Min-plus Algebra

In min-plus algebra, the arithmetic addition of the conventional algebra is replaced by the point-wise minimization — denoted here by the notation $\oplus$ — and the arithmetic multiplication is replaced by the point-wise addition, represented by $\otimes$. Subtraction, multiplication, and division in the conventional algebra also have corresponding counterparts in the min-plus algebra as represented in Table 1.

Table 1: The relationship between the operators in the conventional algebra and the min-plus algebra.

Conventional Algebra	Min-plus Algebra
$\min\{a, b\}$	$a \oplus b$
$a + b$	$a \otimes b$
$a - b$	$\overset{a}{\underset{b}{-\!\!\circ\!\!-}}$
ab	$a^b = b^a$
$\frac{a}{b}$	$\sqrt[b]{a}$
$\max\{a, b\}$	$a \vee b$

Let the set of all non-decreasing functions over $I\!\!R^+$ be denoted by

$$\mathcal{J} \triangleq \{a(t) | 0 \leq a(t_1) \leq a(t_2), \text{ for } 0 \leq t_1 \leq t_2, \text{ and } a(0) = 0\}. \tag{1}$$

For $a(t), b(t) \in \mathcal{J}$, we have

$$a(t) \oplus b(t) \triangleq \min\{a(t), b(t)\}, \tag{2}$$

$$a(t) \otimes b(t) \triangleq a(t) + b(t). \tag{3}$$

It is straightforward to show that the set $\mathcal{J}$ is closed under the $\oplus$ and $\otimes$ operations.

Definition 1 A set $\mathcal{A}$ supplied with two inner operations $\oplus$ and $\otimes$ is a *commutative dioid* if the following axioms hold:

- *Axiom 1*: (Associativity) $\forall\, a(t), b(t), c(t) \in \mathcal{A}$,

$$[a(t) \oplus b(t)] \oplus c(t) = a(t) \oplus [b(t) \oplus c(t)], \tag{4}$$

$$[a(t) \otimes b(t)] \otimes c(t) = a(t) \otimes [b(t) \otimes c(t)]. \tag{5}$$

- *Axiom 2*: (Commutativity) $\forall\, a(t), b(t) \in \mathcal{A}$,

$$\begin{aligned} a(t) \oplus b(t) &= b(t) \oplus a(t), && (6) \\ a(t) \otimes b(t) &= b(t) \otimes a(t). && (7) \end{aligned}$$

- *Axiom 3*: (Distributivity) $\forall\, a(t), b(t), c(t) \in \mathcal{A}$,

$$[a(t) \oplus b(t)] \otimes c(t) = [a(t) \otimes c(t)] \oplus [b(t) \otimes c(t)]. \tag{8}$$

- *Axiom 4*: (Null and Identity Elements) $\forall\, a(t) \in \mathcal{A}, \exists \epsilon(t), e(t) \in \mathcal{A}:$

$$\begin{aligned} a(t) \oplus \epsilon(t) &= a(t), && (9) \\ a(t) \otimes e(t) &= a(t). && (10) \end{aligned}$$

- *Axiom 5*: (Absorbing Null Element) $\forall\, a(t) \in \mathcal{A}$,

$$a(t) \otimes \epsilon(t) = \epsilon(t). \tag{11}$$

- *Axiom 6*: (Idempotency of Addition) $\forall\, a(t) \in \mathcal{A}$,

$$a(t) \oplus a(t) = a(t). \tag{12}$$

The null and the identity elements in Axiom 4 are defined as $\epsilon(t) = \infty$ for $t > 0$, and $\epsilon(0) = 0$, and $e(t) = 0$ for $t \geq 0$, respectively.

Definition 2 A dioid is *complete* if and only if it is closed for infinite sums, and Axiom 3 extends to infinite sums.

Remark 1 *It is possible to show that* $(\mathcal{J}, \oplus, \otimes)$ *is a complete dioid.*

In a dioid, a partial order is defined as

$$a(t) \leq b(t) \Leftrightarrow a(t) = a(t) \oplus b(t). \tag{13}$$

With this ordering, the dioid is an *inf-semilattice.* Defining the inner operation $\vee$ as

$$a(t) \vee b(t) \triangleq \max\{a(t), b(t)\}, \tag{14}$$

the semilattice becomes a complete *lattice*.

Reference

For further reading on min-plus algebra refer to:
Synchronization and Linearity, An Algebra for Discrete Event Systems, by F. Baccelli, G. Cohen, G. J. Olster, and J. P. Quadrat, New York: Wiley, 1992.