are each of the following systems invertible?

1. $y(t) = A\,x(t)$, note: $A \neq 0$
2. $y(t) = A\,x(t) + B$, note: $A, B \neq 0$
3. $y[n] = n\,x[n]$
4. $y(t) = \frac{1}{L}\int_{-\infty}^{t} x(\tau)d\tau$
5. $y[n] = x[-n]$
6. $y(t) = x^2(t-1)$
7. $y[n] = \sum_{k=-\infty}^{n} x[k]$
8. $y(t) = e^{3x(t)}$

Ans: Y, Y, N, Y, Y, N, Y, Y





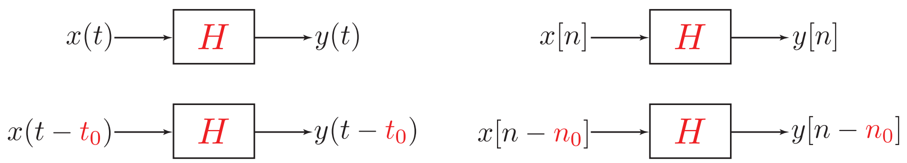
# Invertibility

Examples: The associated inverse systems are:

1. $y(t) = \frac{x(t)}{A}$, note: $A \neq 0$
2. $y(t) = \frac{x(t)-B}{A}$, note: $A, B \neq 0$
3. N/A; $x_1[n] = \delta[n]$ and $x_2[n] = 2\delta[n]$ give the same output $y[n] = 0$
4. $y(t) = \frac{dx(t)}{dt}$
5. $y[n] = x[-n]$
6. N/A; $x_1(t) = 1$ and $x_2(t) = -1$ give the same output $y(t) = 1$
7. $y[n] = x[n] - x[n-1]$
8. $y(t) = \frac{\ln(x(t))}{3}$

# Time-invariance

- Time-invariant system: a time delay or time advance of the input signal leads to an identical time shift in the output signal;








# Time-invariance

- The characteristics of $H$ do not change with time.
- a cts-time system $H$ is time-invariant iff

$$y(t) = H\{x(t)\} \implies y(t-t_0) = H\{x(t-t_0)\}$$

for every input $x(t)$ and every time shift $t_0$.

- a dst-time system $H$ is time-invariant iff

$$y[n] = H\{x[n]\} \implies y[n-n_0] = H\{x[n-n_0]\}$$

for every input $x[n]$ and every time shift $n_0$.





# Time-invariance

Examples: Are each of the following systems time-invariant?

1. $y(t) = A\, x(t)$
2. $y(t) = A\, x(t) + B$
3. $y[n] = n\, x[n]$
4. $y(t) = x(t)\cos(\omega_c t)$
5. $y[n] = x[-n]$
6. $y(t) = \frac{1}{L}\int_{-\infty}^{t} x(\tau)d\tau$
7. $y[n] = \frac{1}{1-x[n+2]}$
8. $y(t) = e^{3x(t)}$

Ans: Y, Y, N, N, N, Y, Y, Y

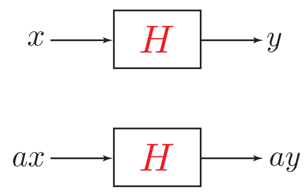




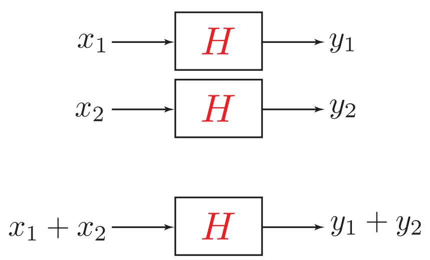
# Linearity

- Linear system: obeys superposition principle
- Linearity = Homogeniety + Additivity

Homogenous system:



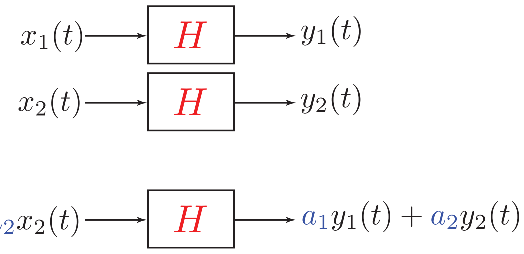

Additive system:



# Linearity

- a cts-time system $H$ is linear iff

$$\begin{array}{rcl} y_1(t) & = & H\{x_1(t)\} \\ y_2(t) & = & H\{x_2(t)\} \end{array} \implies a_1 y_1(t) + a_2 y_2(t) = H\{a_1 x_1(t) + a_2 x_2(t)\}$$



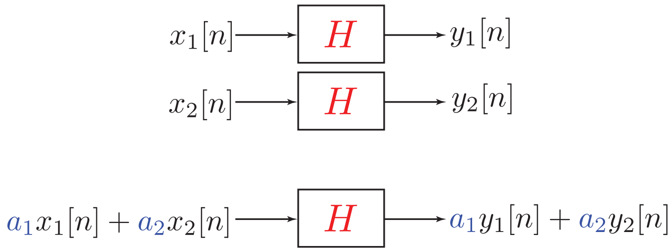





# Linearity

- a dst-time system $H$ is linear iff

$$\begin{array}{rcl} y_1[n] & = & H\{x_1[n]\} \\ y_2[n] & = & H\{x_2[n]\} \end{array} \implies a_1 y_1[n] + a_2 y_2[n] = H\{a_1 x_1[n] + a_2 x_2[n]\}$$








# Linearity

Examples: Are each of the following systems linear?

1. $y(t) = A\,x(t)$
2. $y(t) = A\,x(t) + B$, $B \neq 0$
3. $y[n] = n\,x[n]$
4. $y(t) = x(t)\cos(\omega_c t)$
5. $y[n] = x[-n]$
6. $y(t) = x^2(t-1)$
7. $y[n] = \frac{1}{1-x[n+2]}$
8. $y(t) = e^{3x(t)}$

Ans: Y, N, Y, Y, Y, N, N, N





# Final Words

To prove a property, you must show that it holds in general. For instance, for all possible inputs and/or time instants.

To disprove a property, provide a simple counterexample to the definition.

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