EXIT-Chart Properties of the Highest-Rate LDPC Code with Desired Convergence Behavior

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***Abstract*— We consider uni-parametric LDPC decoding schemes, i.e., the class of decoding algorithms for which an extrinsic information transfer (EXIT) chart analysis of the decoder is exact. We treat the general case of code design for a desired convergence behavior and provide necessary conditions and sufficient conditions that the EXIT chart of the maximum rate low-density parity-check code must satisfy. Our results generalize some of the existing results for the binary erasure channel: our results apply to all uni-parametric decoding schemes and they apply to any desired convergence behavior.**

***Index Terms*— LDPC codes, EXIT chart.**

I. Introduction

Irregular low-density parity-check (LDPC) codes can have a significantly better performance compared to regular ones [1], [2], thus the design of irregular LDPC codes has been of great interest, e.g. [1], [3]. For the binary erasure channel (BEC), it is known that a capacity-achieving LDPC code has a truly flat extrinsic information transfer (EXIT) chart, i.e., its first derivative approaches one and its higher derivatives approach zero everywhere [3]. All known capacity-achieving codes have infinitely long degree distributions [4]; indeed, it is known that a truly flat curve cannot always be achieved using finite degree distributions.

In the case of the BEC, the area under the EXIT chart scales with the rate of the code [5]. Although such a nice relation between area and rate does not hold for other channels, it seems that a similar concept is still valid. For example, Fig. 1 compares the EXIT chart of two irregular codes under sum-product decoding on the same AWGN channel. One can see that the higher rate code has a more "flat" EXIT chart.

In this work we relate the EXIT chart of an irregular code to its rate without considering a specific decoding rule. Inevitably the results cannot be as strong as the existing results for BEC, but their general nature is their advantage.

Following [2], we specify an irregular LDPC code by its variable and check degree distributions $\Lambda = \{\lambda_2, \lambda_3, \ldots, \lambda_I\}$ and $\mathcal{P} = \{\rho_2, \rho_3, \ldots \rho_J\}$, where λ_i (ρ_i) shows the fraction of edges connected to variable (check) nodes of degree i.

II. Problem Definition

When the density of messages in a message passing iterative decoder can be described by a single parameter, density

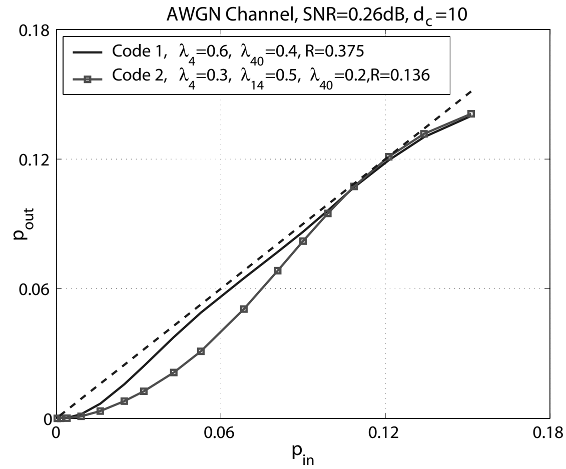

Fig. 1. Comparison of EXIT charts for two irregular codes.

evolution [1] is equivalent to tracking the evolution of that parameter, hence, a single-parameter analysis of the decoder is an exact analysis. We refer to such decoding schemes, which are the focus of this paper, as "uni-parametric" decoding schemes. Belief propagation in the BEC is one of the most famous examples of a uni-parametric decoding scheme. Another important case is when the messages are binary valued, e.g., Gallager's decoding algorithm A and B [1].

Even when the decoding scheme is not uni-parametric, a single-parameter analysis can be used to approximate the behavior of such decoders, e.g., [6]–[8]. Although the results of this paper are strictly valid only for uni-parametric decoding schemes, they can still be used within the framework of any single-parameter approximation of other decoding schemes.

A. EXIT Charts

Under the symmetry assumption [1], which is required for density evolution, a message error rate can be defined [7]. We assume a 1-to-1 correspondence between message error rate and the parameter defining the message distribution.

We are most interested in p_{in} vs. p_{out} EXIT charts [1], where p_{in} and p_{out} are the input and the output message error rate at each iteration. This is because, for a fixed $\mathcal{P}$, the p_{in} vs. p_{out} EXIT chart of an irregular code is a linear combination of the p_{in} vs. p_{out} EXIT charts corresponding to fixed variable degree codes [7]. The weights of this linear combination are determined by Λ. Hence, using p_{in} vs. p_{out} EXIT charts makes

[1]This is a generalization of the term EXIT chart, which is usually used when mutual information is the parameter whose evolution is tracked.

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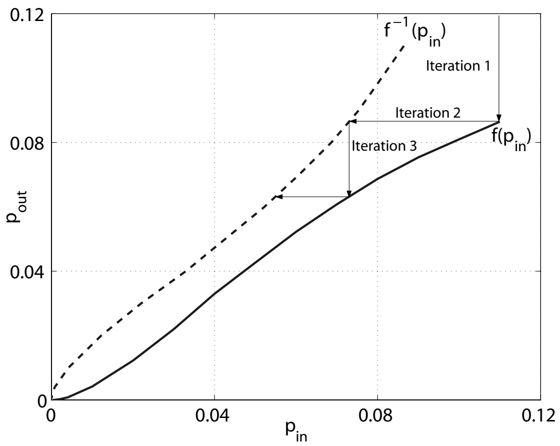

Fig. 2. Shows the concept of EXIT charts.

the analysis and design of irregular codes more insightful. Thus, in the remainder of this paper an EXIT chart is defined as a curve that, for a fixed channel condition, represents p_{out} as a function of p_{in}, i.e., $p_{out} = f(p_{in})$.

Usually both f and its inverse are plotted. In this way one can track the decoding as shown in Fig. 2. Note that successful decoding can only occur if $p_{out} < p_{in}$ for all $p_{in} \leq p_0$, where p_0 is the error rate from the channel.

B. Problem Formulation

In this work we treat the code design problem for a fixed $\mathcal{P}$. We refer to the EXIT chart of a code with a fixed variable degree i by $f_i(x)$ and we call it an elementary EXIT chart (as opposed to the EXIT chart of an irregular code which is a combination of elementary EXIT charts). The EXIT chart of an irregular code with the variable degree distribution $\Lambda = \{\lambda_i, i \in \mathcal{I}\}$, $\mathcal{I} = \{2, \ldots, I\}$, can be written as $f(x) = \sum_{i \in \mathcal{I}} \lambda_i f_i(x)$.

Unlike previous work on irregular code design that only considers convergence to zero error rate, we treat here the general case of code design with a desired convergence behavior. This allows for the tradeoff of code rate for convergence performance. To see this, suppose the EXIT chart of a code $f(x)$ is very close to x, for example $ax < f(x) < x$, and $a \to 1^-$, as would be the case for capacity-approaching sequences for the BEC [3]. It is clear that after n iterations a lower bound on the message error rate would be $p_0 a^n$, and so a large number of iterations, on the order of $\frac{1}{\log(a)}$, is required to achieve a small message error rate. Hence, in general one might be interested in trading code rate for decoding complexity. Thus, we will be interested in the maximum rate code while guaranteeing a specific convergence behavior described by $h(x)$.

The design problem is, then, equivalent to shaping an EXIT chart out of a group of elementary EXIT charts to maximize the rate of the code while having the EXIT chart of the irregular code below $h(x)$, i.e., satisfying $f(x) \leq h(x)$ for all x. This guarantees a performance at least as good as the required one. If convergence to zero error rate is required then $h(x)$ must satisfy $h(x) < x$ for all x in the convergence region $(0, p_0]$, where p_0 is the intrinsic message error rate.

The design rate of the code is $R = 1 - \frac{\sum \rho_j / j}{\sum \lambda_i / i}$ and hence for a fixed set $\mathcal{P}$, the design problem can be formulated as the following linear program:

$$\begin{array}{ll} \text{maximize} & \sum_{i \in \mathcal{I}} \lambda_i / i \\ \text{subject to} & \lambda_i \geq 0,\ \sum_{i \in \mathcal{I}} \lambda_i = 1, \text{ and} \\ & \forall p_{in} \in [0, p_0] \left(\sum_{i \in \mathcal{I}} \lambda_i f_i(p_{in}) \leq h(p_{in}) \right). \end{array}$$

Letting $x = p_{in}/p_0$, the region of interest for x is $[0, 1]$.

In the rest of this paper we make the following assumptions: $f_i(x)$, $h(x)$ and $f_i(x)/h(x)$ are continuous functions over [0,1] and $\exists x \in [0, 1]$ such that $f_2(x) > h(x)$. The latter assumption is made to avoid a trivial problem. Otherwise, the highest rate code uses only degree two variable nodes.

C. Monotonic Decoders

Definition 1: A decoder is called monotonic if for any $j > i$, $f_j(x) \leq f_i(x)$ for $x \in [0, 1]$.

This definition is justified because any reasonable message passing scheme should consider all the information it receives in each variable node and make the best estimate out of it. Suppose that at a degree i node, the input-output relation is given by $p_{out} = f_i(p_{in})$. At a degree $j > i$ node we can have the same input-output relation by throwing away $j - i$ input messages and using a similar message passing rule. Hence, for a reasonable decoder, the input-output relation for a variable node of degree j should be at least as good as any lower variable degree. Thus, $f_j(p_{in})$ should be less than or equal to $f_i(p_{in})$ for all p_{in}, where $j > i$. Therefore, in this work, we are only interested in those decoders which are monotonic.

III. Results

Theorem 1: Among all codes satisfying a convergence behavior $h(x)$, i.e., $\left(\sum_{i \in \mathcal{I}} \lambda_i f_i(p_{in}) \leq h(p_{in}), \right)$ for all $p_{in} \in [0, p_0]$ there exists a maximal rate code.

Proof: It is well known that any continuous function over a closed bounded set achieves a maximum. The set of "admissible" variable degree distributions

$$\left\{ \begin{array}{l} \Lambda : \lambda_i \geq 0,\ \sum_{i \in \mathcal{I}} \lambda_i = 1 \\ \forall x \in [0, 1]\ \sum_{i \in \mathcal{I}} \lambda_i f_i(x) \leq h(x) \end{array} \right\}$$

is a closed and bounded set, and $\sum \lambda_i / i$ (or equivalently the rate) is continuous. Hence, the highest rate is achievable. ■

Definition 2: Let $\{f_i(x) : i \in \mathcal{I}\}$ be the set of elementary EXIT charts and $h(x)$ be the desired convergence behavior curve. A variable degree distribution $\Lambda = \{\lambda_i : i \in \mathcal{I}\}$ is called *critical* with respect to $h(x)$ if there exists $x \in [0, 1]$ such that $\sum_{i \in \mathcal{I}} \lambda_i \frac{f_i(x)}{h(x)} = 1$.

Notice that $\sum_{i \in \mathcal{I}} \lambda_i \frac{f_i(x)}{h(x)} = 1$ is a stronger statement than $\sum_{i \in \mathcal{I}} \lambda_i f_i(x) = h(x)$. For example, when $\sum_{i \in \mathcal{I}} \lambda_i f_i(x) = h(x) = 0$, the former statement requires that their first derivatives at x (if they exist) should be equal to each other.

Theorem 2: Let $\{f_i(x) : i \in \mathcal{I}\}$ be the set of elementary EXIT charts and $h(x)$ be the desired convergence-behavior curve. Suppose $\Lambda = \{\lambda_i : i \in \mathcal{I}\}$ is an admissible variable degree distribution achieving the highest rate. Then Λ is critical with respect to $h(x)$.

Proof: Since Λ is an admissible variable degree distribution, $\sum_{i\in\mathcal{I}} \lambda_i f_i(x) \leq h(x)$ for $x \in [0,1]$. As a result, $\sum_{i\in\mathcal{I}} \lambda_i \frac{f_i(x)}{h(x)} \leq 1$. Suppose, to the contrary that Λ is not critical. Then $\sum_{i\in\mathcal{I}} \lambda_i \frac{f_i(x)}{h(x)} < 1$ for all $x \in [0,1]$. By the continuity of $\sum_{i\in\mathcal{I}} \lambda_i \frac{f_i(x)}{h(x)}$ over the interval $[0,1]$, there exists $\epsilon > 0$ such that $\sum_{i\in\mathcal{I}} \lambda_i \frac{f_i(x)}{h(x)} \leq 1-\epsilon$. Now, we are going to construct another admissible variable degree distribution which has a higher rate than Λ, causing a contradiction.

Let $\kappa = \max_{x\in[0,1]} \frac{f_2(x)}{h(x)}$. As Λ is admissible, there exists $n > 2$ such that $\lambda_n > 0$. Define a variable degree distribution $\Psi = \{\psi_i : i \in \mathcal{I}\}$ as follows:

$$\begin{cases} \psi_2 &= \lambda_2 + \min(\frac{\epsilon}{\kappa}, \lambda_n) \\ \psi_n &= \lambda_n - \min(\frac{\epsilon}{\kappa}, \lambda_n) \\ \psi_i &= \lambda_i \text{ otherwise.} \end{cases}$$

Clearly $\sum_{i\in\mathcal{I}} \psi_i = 1$ and $\psi_i \geq 0$. It can also be verified that

$$\sum_{i\in\mathcal{I}} \psi_i \frac{f_i(x)}{h(x)} \leq \sum_{i\in\mathcal{I}} \lambda_i \frac{f_i(x)}{h(x)} + \min(\frac{\epsilon}{\kappa}, \lambda_n)\frac{f_2(x)}{h(x)} \leq 1$$

Therefore, Ψ is admissible. In addition, it has a rate strictly greater than rate of Λ because $\sum_{i\in\mathcal{I}} \frac{\psi_i}{i} = \sum_{i\in\mathcal{I}} \frac{\lambda_i}{i} + \min(\frac{\epsilon}{\kappa}, \lambda_n)(\frac{1}{2} - \frac{1}{n})$. Hence, a contradiction occurs. ∎

Now, consider a scenario where there are two variable degree distributions Λ and Ψ such that the rate of Λ is greater than the rate of Ψ and the EXIT chart of Λ is always below the chart of Ψ. This means that Λ is better than Ψ in all aspects. A good code should avoid using degree distributions which offer a lower rate and a tighter EXIT chart at the same time. To be more specific we make the following definition.

Definition 3: Let $\mathcal{J}$ be a subset of $\mathcal{I}$. If for every set of real numbers $\{\delta_j : j \in \mathcal{J}\}$, which satisfies $\sum_{j\in\mathcal{J}} \delta_j = 0$ and $\sum_{j\in\mathcal{J}} \delta_j f_j(x) \leq 0$ for all $x \in [0,1]$, we have $\sum_{j\in\mathcal{J}} \frac{\delta_j}{j} \leq 0$, we say $\mathcal{J}$ is *consistent* with respect to the set of elementary EXIT charts $\mathcal{F} = \{f_i(x) : i \in \mathcal{I}\}$. If $\mathcal{F}$ is known from the context, we simply say $\mathcal{J}$ is consistent.

If $\mathcal{J}$ is inconsistent then there exists a set of real numbers $\{\delta_j : j \in \mathcal{J}\}$ such that $\sum_{j\in\mathcal{J}} \delta_j = 0$, $\sum_{j\in\mathcal{J}} \delta_j f_j(x) \leq 0$ for all $x \in [0,1]$ and $\sum_{j\in\mathcal{J}} \frac{\delta_j}{j} > 0$.

For any admissible variable degree distribution $\Lambda = \{\lambda_i : i \in \mathcal{I}\}$, its support $\mathcal{J}$ is defined by $\{j \in \mathcal{I} : \lambda_j > 0\}$.

Theorem 3: Let $\Lambda = \{\lambda_i : i \in \mathcal{I}\}$ be the highest rate admissible degree distribution. Then its support is consistent.

Proof: Let $\mathcal{J}$ be the support of Λ. If $\mathcal{J}$ is not consistent, then by definition, there exists $\{\delta_j : j \in \mathcal{J}\}$ such that $\sum_{j\in\mathcal{J}} \delta_j = 0$, $\sum_{j\in\mathcal{J}} \delta_j f_j(x) \leq 0$ for all $x \in [0,1]$ and $\sum_{j\in\mathcal{J}} \frac{\delta_j}{j} > 0$. Let $\epsilon = \max_{j\in\mathcal{J}} |\delta_j|$, $\tau = \min_{j\in\mathcal{J}} |\lambda_j|$. Let $\Psi = \{\psi_i : i \in \mathcal{I}\}$ be a variable degree distribution where ψ_j equals to $\lambda_j + \frac{\tau}{\epsilon}\delta_j$ for $j \in \mathcal{J}$ and equals to zero otherwise. It is obvious that $\sum_{i\in\mathcal{I}} \psi_i = \sum_{i\in\mathcal{I}} \lambda_i = 1$. For any $j \in \mathcal{J}$,

$$\psi_j = \lambda_j + \frac{\tau}{\epsilon}\delta_j \geq \lambda_j - \frac{\tau}{\epsilon}|\delta_j| \geq 0.$$

On the other hand, by definition,

$$\sum_{j\in\mathcal{J}} \psi_j f_j(x) = \sum_{j\in\mathcal{J}} \lambda_j f_j(x) + \frac{\tau}{\epsilon}\sum_{j\in\mathcal{J}} \delta_j f_j(x) \leq h(x).$$

Hence, Ψ is admissible. This causes a contradiction, because the rate associated with Ψ is greater than the rate of Λ since

$$\sum_{j\in\mathcal{J}} \frac{\psi_j}{j} = \sum_{j\in\mathcal{J}} \frac{\lambda_j}{j} + \frac{\tau}{\epsilon}\sum_{j\in\mathcal{J}} \frac{\delta_j}{j} > \sum_{j\in\mathcal{J}} \frac{\lambda_j}{j}.$$

∎

Let $\Psi = \{\psi_i : i \in \mathcal{I}\}$ and $\Lambda = \{\lambda_i : i \in \mathcal{I}\}$ be two variable degree distributions. We define a relation "$\succ$" as follows: $\Lambda \succ \Psi$ if and only if the union of the support of Λ and Ψ is consistent, and for all $x \in [0,1]$, $\sum_{i\in\mathcal{I}} \lambda_i f_i(x) \geq \sum_{i\in\mathcal{I}} \psi_i f_i(x)$. When $\Lambda \succ \Psi$, we say Λ dominates Ψ.

Corollary 1: Suppose $\mathcal{J}$ is consistent. Let Λ and Ψ be two admissible variable degree distributions with support being subsets of $\mathcal{J}$. If $\Lambda \succ \Psi$, then the rate of Λ is greater than or equal to the rate of Ψ.

Proof: Let $\Gamma = \{\gamma_i = \psi_i - \lambda_i : i \in \mathcal{I}\}$. Then $\sum_{j\in\mathcal{J}} \gamma_j = 0$, and $\sum_{j\in\mathcal{J}} \gamma_j f_j(x) \leq 0$. Thus, by the consistency of $\mathcal{J}$, $\sum_{j\in\mathcal{J}} \frac{\gamma_j}{j} \leq 0$, or equivalently $\sum_{j\in\mathcal{J}} \frac{\lambda_j}{j} \geq \sum_j \frac{\psi_j}{j}$. ∎

IV. Discussion

Let us compare our results with the existing results for the BEC channel. For the BEC we know that the maximum rate code has a flat EXIT chart [3]. We showed that in the general case, the EXIT chart of the maximum rate code has to meet the desired convergence behavior curve at least at one point. For the BEC we know that the area under the EXIT chart scales with the rate of the code [5]. We showed that for two irregular codes $\mathcal{C}_1$ and $\mathcal{C}_2$, whose supports are subsets of a consistent set $\mathcal{J}$, if the EXIT chart of $\mathcal{C}_1$ dominates that of $\mathcal{C}_2$ then $\mathcal{C}_1$ has a higher rate. As a result, if the EXIT chart of a code $\mathcal{C}$ is equal to $h(x)$ it is the highest rate code which guarantees this convergence behavior among all codes whose support is a subset of $\mathcal{J}$.

Choosing $h(x) = x$ results in the highest code rate, but has impractical decoding complexity. Hence, here we considered general $h(x)$. Relaxing $h(x)$ reduces the complexity at the expense of rate loss. Given a target error rate and a maximum affordable complexity, one interesting open question is to find the best $h(x)$ which results in the highest code rate.

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