APPENDIX A
ANALYZING *District-D* WITH UNIT-DEMAND BUYERS

In this section, we extend Myerson's revenue characterization theorem to double auctions and show that Algorithm 2 is bid monotonic while Algorithm 4 returns the critical submission for every winning seller. We start by proving Lemma 2.

Proof to Lemma 2: We use the following notation for convenience. Given $\mathbf{v}$, let $\mathbf{v}^a_{-m} = (a_1, \ldots, a_{m-1}, a_{m+1}, \ldots, a_M, b_1, \ldots, b_N)$ be the vector where the mth ask is removed. Similarly, let $\mathbf{v}^b_{-n}$ be the vector where the nth bid is removed. Then we have

$$\begin{aligned}\mathbf{E}_{\mathbf{v}}[\gamma(\mathbf{v})] &= \mathbf{E}_{\mathbf{v}}\left[\sum_n c_n(\mathbf{v}) - \sum_m p_m(\mathbf{v})\right] \qquad (12)\\ &= \sum_n \mathbf{E}_{\mathbf{v}^b_{-n}} \mathbf{E}_{b_n}[c_n(\mathbf{v})] - \sum_m \mathbf{E}_{\mathbf{v}^a_{-m}} \mathbf{E}_{a_m}[p_m(\mathbf{v})] \ .\end{aligned}$$

Now consider $\mathbf{E}_{a_m}[p_m(\mathbf{v})]$. It has been shown in [22] and [12] that, when $\mathbf{v}^a_{-m}$ is fixed, a truthful mechanism $\mathcal{M}$ always offers a *take-it-or-leave-it* payment[5], say p, for the seller m . That is, the seller m wins the auction if and only if its submitted ask does not exceed p. Therefore,

$$\begin{aligned}\mathbf{E}_{a_m}[p_m(\mathbf{v})] &= p \cdot F^s_m(p)\\ &= \int_0^p \left(z + \frac{F^s_m(z)}{f^s_m(z)}\right) \cdot f^s_m(z)\mathrm{d}z\\ &= \mathbf{E}_{a_m}[\psi_m(a_m) \cdot y_m(\mathbf{v})] \ , \qquad (13)\end{aligned}$$

where the second equality can be verified by performing integration by parts on the right hand side.

Similar argument also applies to buyers. When $\mathbf{v}^a_{-m}$ is given, the truthful $\mathcal{M}$ induces a take-it-or-leave-it price p for buyer n, who wins when bidding higher than p. Therefore,

$$\begin{aligned}\mathbf{E}_{b_n}[c_n(\mathbf{v})] &= p \cdot (1 - F^b_n(p))\\ &= \int_p^\infty \left(z - \frac{1 - F^b_n(z)}{f^b_n(z)}\right) \cdot f^b_n(z)\mathrm{d}z\\ &= \mathbf{E}_{b_n}[\phi_n(b_n) \cdot x_n(\mathbf{v})] \ . \qquad (14)\end{aligned}$$

Substituting (13) and (14) back to (12), we have

$$\mathbf{E}_{\mathbf{v}}[\gamma(\mathbf{v})] = \mathbf{E}_{\mathbf{v}}\left[\sum_n \phi_n(b_n)x_n(\mathbf{v}) - \sum_m \psi_m(a_m)y_m(\mathbf{v})\right].$$

□

We now show that Algorithm 2 is bid monotonic by proving Proposition 5.

Proof to Proposition 5: We prove the buyer's case by contradiction. Suppose by submitting b_n (Bid 1), n wins in the kth iteration of Algorithm 2, while by submitting $b'_n > b_n$ (Bid 2), n loses.

For notational convenience, for Bid 2, denote the other buyers' bids by $b'_j = b_j, j \neq n$. Let $\Delta\phi_j = \phi_j(b'_j) - \phi_j(b_j)$. We see that $\Delta\phi_j = 0$ for all $j \neq n$ while $\Delta\phi_n \geq 0$. By (8), we have

$$\Delta_{i,j}(\mathcal{T}, a_i, b'_j) - \Delta_{i,j}(\mathcal{T}, a_i, b_j) = \Delta\phi_j \ . \qquad (15)$$

Let ($\mathcal{T}_l$, γ_l) and ($\mathcal{T}'_l$, γ'_l) be the vectors containing the transactions and total revenue after the lth iteration of Algorithm 2 with Bid 1 and Bid 2, respectively. Since buyer n does not win in the first $k-1$ iterations in either case, $\mathcal{T}_l = \mathcal{T}'_l, \gamma_l = \gamma'_l, l = 0, \ldots, k-1$. Now for any feasible transaction pair (i, j) in the kth iteration with Bid 2, its marginal revenue is

$$\begin{aligned}\Delta_{i,j}(\mathcal{T}'_{k-1}, a_i, b'_j) &= \Delta_{i,j}(\mathcal{T}_{k-1}, a_i, b'_j)\\ &= \Delta_{i,j}(\mathcal{T}_{k-1}, a_i, b_j) + \Delta\phi_j \ , \qquad (16)\end{aligned}$$

where the second equality holds because of (15).

For Bid 1, suppose n trades with m in the kth iteration. Then, (m, n) is of the maximum marginal revenue and maintains the budget balance, i.e.,

$$\Delta_{m,n}(\mathcal{T}_{k-1}, a_m, b_n) = \max_{i\in\mathbf{S}, j\in\mathbf{B}} \Delta_{i,j}(\mathcal{T}_{k-1}, a_i, b_j), \qquad (17)$$

$$\gamma_{k-1} + \Delta_{m,n}(\mathcal{T}_{k-1}, a_m, b_n) \geq 0 \ . \qquad (18)$$

Now inspect the marginal revenue of the same transaction pair (m, n) in the kth round with Bid 2. By (16), we have:

$$\begin{aligned}\Delta_{m,n}(\mathcal{T}'_{k-1}, a_m, b'_n) &= \Delta_{m,n}(\mathcal{T}_{k-1}, a_m, b_n) + \Delta\phi_n \qquad (19)\\ &= \max_{i,j} \Delta_{i,j}(\mathcal{T}_{k-1}, a_i, b_j) + \Delta\phi_n\\ &= \max_{i,j} \Delta_{i,j}(\mathcal{T}'_{k-1}, a_i, b'_j) \ . \qquad (20)\end{aligned}$$

Here, the the third equality is derived from (16).

Besides, based on (18) and (19), we see that

$$\gamma_{k-1} + \Delta_{m,n}(\mathcal{T}'_{k-1}, a_m, b'_n) \geq 0 \ . \qquad (21)$$

Thus, we conclude that (m, n) should also be selected in the kth iteration in Bid 2, because it generates the maximum marginal revenue and maintains budget balance. This contradicts the assumption that n loses with Bid 2.

With a similar argument, we see that the statement also holds for the seller's case. □

We finally prove that Algorithm 4 returns the critical submission for every winning seller.

Proof of Proposition 7: We first show that m wins by asking for lower than p_m. It suffices to consider two cases below:

Case 1: $p_m = a^l = a^l_{(j)}$ for some $l = 1, \ldots, k$ and some $j \in \mathbf{B}$. By asking for $a_m < p_m$, the worst case for m is that it loses in the first $l-1$ rounds, but in the lth round, we have $\Delta_{m,j}(\mathcal{T}_{l-1}, a_m, b_j) \geq \Delta_{m,j}(\mathcal{T}_{l-1}, a^l_{(j)}, b_j) = \Delta_{i_l,j_l}(\mathcal{T}_{l-1}, a_{i_l}, b_{j_l})$, which implies that assigning (m, j_l) would generate more marginal revenue than assigning (i_l, j_l). Noticing that (i_l, j_l) is already of the maximum marginal revenue in the lth round when m is absent, we know that m will be selected and trade with buyer j when it joins the auction.

Case 2: $p_m = a^{k+1} = a^{k+1}_{(j)}$ for some $j \in \mathbf{B}$. In this case,

5. It can be proved that this payment relies on $\mathcal{M}$ and $\mathbf{v}^a_{-m}$, but is independent of a_m.

we have $\Delta_{m,j}(\mathcal{T}_k, a_m, b_j) \geq \Delta_{m,j}(\mathcal{T}_k, a^l_{(j)}, b_j) = -\gamma$, which implies that adding the transaction (m, j) does not incur budget deficit. Therefore, m wins and trades with buyer j.

We next show that m loses by asking for more than p_m. First we see m loses in the first k rounds. For any $l = 1, \ldots, k$ and any $j \in \mathbf{B}$, transaction (m, j) is not the most profitable and will not be selected: $\Delta_{m,j}(\mathcal{T}_{l-1}, a_m, b_j) \leq \Delta_{m,j}(\mathcal{T}_{l-1}, p_m, b_j) \leq \Delta_{m,j}(\mathcal{T}_{l-1}, a^l_{(j)}, b_j) = \Delta_{i_l,j_l}(\mathcal{T}_{l-1}, a_{i_l}, b_{j_l})$. Moreover, even after the first k rounds, m cannot be selected to trade with any buyer $j \in \mathbf{B}$. Otherwise, the budget deficit arises: $\Delta_{m,j}(\mathcal{T}_k, a_m, b_j) \leq \Delta_{m,j}(\mathcal{T}_k, a^{k+1}, b_j) \leq \Delta_{m,j}(\mathcal{T}_{l-1}, a^{k+1}_{(j)}, b_j) = -\gamma$. □

Appendix B
Analyzing Extended *District-D* with Multi-Demand Buyers

In this section, we analyze the economic properties and computational complexity of the extended *District-D* with multi-demand buyers.

B.1 Economic Properties

For clarity, we present a complete winner determination algorithm described in Sec. 6.1.3 in Algorithm 7.

Algorithm 7 Extended *District-D* Winner Determination

```
1.  Initialization: γ ← 0, T ← ∅, and stop ← false.
2.  while stop = false do
3.     Δ_{S,n} ← MaxMarginalRev(T)
4.     if γ + Δ_{S,n} ≥ 0 then
5.        γ ← γ + Δ_{S,n}
6.        Add (S, n) to T.
7.     else
8.        stop ← true
9.     end if
10. end while
11. return T
```

It is easy to see that the extension above is budget balanced, as stated below.

Proposition 8: Extended *District-D* is *ex ante* budget balanced.

Proof: Given conflict graph G, for all asks $\mathbf{a}$ and all bids $\mathbf{b}$, Algorithm 7 ensures that $\gamma \geq 0$. Therefore $\mathbf{E}_{\mathbf{v},G}[\gamma] = \mathbf{E}_{\mathbf{v},G}[\sum_{n \in W_b} r_n \phi_n(b_n) - \sum_{m \in W_s} \psi_m(a_m)] \geq 0$, where W_b and W_s are sets of winning buyers and winning sellers, respectively. □

Besides, the winner determination is also bid monotonic. The proof is similar to Proposition 5.

Proposition 9: Extended *District-D* is bid monotonic.

Proof: We prove the buyer's case by contradiction. Suppose by submitting b_n (Bid 1), n wins in the kth iteration of Algorithm 7, while by submitting $b'_n > b_n$ (Bid 2), n loses.

For notational convenience, for Bid 2, denote the other buyers' bids by $b'_j = b_j, j \neq n$. Also let $\Delta\phi_j = \phi_j(b'_j) - \phi_j(b_j)$. We see that $\Delta\phi_j = 0$ for all $j \neq n$ while $\Delta\phi_n \geq 0$. By (11), we have

$$\Delta_{S,j}(\mathcal{T}, b'_j) - \Delta_{S,j}(\mathcal{T}, b_j) = \Delta\phi_j. \tag{22}$$

Let ($\mathcal{T}_l$, γ_l) and ($\mathcal{T}'_l$, γ'_l) be the vectors containing the transactions and total revenue after the lth iteration of Algorithm 7 with Bid 1 and Bid 2, respectively. Since buyer n does not win in the first $k-1$ iterations in either case, $\mathcal{T}_l = \mathcal{T}'_l, \gamma_l = \gamma'_l, l = 0, \ldots, k-1$. Now for any feasible transactions (S, j) in the kth iteration with Bid 2, its marginal revenue is

$$\begin{aligned}\Delta_{S,j}(\mathcal{T}'_{k-1}, b'_j) &= \Delta_{S,j}(\mathcal{T}_{k-1}, b'_j) \\ &= \Delta_{S,j}(\mathcal{T}_{k-1}, b_j) + \Delta\phi_j,\end{aligned} \tag{23}$$

where the second equality holds because of (22).

For Bid 1, suppose n trades with sellers S^* in the kth iteration. Then, (S^*, n) is of the maximum marginal revenue and maintains the budget balance, i.e.,

$$\Delta_{S^*,n}(\mathcal{T}_{k-1}, b_n) = \max_{S \in \mathbf{S}, j \in \mathbf{B}} \Delta_{S,j}(\mathcal{T}_{k-1}, b_j), \tag{24}$$

$$\gamma_{k-1} + \Delta_{S^*,n}(\mathcal{T}_{k-1}, b_n) \geq 0 \ . \tag{25}$$

Now inspect the marginal revenue of the same transactions (S^*, n) in the kth round with Bid 2. By (23), we have:

$$\Delta_{S^*,n}(\mathcal{T}'_{k-1}, b'_n) = \Delta_{S^*,n}(\mathcal{T}_{k-1}, b_n) + \Delta\phi_n \tag{26}$$

$$\begin{aligned}&= \max_{S \in \mathbf{S}, j \in \mathbf{B}} \Delta_{S,j}(\mathcal{T}_{k-1}, b_j) + \Delta\phi_n \\ &= \max_{S \in \mathbf{S}, j \in \mathbf{B}} \Delta_{S,j}(\mathcal{T}'_{k-1}, b'_j) \ .\end{aligned} \tag{27}$$

Here, the third equality is derived from (23).

Besides, based on (25) and (26), we see that

$$\gamma_{k-1} + \Delta_{S^*,n}(\mathcal{T}'_{k-1}, b'_n) \geq 0 \ . \tag{28}$$

Thus, we conclude that (S^*, n) should also be selected in the kth iteration in Bid 2, because it generates the maximum marginal revenue and maintains budget balance. This contradicts the assumption that n loses with Bid 2.

With a similar argument, we see that the statement also holds for the seller's case. □

Next we show that the extended buyer pricing charges a critical bid to every winning buyer.

Proposition 10: For every winning buyer n, Algorithm 5 returns its critical bid c_n.

Proof: We first prove that n wins by bidding higher than c_n, i.e., $b_n > c_n$. It suffices to consider two cases:

Case 1: c_n is finalized in the first k loops, i.e., $c_n = b^l = b^l_{(S)}$ for some $l \leq k$ and $S \subset \mathbf{S}$. For n bidding $b_n > c_n = b^l_{(S)}$, the worst case is that it loses in the first $l-1$ rounds. But in the lth round, $\Delta_{S,n}(\mathcal{T}_{l-1}, b_n) > \Delta_{S,n}(\mathcal{T}_{l-1}, b^l_{(S)}) = \Delta_{S_l,j_l}(\mathcal{T}_{l-1}, b_{j_l})$, where the first inequality holds due to the increasing $\phi_n(\cdot)$ and (11). This implies that making the transactions (S, n) would generate more marginal revenue than selecting (S_l, j_l). Since (S_l, j_l) is already the most profitable transactions when n is absent, we conclude that (S, n) maximizes the marginal revenue. Therefore, n wins by being selected to trade with a set of sellers S.

Case 2: $c_n = b^{k+1} = b^{k+1}_{(S)}$ for some $S \subset \mathbf{S}$. For n bidding $b_n > c_n = b^{k+1}_{(S)}$, the worst case is that it loses in the first k rounds. However, n can still trade with sellers S after the first k rounds, with the marginal revenue $\Delta_{S,n}(\mathcal{T}_k, b_n) > \Delta_{S,n}(\mathcal{T}_k, b^{k+1}_{(S)}) = -\gamma$. Therefore, adding transactions (S, n) to the auction results makes the total revenue remain positive, i.e., $\Delta_{S,n}(\mathcal{T}_k, b_n) + \gamma > 0$. Based on the winner determination algorithm (Algorithm 7), n wins and trades with sellers S.

Next, if n bids less than c_n (i.e., $b_n < c_n$), then it loses in the first k rounds. For any $l = 1, \ldots, k$ and any set of sellers $S \subset \mathbf{S}$, we have $\Delta_{S,n}(\mathcal{T}_{l-1}, b_n) < \Delta_{S,n}(\mathcal{T}_{l-1}, c_n) \le \Delta_{S,n}(\mathcal{T}_{l-1}, b^l) \le \Delta_{S,n}(\mathcal{T}_{l-1}, b^l_{(S)}) = \Delta_{S_l,j_l}(\mathcal{T}_{l-1}, b_{j_l})$, where the third inequality holds because of line 4. This essentially indicates that buyer n loses in the first k rounds as any transactions involving it does not generate the optimal marginal revenue.

Moreover, even if $b^{k+1} < \infty$ after k rounds, n loses and cannot trade with any set of sellers $S \subset \mathbf{S}$. Otherwise, the total revenue would become negative: $\Delta_{S,n}(\mathcal{T}_k, b_n) + \gamma < \Delta_{S,n}(\mathcal{T}_k, b^{k+1}) + \gamma \le \Delta_{S,n}(\mathcal{T}_k, b^{k+1}_{(S)}) + \gamma = 0$. □

Also, the extended *District-D* pays critical submissions to every winning seller. Before we present the formal proof, we need the following technical lemma.

Lemma 3: Given a transaction list $\mathcal{T}$, for any seller $m \in \mathbf{S}$ and buyer $j \in \mathbf{B}$, there exists a set of sellers S^* such that S^* solves the following equation

$$\max_{S \in \mathbf{S}, |S| = r_j, m \in S} \Delta_{S,j}(\mathcal{T}) \tag{29}$$

and is independent of m's ask a_m and j's bid b_j.

Proof: By (11), we have

$$\begin{aligned}\Delta_{S,j}(\mathcal{T}) &= r_j \phi_j(b_j) - \psi_m(a_m) I_{m \notin \mathcal{T}^s} - \sum_{i \ne m, i \in S} \psi_i(a_i) I_{i \notin \mathcal{T}^s} \\ &= r_j \phi_j(b_j) - \psi_m(a_m) I_{m \notin \mathcal{T}^s} - \sum_{i \ne m, i \in S - \mathcal{T}^s} \psi_i(a_i) \ .\end{aligned} \tag{30}$$

Therefore, maximizing $\Delta_{S,j}(\mathcal{T})$ is equivalent to minimizing $\sum_{i \ne m, i \in S - \mathcal{T}^s} \psi_i(a_i)$. To achieve this objective, we need the following two steps. *First*, buyer j has to trade with as many sellers in $\mathcal{T}^s$ as possible, because trading with those sellers does not incur additional costs (i.e., $\psi_i(a_i) I_{i \notin \mathcal{T}^s} = 0$), generating more marginal revenue to the auctioneer. *Second*, if j has to trade with those sellers not in $\mathcal{T}^s$, the selected sellers should be those who asks for the least price for their channels.

Given $\mathcal{T}$, let S^j be the set of tradable sellers of buyer j. That is, for every seller $i \in S^j$, transaction (i, j) is feasible given the already made transactions $\mathcal{T}$. Algorithm 8 constructs the optimal sellers S^* by formalizing the aforementioned two steps, from which we see that S^* does not depend on the specific submissions of seller m and buyer j. □

Lemma 3 simplifies the computation of $a^l_{(j)}$ and $a^{k+1}_{(j)}$ in line 4 and line 8 of Algorithm 7, respectively. Since both m and j are given, the auctioneer first runs Algorithm 8 to calculate the corresponding S^*. It then solves the equation $\Delta_{S^*,j}(\mathcal{T}_{l-1}, a^l_{(j)}) = \Delta_{S_l,j_l}(\mathcal{T}_{l-1})$ to obtain $a^l_{(j)}$.

Algorithm 8 Construct S^*

```
X ← S^j ∩ T^s.
if |X| ≥ r_j − 1 then
    Y ← an arbitrary subset of X with r_j − 1 sellers
    S* ← Y ∪ {m}
else
    Z ← S^j − X
    Y ← a set of r_j − |X| − 1 sellers in Z with the least
    asks
    S* ← X ∪ Y ∪ {m}
end if
return S*
```

We are now ready to prove that Algorithm 6 returns a critical submission of seller m.

Proposition 11: For every winning seller m, Algorithm 6 returns its critical ask p_m.

Proof: We first show that m wins by asking for lower than p_m. It suffices to consider two cases below:

Case 1: $p_m = a^l = a^l_{(j)}$ for some $l = 1, \ldots, k$ and some $j \in \mathbf{B}$. With m and j as input, we run Algorithm 8 to calculate S^*. We then have

$$\begin{aligned}\Delta_{S^*,j}(\mathcal{T}_{l-1}, a^l_{(j)}) &= \max_{S \subset \mathbf{S}, |S| = r_j, m \in S} \{\Delta_{S,j}(\mathcal{T}_{l-1}, a^l_{(j)})\} \\ &= \Delta_{S_l,j_l}(\mathcal{T}_{l-1}) \ .\end{aligned} \tag{31}$$

Now by asking for $a_m < p_m$, the worst case for m is that it loses in the first $l - 1$ rounds, but in the lth round, we have

$$\begin{aligned}\Delta_{S^*,j}(\mathcal{T}_{l-1}, a_m) &\ge \Delta_{S^*,j}(\mathcal{T}_{l-1}, a^l_{(j)}) \\ &= \Delta_{S_l,j_l}(\mathcal{T}_{l-1}) \ ,\end{aligned}$$

where the second equality is derived from (31). Given that (S_l, j_l) is already of the maximum marginal revenue in the lth round when m is absent, we know that facilitating (S^*, j_l) generates the highest marginal revenue when m joins the auction. Based on the winner determination (Algorithm 7), (S^*, j_l) would be selected and m wins because $m \in S^*$.

Case 2: $p_m = a^{k+1} = a^{k+1}_{(j)}$ for some $j \in \mathbf{B}$. With m and j as input, we run Algorithm 8 to calculate the corresponding S^*. We have

$$\begin{aligned}\Delta_{S^*,j}(\mathcal{T}_k, a^{k+1}_{(j)}) &= \max_{S \subset \mathbf{S}, |S| = r_j, m \in S} \{\Delta_{S,j}(\mathcal{T}_k, a^{k+1}_{(j)})\} \\ &= -\gamma \ .\end{aligned}$$

Now by asking for $a_m < p_m$, the worst case for m is that it loses in the first k rounds. But after that, we have $\Delta_{S^*,j}(\mathcal{T}_k, a_m) \ge \Delta_{S^*,j}(\mathcal{T}_k, a^l_{(j)}) = -\gamma$, which implies that adding transactions (S^*, j) does not incur budget deficit. Based on the winner determination (Algorithm 7), m wins and trades with buyer j as $m \in S^*$.

We next show that m loses by asking for more than p_m. First we see m loses in the first k rounds. For any $l =$

$1, \ldots, k$, any $j \in \mathbf{B}$, and any $S \subset \mathbf{S}$ with $|S| = r_j$ and $m \in S$, transactions (S, j) will not be selected as they are less profitable than transanctions (S_l, j_l):

$$\begin{aligned}\Delta_{S,j}(\mathcal{T}_{l-1}, a_m) &< \Delta_{S,j}(\mathcal{T}_{l-1}, p_m) \\ &\le \Delta_{S,j}(\mathcal{T}_{l-1}, a^l_{(j)}) \\ &\le \Delta_{S^*,j}(\mathcal{T}_{l-1}, a^l_{(j)}) \\ &= \Delta_{S_l,j_l}(\mathcal{T}_{l-1}) \ ,\end{aligned}$$

where S^* is calculated by Algorithm 8 with input m and j.

Moreover, even after the first k rounds, m cannot be selected to trade with any leftover buyer $j \in \mathbf{B}$. Otherwise, the budget deficit arises. To see this, consider any feasible transanctions (S, j) where $S \subset \mathbf{S}$, $|S| = r_j$ and $m \in S$. We have

$$\begin{aligned}\Delta_{S,j}(\mathcal{T}_k, a_m) &< \Delta_{S,j}(\mathcal{T}_k, a^{k+1}) \\ &\le \Delta_{S,j}(\mathcal{T}_k, a^{k+1}_{(j)}) \\ &\le \Delta_{S^*,j}(\mathcal{T}_k, a^{k+1}_{(j)}) \\ &= -\gamma \ , \end{aligned} \tag{32}$$

where S^* is calculated by Algorithm 8 with input m and j. Therefore, (S, j) will not be selected. We see that m loses in the end. □

B.2 Computational Complexity

This section analyzes the computational complexity of the extended *District-D*. For the winner determination (Algorithm 7), we see that one buyer is assigned to trade with a set of sellers in each round of the loop, and the loop runs at most N rounds for N buyers. The complexity of each round is dominated by *MaxMarginalRev*$(\mathcal{T})$. Note that *MaxMarginalRev*$(\mathcal{T}) = \max_{n \in \mathbf{B}, S \subset \mathbf{S}, |S| = r_n} \Delta_{S,n}(\mathcal{T}) = \max_{n \in \mathbf{B}} \Delta_{S^*,n}(\mathcal{T})$, where $\Delta_{S^*,n}(\mathcal{T}) = \max_{S \subset \mathbf{S}, |S| = r_n} \Delta_{S,n}(\mathcal{T})$. Following the idea of Algorithm 8, one can easily see that S^* can be similarly calculated by Algorithm 9, which takes $O(M^2)$ time. Therefore, *MaxMarginalRev*$(\mathcal{T})$ can be finalized within $O(M^2 N)$ time. We hence need $O(M^2 N^2)$ time for Algorithm 7. Note that Algorithm 7 also dominates the complexity of Algorithm 5 and 6, where the former runs N times while the later runs M times to calculate prices for all winners. We conclude that the extended *District-D* completes within $O(M^2 N^3 + M^3 N^2)$ time.

Appendix C Profit-Driven Auctioneer with Multi-Demand Buyers

C.1 The Complete Design

This section presents a complete design of *District-D* for a profit-driven auctioneer with multi-demand buyers. Algorithm 10, 11, and 12 illustrate the winner determination, buyer pricing, and seller pricing, respectively.

Algorithm 9 Construct S^*

$X \leftarrow S^n \cap \mathcal{T}^s$.
if $|X| \ge r_n$ **then**
    $S^* \leftarrow$ an arbitrary subset of X with r_n sellers
else
    $Z \leftarrow S^n - X$
    $Y \leftarrow$ a set of $r_n - |X|$ sellers in Z with the least asks
    $S^* \leftarrow X \cup Y$
end if
return S^*

Algorithm 10 Profit-Driven *District-D* Winner Determination

Initialization: $\gamma \leftarrow 0$, $\mathcal{T} \leftarrow \emptyset$, and *stop* $\leftarrow$ *false*.
while *stop* $=$ *false* **do**
    $\Delta_{S,n} \leftarrow$ *MaxMarginalRev*$(\mathcal{T})$
    if $\Delta_{S,n} \ge 0$ **then**
        $\gamma \leftarrow \gamma + \Delta_{S,n}$
        Add (S, n) to $\mathcal{T}$.
    else
        stop $\leftarrow$ *true*
    end if
end while
return $\mathcal{T}$

C.2 Economic Properties

This section shows that the profit-driven *District-D* is truthful and individually rational, which is equivalent to proving that Algorithm 10 is bid monotonic while Algorithm 11 and 12 return critical submissions for winning buyers and winning sellers, respectively. The proofs are similar to those in Proposition 9, 10, and 11.

Proposition 12: Profit-driven *District-D* is bid monotonic.

Proof: We prove the buyer's case by contradiction. Suppose by submitting b_n (Bid 1), n wins in the kth iteration of Algorithm 10, while by submitting $b'_n > b_n$ (Bid 2), n loses.

Algorithm 11 Profit-Driven *District-D* Pricing for a Winning Buyer n

Remove n and run Algorithm 10 to obtain the transaction list $\mathcal{T} = \{(S_1, j_1), \ldots, (S_k, j_k)\}$.
$c_n \leftarrow \infty$, and $\gamma \leftarrow 0$
for $l = 1$ to k **do**
    $b^l \leftarrow \min_{S \subset \mathbf{S}, |S| = r_n} b^l_{(S)}$, where $b^l_{(S)}$ solves the equation $\Delta_{S,n}(\mathcal{T}_{l-1}, b^l_{(S)}) = \Delta_{S_l,j_l}(\mathcal{T}_{l-1}, b_{j_l})$.
    $c_n \leftarrow \min\{c_n, b^l\}$
    $\gamma \leftarrow \gamma + \Delta_{S_l,j_l}(\mathcal{T}_{l-1}, b_{j_l})$
end for
$b^{k+1} \leftarrow \min_{S \subset \mathbf{S}, |S| = r_n} b^{k+1}_{(S)}$, where $b^{k+1}_{(S)}$ solves the equation $\Delta_{S,n}(\mathcal{T}_k, b^{k+1}_{(S)}) = 0$
$c_n \leftarrow \min\{c_n, b^{k+1}\}$
return c_n

Algorithm 12 Profit-Driven *District-D* Pricing for a Winning Seller m

Remove m and run Algorithm 10 to obtain the transaction list $\mathcal{T} = \{(S_1, j_1), \ldots, (S_k, j_k)\}$.
$p_m \leftarrow -\infty$, and $\gamma \leftarrow 0$
for $l = 1$ to k **do**
  $a^l \leftarrow \max_{j \in \mathbf{B}} a^l_{(j)}$, where $a^l_{(j)}$ solves the equation $\max_{S \subset \mathbf{S}, |S| = r_j, m \in S} \{\Delta_{S,j}(\mathcal{T}_{l-1}, a^l_{(j)})\} = \Delta_{S_l, j_l}(\mathcal{T}_{l-1})$
  $p_m \leftarrow \max\{p_m, a^l\}$
  $\gamma \leftarrow \gamma + \Delta_{S_l, j_l}(\mathcal{T}_{l-1})$
end for
$a^{k+1} \leftarrow \max_{j \in \mathbf{B}} a^{k+1}_{(j)}$, where $a^{k+1}_{(j)}$ solves the equation $\max_{S \subset \mathbf{S}, |S| = r_j, m \in S} \{\Delta_{S,j}(\mathcal{T}_k, a^{k+1}_{(j)})\} = 0$
$p_m \leftarrow \max\{p_m, a^{k+1}\}$
return p_m

For notational convenience, for Bid 2, denote the other buyers' bids by $b'_j = b_j, j \neq n$. Also let $\Delta\phi_j = \phi_j(b'_j) - \phi_j(b_j)$. We see that $\Delta\phi_j = 0$ for all $j \neq n$ while $\Delta\phi_n \geq 0$.

Let ($\mathcal{T}_l$, γ_l) and ($\mathcal{T}'_l$, γ'_l) be the vectors containing the transactions and total revenue after the lth iteration of Algorithm 10 with Bid 1 and Bid 2, respectively. Since buyer n does not win in the first $k-1$ iterations in either case, $\mathcal{T}_l = \mathcal{T}'_l, \gamma_l = \gamma'_l, l = 0, \ldots, k-1$. Now for any feasible transactions (S, j) in the kth iteration with Bid 2, its marginal revenue is

$$\begin{aligned}\Delta_{S,j}(\mathcal{T}'_{k-1}, b'_j) &= \Delta_{S,j}(\mathcal{T}_{k-1}, b'_j) \\ &= \Delta_{S,j}(\mathcal{T}_{k-1}, b_j) + \Delta\phi_j \ . \qquad (33)\end{aligned}$$

For Bid 1, suppose n trades with sellers S^* in the kth iteration. Then, (S^*, n) is profitable and of the maximum marginal revenue, i.e.,

$$\Delta_{S^*,n}(\mathcal{T}_{k-1}, b_n) = \max_{S \in \mathbf{S}, j \in \mathbf{B}} \Delta_{S,j}(\mathcal{T}_{k-1}, b_j), \qquad (34)$$

$$\Delta_{S^*,n}(\mathcal{T}_{k-1}, b_n) \geq 0 \ . \qquad (35)$$

Now inspect the marginal revenue of the same transactions (S^*, n) in the kth round with Bid 2. By (33), we have:

$$\begin{aligned}\Delta_{S^*,n}(\mathcal{T}'_{k-1}, b'_n) &= \Delta_{S^*,n}(\mathcal{T}_{k-1}, b_n) + \Delta\phi_n \qquad (36) \\ &= \max_{S \in \mathbf{S}, j \in \mathbf{B}} \Delta_{S,j}(\mathcal{T}_{k-1}, b_j) + \Delta\phi_n \\ &= \max_{S \in \mathbf{S}, j \in \mathbf{B}} \Delta_{S,j}(\mathcal{T}'_{k-1}, b'_j) \ . \qquad (37)\end{aligned}$$

Here, the third equality is derived from (33).

Besides, based on (35) and (36), we see that

$$\Delta_{S^*,n}(\mathcal{T}'_{k-1}, b'_n) \geq 0 \ . \qquad (38)$$

Thus, we conclude that (S^*, n) should also be selected in the kth iteration in Bid 2, because it generates the maximum positive marginal revenue and remains profitable. This contradicts the assumption that n loses with Bid 2.

With a similar argument, we see that the statement also holds for the seller's case. □

Proposition 13: For every winning buyer n, Algorithm 11 returns its critical bid c_n.

Proof: We first prove that n wins by bidding higher than c_n, i.e., $b_n > c_n$. It suffices to consider two cases:

Case 1: c_n is finalized in the first k loops. In this case, the proof is exactly the same as the analysis of Proposition 10 (see Case 1 in the proof).

Case 2: $c_n = b^{k+1} = b^{k+1}_{(S)}$ for some $S \subset \mathbf{S}$. For n bidding $b_n > c_n = b^{k+1}_{(S)}$, the worst case is that it loses in the first k rounds. However, n can still trade with sellers S after the first k rounds, with the marginal revenue $\Delta_{S,n}(\mathcal{T}_k, b_n) > \Delta_{S,n}(\mathcal{T}_k, b^{k+1}_{(S)}) = 0$, indicating that transactions (S, n) generates nonnegative marginal revenue to the auctioneer. Based on the winner determination algorithm (Algorithm 10), n wins and trades with sellers S.

Next, if n bids less than c_n (i.e., $b_n < c_n$), then it loses in the first k rounds. The proof is exactly the same as the analysis of Proposition 10.

Moreover, even if $b^{k+1} < \infty$ after k rounds, n loses and cannot trade with any set of sellers $S \subset \mathbf{S}$, as transactions (S, n) generates negative revenue to the auctioneer: $\Delta_{S,n}(\mathcal{T}_k, b_n) < \Delta_{S,n}(\mathcal{T}_k, b^{k+1}) \leq \Delta_{S,n}(\mathcal{T}_k, b^{k+1}_{(S)}) = 0$. □

Proposition 14: For every winning seller m, Algorithm 12 returns its critical ask p_m.

Proof: We first show that m wins by asking for lower than p_m. It suffices to consider two cases below:

Case 1: $p_m = a^l = a^l_{(j)}$ for some $l = 1, \ldots, k$ and some $j \in \mathbf{B}$. In this case, the proof is exactly the same as the analysis of Proposition 11 (see Case 1 in the proof).

Case 2: $p_m = a^{k+1} = a^{k+1}_{(j)}$ for some $j \in \mathbf{B}$. With m and j as input, we run Algorithm 8 to calculate the corresponding S^*. We have

$$\begin{aligned}\Delta_{S^*,j}(\mathcal{T}_k, a^{k+1}_{(j)}) &= \max_{S \subset \mathbf{S}, |S| = r_j, m \in S} \{\Delta_{S,j}(\mathcal{T}_k, a^{k+1}_{(j)})\} \\ &= 0 \ .\end{aligned}$$

Now by asking for $a_m < p_m$, the worst case for m is that it loses in the first k rounds. But after that, we have $\Delta_{S^*,j}(\mathcal{T}_k, a_m) \geq \Delta_{S^*,j}(\mathcal{T}_k, a^l_{(j)}) = 0$, which implies that adding transactions (S^*, j) does not incur budget deficit. Based on the winner determination (Algorithm 10), m wins and trades with buyer j as $m \in S^*$.

We next show that m loses by asking for more than p_m. First we see m loses in the first k rounds. The proof is exactly the same the analysis of Proposition 11.

Moreover, even after the first k rounds, m cannot be selected to trade with any leftover buyer $j \in \mathbf{B}$, as the trade generates negative profit to the auctioneer. To see this, consider any feasible transactions (S, j) where $S \subset \mathbf{S}$, $|S| = r_j$ and $m \in S$. We have

$$\begin{aligned}\Delta_{S,j}(\mathcal{T}_k, a_m) &< \Delta_{S,j}(\mathcal{T}_k, a^{k+1}) \\ &\leq \Delta_{S,j}(\mathcal{T}_k, a^{k+1}_{(j)}) \\ &\leq \Delta_{S^*,j}(\mathcal{T}_k, a^{k+1}_{(j)}) \\ &= 0 \ , \qquad (39)\end{aligned}$$

where S^* is calculated by Algorithm 8 with input m and j. Therefore, (S, j) will not be selected. We see that m loses in the end. $\square$

Based on Proposition 12, 13, and 14, we have

Theorem 6: Profit-driven *District-D* designed for multi-demand buyers is truthful and individually rational.