Homework #1

2-6. Show that if S consists of a countable number of elements ζ_i and each subset $\{\zeta_i\}$ is an event, then all subsets of S are events.

2-7. If $S = \{1, 2, 3, 4\}$, find the smallest field that contains the sets $\{1\}$ and $\{2, 3\}$.

2-10. (*Chain rule*) Show that

$$P(A_n \cdots A_1) = P(A_n|A_{n-1} \cdots A_1) \cdots P(A_2|A_1)P(A_1)$$

2-27 We have two coins; the first is fair and the second two-headed. We pick one of the coins at random, we toss it twice and heads shows both times. Find the probability that the coin picked is fair.

3-10 Refer to Example 3-15 (Gambler's ruin problem). Let N_a denote the average duration of the game for player A starting with capital a. Show that

$$N_a = \begin{cases} \frac{b}{2p-1} - \frac{a+b}{2p-1} \frac{1-\left(\frac{p}{q}\right)^b}{1-\left(\frac{p}{q}\right)^{a+b}} & p \neq q, \\ ab & p = q = \frac{1}{2}. \end{cases}$$

(*Hint:* Show that N_k satisfies the iteration $N_k = 1 + pN_{k+1} + qN_{k-1}$ under the initial conditions $N_0 = N_{a+b} = 0$.)

Example 3-15: Two players A and B play a game consecutively till one of them loses all his capital. Suppose A starts with a capital of $\$a$ and B with a capital of $\$b$ and the loser pays $\$1$ to the winner in each game. Let p represent the probability of winning each game for A and $q = 1 - p$ for player B. Find the probability of ruin for each player if no limit is set for the number of games.

3-12. Three dice are rolled and the player may bet on any one of the face values 1, 2, 3, 4, 5, and 6. If the player's number appears on one, two, or all three dice, the player receives respectively one, two, or three times his original stake plus his own money back. Determine the expected loss per unit stake for the player.

4-16. Show that if $X(\zeta) \leq Y(\zeta)$ for every $\zeta \in S$, then $F_X(\omega) \geq F_Y(\omega)$ for every ω.

4-20. Show that if $P(A|X = x) = P(B|X = x)$ for every $x \leq x_0$, then $P(A|X \leq x_0) = P(B|X \leq x_0)$.

(*Hint:* Replace in (4-80) $P(A)$ and $f(x)$ by $P(A|X \leq x_0)$ and $f(x|X \leq x_0)$.)

Equation (4-80) (continuous version of the total probability theorem):

$$\int_{-\infty}^{\infty} P(A|X = x)f(x)dx = P(A) .$$

4-33. Players X and Y roll dice alternately starting with X. The player that rolls *eleven* wins. Show that the probability p that X wins equals 18/35.

(*Hint:* Show that

$$P(A) = P(A|M)P(M) + P(A|\overline{M})P(\overline{M}).$$

Set $A = \{X \text{wins}\}$, $M = \{$*eleven* shows at first try$\}$. Note that $P(A) = p, P(A|M) = 1$, $P(M) = 2/36$, $P(A|\overline{M}) = 1 - p$.)

4-36 We place at random 200 points in the interval $(0, 100)$. Find the probability that in the interval $(0, 2)$ there will be one and only one point ***(a)*** exactly and ***(b)*** using the Poisson approximation.

5-18 Let $X \sim U(0,1)$. Show that $Y = -2logX$ is $\chi^2(2)$.

5-37 Show that (a) if $f(x)$ is a Cauchy density, then $\Phi(\omega) = e^{-a|\omega|}$; (b) if $f(x)$ is a Laplace density, then $\Phi(\omega) = \alpha^2/(\alpha^2 + \omega^2)$.

5-42 Show that if $E\{X\} = \eta$, then

$$E\{e^{sX}\} = e^{s\eta} \sum_{n=0}^{\infty} \mu_n \frac{s^n}{n!}\ , \quad \mu_n = E\{(X - \eta)^n\}$$

6-25. Let X be the lifetime of a certain electric bulb, and Y that of its replacement after the failure of the first bulb. Suppose X and Y are independent with common exponential density function with parameter λ. Find the probability that the combined lifetime exceeds 2λ. What is the probability that the replacement outlasts the original component by λ?

6-44. X and Y are independent, identically distributed binomial random variables with parameters n and p. Show that $Z = X + Y$ is also a binomial random variable. Find its parameters.

6-51. Show that for any X, Y real or complex (a) $|E\{XY\}|^2 \le E\{|X|^2\}E\{|Y|^2\}$; ($b$) (*triangle inequality*) $\sqrt{E\{|X+Y|^2\}} \le \sqrt{E\{|X|^2\}} + \sqrt{E\{|Y|^2\}}$.

6-64 X and Y are jointly normal with parameters $N(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho_{xy})$. Find ($a$) $E\{Y|X = x\}$, and (b) $E\{X^2|Y = y\}$.