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Iterative Water-Filling for Gaussian Vector Multiple-Access Channels

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***Abstract*—This correspondence proposes an efficient numerical algorithm to compute the optimal input distribution that maximizes the sum capacity of a Gaussian multiple-access channel with vector inputs and a vector output. The numerical algorithm has an iterative water-filling interpretation. The algorithm converges from any starting point, and it reaches within 1/2 nats per user per output dimension from the sum capacity after just one iteration. The characterization of sum capacity also allows an upper bound and a lower bound for the entire capacity region to be derived.**

***Index Terms*—Channel capacity, convex optimization, Gaussian channels, multiuser channels, multiple-access channels, multiple-access communications, multiple-antenna systems, optimization methods, power control, water-filling.**

I. Introduction

A communication situation where multiple uncoordinated transmitters send independent information to a common receiver is referred to as a multiple-access channel. Fig. 1 illustrates a two-user multiple-access channel, where X_1 and X_2 are uncoordinated transmitters encoding independent messages W_1 and W_2, respectively, and the receiver is responsible for decoding both messages at the same time. An $(n, 2^{nR_1}, 2^{nR_2})$ codebook for a multiple-access channel consists of encoding functions $X_1^n(W_1)$, $X_2^n(W_2)$, where $W_1 \in \{1, \ldots, 2^{nR_1}\}$ and $W_2 \in \{1, \ldots, 2^{nR_2}\}$, and decoding functions $\hat{W}_1(Y^n), \hat{W}_2(Y^n)$. An error occurs when $W_1 \neq \hat{W}_1$ or $W_2 \neq \hat{W}_2$. A rate pair (R_1, R_2) is achievable if there exists a sequence of $(n, 2^{nR_1}, 2^{nR_2})$ codebooks for which the average probability of error $P_e^n \to 0$ as $n \to \infty$. The capacity region of a multiple-access channel is the set of all achievable rate pairs.

The capacity region for the multiple-access channel has the following well-known single-letter characterization [1], [2]. Consider a discrete-time memoryless multiple-access channel with a channel transition probability $p(y|x_1, x_2)$. For each fixed input distribution $p_1(x_1)p_2(x_2)$, the following pentagon rate region is achievable:

$$\begin{aligned} R_1 &\le I(X_1; Y|X_2) \\ R_2 &\le I(X_2; Y|X_1) \\ R_1 + R_2 &\le I(X_1, X_2; Y) \end{aligned} \tag{1}$$

where the mutual information expressions are computed with respect to the joint distribution $p(y|x_1, x_2)p_1(x_1)p_2(x_2)$. When the input distribution is not fixed, but constrained in some ways, the capacity region is the convex hull of the union of all capacity pentagons whose corresponding input distributions satisfy the input constraint after the convex hull operation [3], [4]. Since the input signals in a multiple-access channel are independent, the input distribution must take a product form $p_1(x_1)p_2(x_2)$. This product constraint is not a convex constraint, so the problem of finding the optimal input distribution for a multiple-access channel is in general nontrivial [5]. The aim of this correspondence is to provide a numerical solution to this input optimization problem for a particular type of multiple-access channel: the Gaussian vector multiple-access channel.

A Gaussian multiple-access channel refers to a multiple-access channel in which the law of the channel transition probability $p(y|x_1, x_2)$ is Gaussian. When a Gaussian multiple-access channel is memoryless and when X_1 and X_2 are scalars, the input optimization problem has a simple solution. Let the power constraints on X_1 and X_2 be P_1 and P_2, respectively. Gaussian independent distributions $X_1 \sim \mathcal{N}(0, P_1)$ and $X_2 \sim \mathcal{N}(0, P_2)$ are optimal for every boundary point of the capacity region. In fact, for scalar Gaussian channels, the union and the convex hull operations are superfluous, and the capacity region is just a simple pentagon [6]. However, the input optimization problem becomes more difficult when the Gaussian multiple-access channel has vector inputs. In this case, different points in the capacity region may correspond to different input distributions, and a characterization of the capacity region involves an optimization over vector random variables.

The input optimization problem for the vector Gaussian multiple-access channel has been studied in the literature for the special cases of intersymbol interference (ISI) channels and scalar fading channels. The capacity region of the Gaussian multiple-access channel with ISI was characterized by Cheng and Verdú [7]. For the scalar ISI multiple-access channel, the input optimization problem can be formulated as a problem of optimal power allocation over frequencies. An analogous problem of finding the ergodic capacity of the scalar independent and identically distributed (i.i.d.) fading channels was studied by Knopp and Humblet [8] and Tse and Hanly [9], where the optimal power allocation over fading states was characterized. Both the scalar ISI channel and the scalar i.i.d. fading channel are special cases of the vector multiple-access channel considered in this correspondence. In both cases, individual channels in the multiple-access channel can be simultaneously decomposed into parallel independent scalar subchannels. For the ISI channel, a cyclic prefix can be appended to the input sequence so that the channel can be diagonalized in the frequency domain by a discrete Fourier transform. For the i.i.d. fading channel, the independence among the subchannels in time is explicitly assumed. In both cases, the optimal signaling direction is just the direction of the simultaneous diagonalization, and the input optimization problem is reduced to the power allocation problem among the scalar subchannels.

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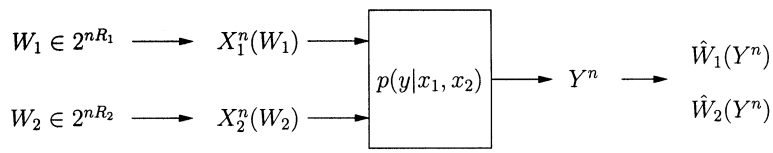

Fig. 1. Multiple-access channel.

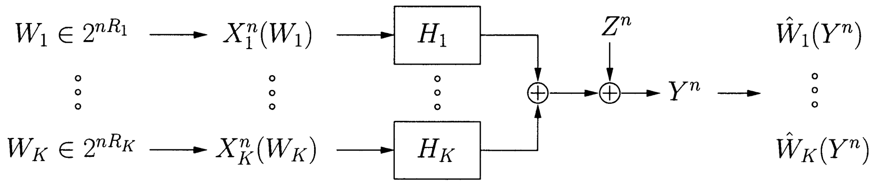

Fig. 2. Gaussian vector multiple-access channel.

The situation is more complicated if simultaneous diagonalization is not possible. This more general setting corresponds to a multiple-access situation where both the transmitters and the receiver are equipped with multiple antennas. In the spatial domain, the channel gain between a transmit antenna and a receive antenna can be arbitrary, so the channel matrix can have an arbitrary structure. In general, it is not possible to simultaneously decompose an arbitrary set of matrix channels into parallel independent scalar subchannels. Unlike the ISI channels, where the time-invariance property leads to a Toeplitz structure for the channel matrix, a multiple-antenna channel does not follow spatial invariance. Consequently, the equivalence of a cyclic prefix does not exist in the spatial domain, and the transmitter optimization problem becomes a combination of choosing the optimal signaling directions for each user and allocating a correct amount of power in each signaling direction. Such a joint optimization strikes a compromise between maximizing each user's data rate and minimizing its interference to other users. Fortunately, it can be shown that the input optimization problem for Gaussian vector multiple-access channels is a convex optimization problem [10]. So, the optimization problem is tractable in theory, and general-purpose convex programming routines, such as the interior-point method [11], are applicable. However, for a large-dimensional problem, the optimization may still be computationally intensive, because the optimization is performed in the space of positive semidefinite matrices, and the number of scalar variables grows quadratically with the number of input dimensions. In the literature, the input optimization problem for a multiple-antenna multiple-access fading channel is considered in [12] where asymptotic results in the limit of infinite number of users and infinite number of antennas have been reported. A similar problem exists for the CDMA systems, where the matrix channel is determined by the spreading sequences. Recent results in this area have been obtained in [13]–[15].

The main contribution of this correspondence is a numerical algorithm that can be used to efficiently compute the sum capacity achieving input distribution for a Gaussian vector multiple-access channel. It is shown that the joint optimization of signaling power and signaling directions for a vector multiple-access channel can be performed by a generalization of the single-user input optimization. In a single-user vector channel, the optimal signaling directions are the eigenmodes of the channel matrix, and the optimal power allocation is the so-called water-filling allocation [6]. For a vector multiple-access channel, although each user has a different channel and experiences a different interference structure, it is possible to apply single-user water-filling iteratively to reach a compromise among the signaling strategies for different users. This iterative water-filling procedure always converges, and it converges to the sum capacity of a vector multiple-access channel.

The rest of this correspondence is organized as follows. In Section II, the input optimization problem for the Gaussian vector multiple-access channel is formulated in a convex programming framework. In Section III, an optimization condition for the sum rate maximization problem is presented. In Section IV, the iterative water-filling algorithm is derived, and its convergence property is studied. Section V provides capacity region bounds based on the sum rate points. Section VI contains concluding remarks.

After this correspondence was initially submitted for review, the authors learned that a similar alternate input optimization procedure was suggested by Médard in the context of single-antenna multipath fading channels [16]. The algorithm presented in this correspondence is based on the same principle. The treatment here includes a more complete proof and a novel convergence analysis of the algorithm.

II. Problem Formulation

A memoryless K-user Gaussian vector multiple-access channel can be represented as follows (see Fig. 2):

$$\boldsymbol{Y} = \sum_{i=i}^{K} H_i \boldsymbol{X}_i + \boldsymbol{Z} \tag{2}$$

where $\boldsymbol{X}_i$ is the input vector signal, $\boldsymbol{Y}$ is the output vector signal, $\boldsymbol{Z}$ is the additive Gaussian noise vector with a covariance matrix denoted as S_z, and H_i is the time-invariant channel matrix. The channels are assumed to be known to both the transmitters and the receiver. Further, no feedback channel is available between the receiver and the transmitters, thus, transmitter cooperation (beyond time synchronization) is not possible. The input signals are assumed to be independent, with a joint distribution $\Pi_{i=1}^{K} p_i(\boldsymbol{x}_i)$ that satisfy the power constraints $\mathrm{tr}(\boldsymbol{E}[\boldsymbol{X}_i \boldsymbol{X}_i^T]) \le P_i$. Let S_i be the covariance matrices of $\boldsymbol{X}_i$, i.e., $S_i = \boldsymbol{E}[\boldsymbol{X}_i \boldsymbol{X}_i^T]$. Then, the power constraint becomes $\mathrm{tr}(S_i) \le P_i$.

The capacity region for a K-user multiple-access channel is the convex hull of the union of capacity pentagons defined in (2). For Gaussian vector multiple-access channels, the convex hull operation is not needed, and the capacity region can be characterized by maximizing $\sum_{i=1}^{K} \mu_i R_i$, with $\mu_i \ge 0$. The input distribution that maximizes this weighted rate sum is known to be a Gaussian distribution. Without loss of generality, let $\mu_1 \le \cdots \le \mu_K$. The optimal covariance matrices $S_1, \ldots, S_K$ can be found by solving the following optimization problem (see, e.g.,[7], [10]):

$$\begin{aligned} \text{maximize} \quad & \mu_1 \cdot \frac{1}{2} \log \left| \sum_{i=1}^{K} H_i S_i H_i^T + S_z \right| - \mu_K \cdot \frac{1}{2} \log |S_z| \\ & + \sum_{j=2}^{K} (\mu_j - \mu_{j-1}) \cdot \frac{1}{2} \log \left| \sum_{i=j}^{K} H_i S_i H_i^T + S_z \right| \\ \text{subject to} \quad & \operatorname{tr}(S_i) \le P_i, \quad i = 1, \ldots, K \\ & S_i \ge 0, \qquad i = 1, \ldots, K \end{aligned} \tag{3}$$

where $S_i \ge 0$ denotes that S_i is positive semidefinite. When the goal is to maximize the sum rate, the problem can be simplified by setting $\mu_1 = \cdots = \mu_K = 1$ as below

$$\begin{aligned} \text{maximize} \quad & \frac{1}{2} \log \left| \sum_{i=1}^{K} H_i S_i H_i^T + S_z \right| - \frac{1}{2} \log |S_z| \\ \text{subject to} \quad & \operatorname{tr}(S_i) \le P_i \qquad i = 1, \ldots, K \\ & S_i \ge 0. \qquad i = 1, \ldots, K. \end{aligned} \tag{4}$$

The objective function is concave because $\log|\cdot|$ is concave (see, e.g., [17, p. 466], [18, p. 48], or [19]). The constraints are convex in the space of positive semidefinite matrices [11]. Thus, the above optimization problem belongs to a class of convex programming problems for which efficient numerical optimization is possible [12], [11].

III. Sum Capacity

This section focuses on the sum capacity and derives a sufficient and necessary condition for the optimal input distribution that achieves the sum capacity. Toward this end, the single-user transmitter optimization problem is first cast into a convex optimization framework. The single-user optimization problem has a well-known water-filling solution. The water-filling algorithm takes advantage of the problem structure by decomposing the channel into orthogonal modes, which greatly reduces the optimization complexity. It turns out that this idea may be extended to the multiuser case under the sum-rate objective.

A. Single-User Water-Filling

For a single-user Gaussian vector channel, the mutual information maximization problem is

$$\begin{aligned} \text{maximize} \quad & \frac{1}{2} \log |HSH^T + S_z| - \frac{1}{2} \log |S_z| \\ \text{subject to} \quad & \operatorname{tr}(S) \le P \\ & S \ge 0. \end{aligned} \tag{5}$$

The solution to this problem involves two steps. First, since S_z is a symmetric positive definite matrix, its eigenvalue decomposition is of the form $S_z = Q\Delta Q^T$, where Q is an orthogonal matrix $QQ^T = I$, and Δ is a diagonal matrix of eigenvalues $\{\delta_1, \ldots, \delta_m\}$. Define $\hat{H} = \Delta^{-\frac{1}{2}} Q^T H$. The objective can then be rewritten as

$$\text{maximize} \quad \frac{1}{2} \log |\hat{H} S \hat{H}^T + I|. \tag{6}$$

Next, let $\hat{H} = F\Sigma M^T$ be a singular-value decomposition of $\hat{H}$, where F and M are orthogonal matrices, and Σ is a diagonal matrix of singular values $\{h_1, h_2, \ldots h_r\}$, where r is the rank of $\hat{H}$. Consider $\hat{S} = M^T S M$ as the new optimization variable. Since $\operatorname{tr}(S) = \operatorname{tr}(\hat{S})$, the problem is then transformed into

$$\begin{aligned} \text{maximize} \quad & \frac{1}{2} \log |\Sigma \hat{S} \Sigma^T + I| \\ \text{subject to} \quad & \operatorname{tr}(\hat{S}) \le P \\ & \hat{S} \ge 0. \end{aligned} \tag{7}$$

Using Hadamard's inequality [6], it is easy to show that the solution can be obtained by the well-known water-filling algorithm. The optimal $\hat{S}$ is a diagonal matrix $\operatorname{diag}\{p_1, p_2, \ldots p_r\}$ such that

$$p_i + 1/h_i^2 = K_l, \qquad \text{if } 1/h_i^2 < K_l \tag{8}$$

$$p_i = 0, \qquad \text{if } 1/h_i^2 \ge K_l \tag{9}$$

where K_l is known as the water-filling level, and it is a constant chosen so that $\sum_{i=1}^{r} p_i = P$. Finally, the optimal S is $M\hat{S}M^T$.

It can be seen that the optimal input distribution for a single-user Gaussian vector channel is a Gaussian distribution with a covariance matrix that satisfies two conditions. First, the transmit directions must align with the right eigenvectors of the effective channel. This decomposes the vector channel into a set of parallel independent scalar subchannels. Second, the power allocation among the subchannels must be a water-filling power allocation based on the noise-to-channel-gain ratio in each subchannel. Solving the single-user input optimization via water-filling is more efficient than using general-purpose convex programming algorithms, because water-filling takes advantage of the problem structure by decomposing the equivalent channel into its eigenmodes.

Note that although eigenvalues and singular values are unique up to ordering, the matrix decompositions themselves (i.e., matrices M and Q in the above derivation) are not necessarily unique. The nonuniqueness occurs when multiple eigenvalues (or singular values) have the same value. However, the optimal covariance matrix for the optimization problem (5) is unique. A short proof of this fact is provided in the following.

Suppose that S_1 and S_2 are both water-filling covariance matrices. Pick *any* orthogonal matrix Q in the eigenvalue decomposition of S_z and *any* orthogonal matrix M in the singular value decomposition of $\hat{H}$. Define $\hat{S}_1 = M^T S_1 M$ and $\hat{S}_2 = M^T S_2 M$. Clearly, both $\hat{S}_1$ and $\hat{S}_2$ must be diagonal, and they both have to satisfy the water-filling condition $p_i = (K_l - 1/h_i^2)_+$. Thus, $\hat{S}_1$ and $\hat{S}_2$ must be the same. This implies that $S_1 = M\hat{S}_1 M^T = M\hat{S}_2 M^T = S_2$.

B. Multiuser Water-Filling

The idea of water-filling can be generalized to multiple-access channels if the objective is to maximize the sum data rate. The first result toward this direction is a multiuser water-filling condition for the optimal transmit covariance matrices that achieve the sum capacity of a multiple-access channel.

Theorem 1: In a K-user multiple-access channel, $\{S_i\}$ is an optimal solution to the rate-sum maximization problem

$$\begin{aligned} \text{maximize} \quad & \frac{1}{2} \log \left| \sum_{i=1}^{K} H_i S_i H_i^T + S_z \right| - \frac{1}{2} \log |S_z| \\ \text{subject to} \quad & \operatorname{tr}(S_i) \le P_i \qquad i = 1, \ldots, K \\ & S_i \ge 0, \qquad i = 1, \ldots, K \end{aligned} \tag{10}$$

if and only if S_i is the single-user water-filling covariance matrix of the channel H_i with $S_z + \sum_{j=1, j\neq i}^{K} H_j S_j H_j^T$ as noise, for all $i = 1, 2, \ldots, K$.

Proof: The *only if* part is easy. Suppose that at the rate-sum optimum, there is an S_i that does not satisfy the single-user water-filling condition. Fix all other covariance matrices, set S_i to be the water-filling covariance matrix with $S_z + \sum_{j=1, j\neq i}^{K} H_j S_j H_j^T$ as noise. With all other covariance matrices fixed, the single-user optimization problem for S_i differs from the sum rate optimization problem by a constant. Thus, setting S_i to be the water-filling covariance matrix strictly increases the sum rate objective. This contradicts the optimality of $\{S_i\}$. Thus, at the optimum, all S_i's must satisfy the single-user water-filling condition.

The *if* part also holds. The proof relies on standard convex analysis. The constraints of the optimization problem are such that Slater's condition is satisfied. So, the Karush–Kuhn–Tucker (KKT) condition of the optimization problem is sufficient and necessary for optimality. To derive the KKT conditions, form the Lagrangian

$$L(S_i, \lambda_i, \Psi_i) = \log\left|\sum_{i=1}^{K} H_i S_i H_i^T + S_z\right| - \sum_{i=1}^{K} \lambda_i(\mathrm{tr}(S_i) - P_i) + \sum_{i=1}^{K} \mathrm{tr}(S_i \Psi_i). \quad (11)$$

The coefficient $1/2$ and the constant $\log|S_z|$ are omitted for simplicity. Here, $\{\lambda_i\}$ are the scalar dual variables associated with the power constraints, $\{\Psi_i\}$ are the matrix dual variables associated with the positive definiteness constraints. The inner product in the space of semidefinite matrices is the trace of matrix product.

The KKT condition of the optimization problem consists of the condition $\partial L/\partial S_i = 0$, the complementary slackness conditions, and the primal and dual constraints

$$\begin{aligned} \lambda_i I = &\ H_i^T \left(\sum_{j=1}^{K} H_j S_j H_j^T + S_z\right)^{-1} H_i + \Psi_i \\ \mathrm{tr}(S_i) = &\ P_i \\ \mathrm{tr}(\Psi_i S_i) = &\ 0 \\ \Psi_i, S_i, \lambda_i \geq &\ 0 \end{aligned} \quad (12)$$

for all $i = 1, \ldots, K$. Note that the gradient of $\log|X|$ is X^{-1}.

Now, the above KKT condition is also valid for the single-user water-filling problem when K is set to 1. In this case, it is easy to verify that the single-user solution (8), (9) satisfies the condition exactly. But, for each user i, the multiuser KKT condition and the single-user KKT condition differ only by the additional noise term $\sum_{j=1, j\neq i}^{K} H_j S_j H_j^T$. So, if each S_i satisfies the single-user condition while regarding other users' signals as additional noise, then collectively, the set of $\{S_i\}$ must also satisfy the multiuser KKT condition. By the sufficiency of the KKT condition, $\{S_i\}$ must then be the optimal covariance for the multiuser problem. This proves the *if* part of the theorem. □

IV. Iterative Water-Filling

A. Algorithm

At the rate-sum optimum, each user's covariance matrix is a water-filling covariance against the combined noise and all other users' interference. This suggests that the set of rate-sum optimal covariance matrices can be found using an iterative procedure.

Algorithm 1: Iterative water-filling
initialize $S_i = 0$, $i = 1, \ldots K$.
repeat
  for i = 1 to K
    $S_z' = \sum_{j=1, j\neq i}^{K} H_j S_j H_j^T + S_z$;
    $S_i = \arg\max_S \frac{1}{2}\log|H_i S H_i^T + S_z'|$;
  end
until the sum rate converges.

Theorem 2: In the iterative water-filling algorithm, the sum rate converges to the sum capacity, and $(S_1, \ldots, S_K)$ converges to an optimal set of input covariance matrices for the Gaussian vector multiple-access channel.

Proof: At each step, the iterative water-filling algorithm finds the single-user water-filling covariance matrix for each user while regarding all other users' signals as additional noise. Since the single-user rate objective differs from the multiuser rate-sum objective by only a constant, the rate-sum objective is nondecreasing with each water-filling step. The rate-sum objective is bounded above, so the sum rate converges to a limit.

The convergence matrices $S_1, \ldots, S_K$ also converge to a limit. Because the single-user water-filling matrix is unique, each water-filling step in the iterative algorithm must either yield a strict increase of the sum rate or keep the covariance matrices the same. At the limit, all S_i's are simultaneously the single-user water-filling covariance matrices of user i with all other users' signals regarded as additional noise. Then, by Theorem 1, this set of $(S_1, \ldots, S_K)$ must achieve the sum capacity of the Gaussian vector multiple-access channel. □

Note that the proof does not depend on the initial starting point. Thus, the algorithm converges to the sum capacity from any starting values of $(S_1, \ldots, S_K)$. However, although the sum capacity is unique, the optimal covariance matrices themselves may not be. It is possible for the iterative algorithm to converge to two different sets of covariance matrices both giving the same optimal sum rate. The following example illustrates this point. Let $H_1 = H_2 = S_z = I_{2\times 2}$, and $P_1 = P_2 = 2$. Then, $S_1 = S_2 = I_{2\times 2}$, and

$$S_1' = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad S_2' = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

both achieve the same sum capacity.

Fig. 3 gives a graphical interpretation of the algorithm. The capacity region of a two-user vector multiple-access channel is shown in Fig. 3(a). The sum rate $R_1 + R_2$ reaches the maximum on the line segment between *C* and *D*. Initially, the covariance matrices for the two users, $S_1^{(0)}$ and $S_2^{(0)}$, are zero matrices.

1) The first iteration is shown in Fig. 3(b). After a single-user water-filling for $S_1^{(1)}$, the rate pair (R_1, R_2) is at point *F*. Then, treating $S_1^{(1)}$ as noise, a single-user water-filling for $S_2^{(1)}$ moves the rate pair to point *E*.

2) The second iteration is shown in Fig. 3(c). First, note that with fixed covariance matrices $S_1^{(1)}$ and $S_2^{(1)}$, the capacity region is the pentagon *abEFO*. So, by changing the decoding order of user 1 and 2, the rate pair can be moved to point *b* without affecting the rate sum. Once at point *b*, water-filling for $S_1^{(1)}$ while treating $S_2^{(1)}$ as noise gives $S_1^{(2)}$. This increases $I(X_1; Y)$, while keeping $I(X_2; Y|X_1)$ fixed, thus moving the rate pair to point *c*.

3) The capacity pentagon with $(S_1^{(2)}, S_2^{(1)})$ is now represented by *acdeO*. So, the decoding order can again be interchanged to get to the point *d*. Performing another single-user water-filling treating $S_1^{(2)}$ as additional noise gives $S_2^{(2)}$ and the corresponding rate-pair point *f* in Fig. 3(d). The process continues until it converges to points *C* and *D*.

Note that in every step, each user negotiates for itself the best signaling direction as well as the optimal power allocation while regarding the interference generated by all other users as noise. The iterative water-filling algorithm is more efficient than general-purpose convex programming routines, because the algorithm decomposes the multiuser problem into a sequence of single-user problems, each of which is much easier to solve. Further, in each step, the single-user water-filling algorithm takes advantage of the problem structure by performing an eigenmode decomposition. In fact, the convergence is very fast.

B. Convergence Behavior

The iterative procedure arrives at a corner point of some rate region pentagon after the first iteration. The following theorem shows that this

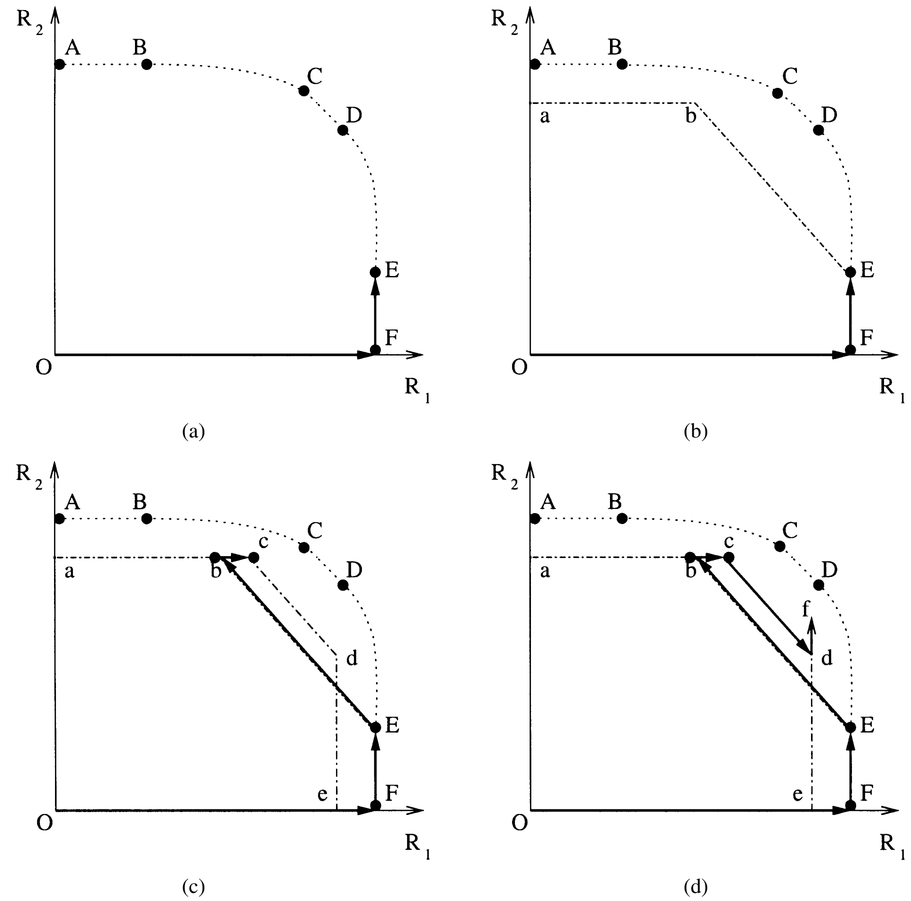

Fig. 3. First two iterations of iterative water-filling algorithm.

corner point is only 1/2 nats per user per output dimension away from the sum capacity.

Theorem 3: After one iteration of the iterative water-filling algorithm, $\{S_i\}$ achieves a total data rate $\sum_{i=1}^{K} R_i$ that is at most $(K-1)m/2$ nats away from the sum capacity.

Proof: The idea is to form the Lagrangian dual of the original optimization problem, and use the fact that the difference between the primal and dual objectives, the so-called duality gap, is a bound on the difference between the primal objective and the optimum.

The first step in deriving the dual problem is to reformulate the optimization problem (10) in the following equivalent form:

$$\begin{array}{lll} \text{minimize} & -\log|T| & \\ \text{subject to} & T \leq \sum_{i=1}^{K} H_i S_i H_i^T + S_z & \\ & \mathrm{tr}(S_i) \leq P_i & i=1,\dots,K \\ & S_i \geq 0, & i=1,\dots,K \end{array} \tag{13}$$

where again the coefficient $1/2$ and the constant $\log|S_z|$ are dropped. The Lagrangian of the new optimization problem is

$$\begin{aligned} &L(\{S_i\}, T, \Gamma, \{\lambda_i\}, \{\Psi_i\}) \\ &\quad = -\log|T| + \mathrm{tr}\left[\Gamma\left(T - \sum_{i=1}^{K} H_i S_i H_i^T - S_z\right)\right] \\ &\qquad + \sum_{i=1}^{K} \lambda_i(\mathrm{tr}(S_i) - P_i) - \sum_{i=1}^{K} \mathrm{tr}(\Psi_i S_i) \\ &\quad = -\log|T| + \mathrm{tr}(\Gamma T) - \mathrm{tr}(\Gamma S_z) \\ &\qquad - \sum_{i=1}^{K} \lambda_i P_i + \sum_{i=1}^{K} \mathrm{tr}[(\lambda_i I - H_i^T \Gamma H_i - \Psi_i) S_i] \end{aligned} \tag{14}$$

where the fact $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ is used. The objective of the dual program is

$$g(\Gamma, \{\lambda_i\}, \{\Psi_i\}) = \inf_{\{S_i\}, T} L(\{S_i\}, T, \Gamma, \{\lambda_i\}, \{\Psi_i\}). \tag{15}$$

At the optimum, $\partial L/\partial S_i = 0$. Thus,

$$\lambda_i I = H_i^T \Gamma H_i + \Psi_i, \qquad i = 1, 2, \dots, K. \tag{16}$$

Further, at the optimum, $\partial L/\partial T = 0$. Thus,

$$\frac{\partial}{\partial T}(-\log|T| + \mathrm{tr}(\Gamma T)) = 0. \tag{17}$$

This implies that

$$T^{-1} = \Gamma. \tag{18}$$

Therefore,

$$g(\Gamma, \{\lambda_i\}, \{\Psi_i\}) = \log|\Gamma| + m - \mathrm{tr}(\Gamma S_z) - \sum_{i=1}^{K} \lambda_i P_i$$

where m is the number of output dimensions. The dual problem of (13) is then

$$\begin{array}{lll} \text{maximize} & \log|\Gamma| + m - \mathrm{tr}(\Gamma S_z) - \sum_{i=1}^{K} \lambda_i P_i & \\ \text{subject to} & \lambda_i I \geq H_i^T \Gamma H_i, & i = 1, \dots, K \\ & \Gamma \geq 0. & \end{array} \tag{19}$$

Note that the only constraints on $\{\Psi_i\}$ are positive semidefinite constraints, so (16) is equivalent to the inequality in (19). Because the primal program is convex, the dual problem achieves a maximum at the minimum value of the primal objective.

The duality gap, denoted as γ, is the difference between the objective of the primal problem (13) and the dual problem (19)

$$\gamma = \mathrm{tr}\left[\left(\sum_{i=1}^{K} H_i S_i H_i^T + S_z\right)^{-1} S_z\right] + \sum_{i=1}^{K} \lambda_i P_i - m. \quad (20)$$

Now, consider the duality gap after one iteration of the algorithm. Starting with $S_i = 0$, the first iteration consists of K water-fillings: S_1 is the single-user water-filling covariance of noise S_z alone, S_2 is the water-filling of noise plus interference from S_1, and so on. S_K is the water-filling of noise plus interference from all other users. The duality gap bound holds for all dual feasible λ_i's. The gap is the tightest when λ_i is chosen to be the smallest nonnegative value satisfying the dual constraints in (19)

$$\lambda_i = \max \mathrm{eig}\left[H_i^T \left(\sum_{j=1}^{K} H_j S_j H_j^T + S_z\right)^{-1} H_i\right], \quad i = 1, \ldots, K. \quad (21)$$

In fact, the duality gap reduces to zero if the primal feasible S_i is the optimal S_i^*, and the dual feasible λ_i's are chosen in the above fashion.

Now, since S_1 is a single-user water-filling, the duality gap for this single-user water-filling must be zero. Thus,

$$\mathrm{tr}[(H_1 S_1 H_1^T + S_z)^{-1} S_z] + \lambda_1' P_1 - m = 0 \quad (22)$$

where

$$\lambda_1' = \max \mathrm{eig}[H_1^T (H_1 S_1 H_1^T + S_z)^{-1} H_1]. \quad (23)$$

More generally, S_i is the single-user water-filling regarding $\sum_{j=1}^{i-1} H_j S_j H_j^T + S_z$ as noise. Thus,

$$\mathrm{tr}\left[\left(\sum_{j=1}^{i} H_j S_j H_j^T + S_z\right)^{-1} \left(\sum_{j=1}^{i-1} H_j S_j H_j^T + S_z\right)\right] + \lambda_i' P_i - m = 0 \quad (24)$$

where

$$\lambda_i' = \max \mathrm{eig}\left[H_i^T \left(\sum_{j=1}^{i} H_j S_j H_j^T + S_z\right)^{-1} H_i\right]. \quad (25)$$

Now, the following three inequalities are needed. First, since $A \geq B$ implies $\mathrm{tr}(A) \geq \mathrm{tr}(B)$, the following must be true:

$$\mathrm{tr}\left(\sum_{j=1}^{K} H_j S_j H_j^T + I\right)^{-1} \leq \mathrm{tr}(H_1 S_1 H_1^T + I)^{-1} \quad (26)$$

Second, since

$$H_i^T \left(\sum_{j=1}^{K} H_j S_j H_j^T + S_z\right)^{-1} H_i \leq H_i^T \left(\sum_{j=1}^{i} H_j S_j H_j^T + S_z\right)^{-1} H_i \quad (27)$$

from (21) and (25)

$$\lambda_i \leq \lambda_i'. \quad (28)$$

Third, since the trace of a positive definite matrix is positive, from (24)

$$\lambda_i' P_i \leq m. \quad (29)$$

Now, putting everything together

$$\begin{aligned}
\gamma &= \mathrm{tr}\left[\left(\sum_{i=1}^{K} H_i S_i H_i^T + S_z\right)^{-1} S_z\right] + \sum_{i=1}^{K} \lambda_i P_i - m & (30)\\
&\leq \mathrm{tr}\left[\left(\sum_{i=1}^{K} H_i S_i H_i^T + S_z\right)^{-1} S_z\right] + \sum_{i=1}^{K} \lambda_i' P_i - m & (31)\\
&= \mathrm{tr}\left[\left(\sum_{i=1}^{K} H_i S_i H_i^T + S_z\right)^{-1} S_z\right] + \lambda_1' P_1 - m + \sum_{i=2}^{K} \lambda_i' P_i & (32)\\
&\leq \sum_{i=2}^{K} \lambda_i' P_i & (33)\\
&\leq (K-1)m & (34)
\end{aligned}$$

where the first inequalities follows from (28), the second inequality follows from (26) and (22), and the last inequality follows from (29). Recall that a factor of $1/2$ was omitted in the statement of the primal and dual problems, (13) and (19). Therefore, the duality gap bound is $(K-1)m/2$ nats. □

The capacity region of a K-user multiple-access channel with fixed input covariance matrices is a polytope. Depending on the order of water-filling, after the first iteration, the iterative water-filling algorithm reaches one of the $K!$ corner points of the capacity polytope. The above theorem asserts that none of these corner points is more than $(K-1)m/2$ nats away from the sum capacity, where m is the number of output dimensions. This result roughly states that the capacity loss per user per output dimension is at most 1/2 nats after just one iteration. This bound is rather general. It works for arbitrary channel matrices, arbitrary power constraints, and arbitrary input dimensions. Numerical simulation on realistic channels suggests that in most cases the actual difference from the capacity is even smaller.

A numerical example with $K = 10$ users is presented below. Each user is equipped with 10 antennas, and the receiver is also equipped with 10 antennas. A fading channel with 10 i.i.d. fading states is considered. Thus, the transmitters and the receiver have effectively 100 dimensions each. The channel matrix is block diagonal, where each block is of size 10×10. The block matrix entries are randomly generated from an i.i.d. zero-mean Gaussian distribution with unit variance. The channel matrix is assumed to be known at both the transmitters and the receiver. The total power constraint for each user is set to 100, and noise variance is set to 1. The ergodic sum capacity iscomputed using the iterative water–computed using the iterative water-filling algorithm. Fig. 4 shows the convergence behavior. The sum capacity of this channel is about 44.5 bits per transmission. Both the duality gap and the difference between the capacity and the achievable rate after each iteration are plotted. Observe that for practical purposes, the algorithm converges after only a few iterations. The convergence appears to be exponentially fast.

V. Capacity Region

The iterative water-filling algorithm can be used to find the set of optimal covariance matrices that achieve the sum capacity of a Gaussian vector multiple-access channel. This set of K covariance matrices gives $K!$ corner points of a capacity pentagon, each corresponding to a different decoding order. Upper and lower bounds on the entire capacity region can be derived from these corner points.

The following two-user multiple-access channel is used as an example. The transmitters and the receiver are equipped with seven antennas each. The power constraint for each user is set to 10. The noise variance is set to 1. The entries of the channel matrices are generated

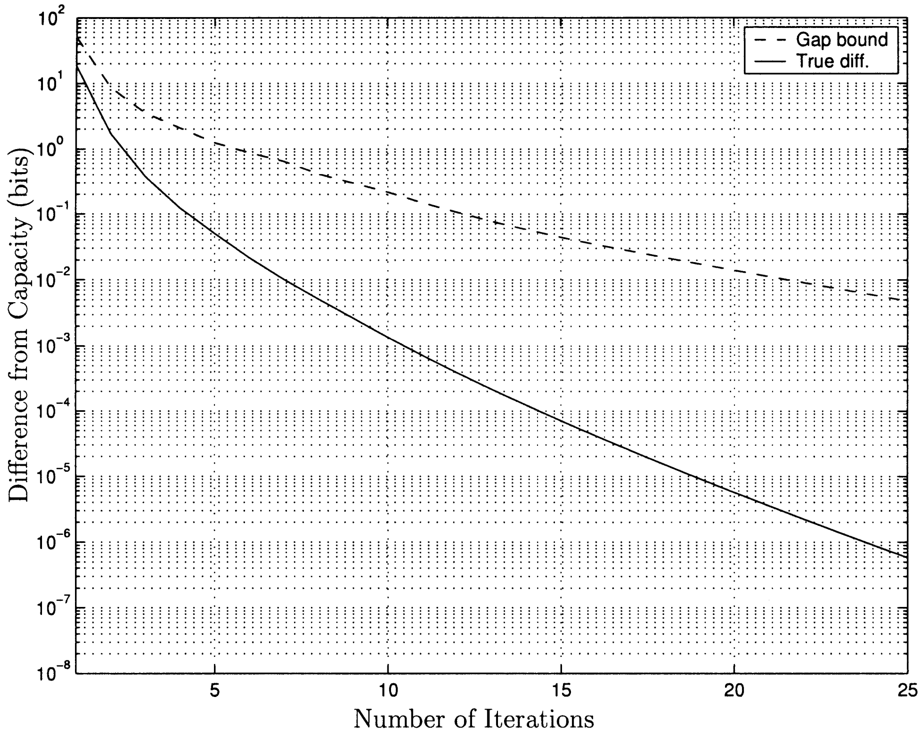

Fig. 4. Convergence of the iterative water-filling algorithm.

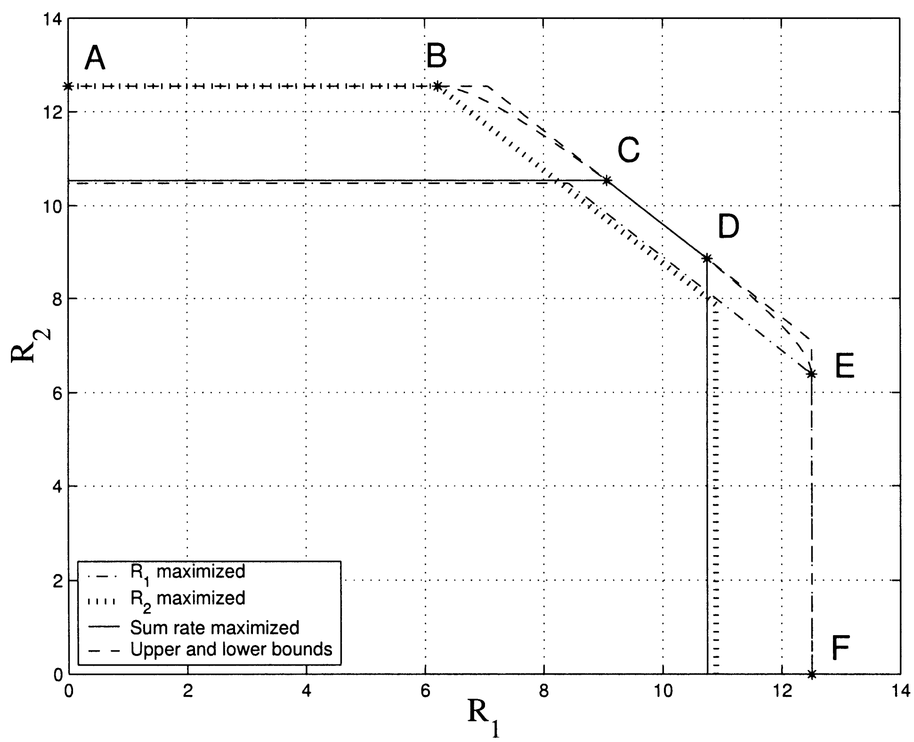

Fig. 5. Lower bound and upper bound of a typical capacity region.

according to an i.i.d. Gaussian random variable with zero mean and unit variance.

In Fig. 5, the points *B* and *E* can be found after one iteration of water-filling. Let their respective input covariance matrices be $S_B = (S_{B,1}, S_{B,2})$ and $S_E = (S_{E,1}, S_{E,2})$. Also, the sum capacity points *C* and *D* are found using iterative water-filling. Denote the sum-capacity achieving covariance matrix as $S_{CD} = (S_{CD,1}, S_{CD,2})$. Note that the line segment between *C* and *D* defines a portion of the capacity boundary. If the optimal sum-capacity covariance matrices happen to be orthogonal, points *C* and *D* collapse to the same point.

A lower bound for the region between *B* and *C* (or *D* and *E*) can be found based on the linear combination of covariance matrices S_B and S_{CD} (or S_E and S_{CD}, respectively). Consider the data rates associated with the covariance matrices $\alpha S_B + (1-\alpha) S_{CD}$ with user 1 decoded first (or $\beta S_E + (1-\beta) S_{CD}$ with user 2 decoded first), where α (or β) ranges from 0 to 1. These rates are achievable, so they are lower bounds. Because the objective is a concave function of the covariance matrices, this lower bound is better than the time-sharing of data rates associated with *B* and *C* (or *D* and *E*). Since the corner points after one iteration (i.e., *B* and *E*) are at most $(K-1)m/2$ nats away from the sum capacity, the lower bound is a close approximation of the capacity region. A typical example is shown in Fig. 5. Extensive numerical simulations show that the lower bound is fairly tight. An upper bound is also plotted by extending the line segments *AB*, *CD*, and *EF*. This is an upper bound because the capacity region is convex.

VI. Conclusion

This correspondence addresses the problem of finding the optimal transmitter covariance matrices that achieve the sum capacity in a Gaussian vector multiple-access channel. The computation of the sum capacity is formulated in a convex optimization framework. A multiuser water-filling condition for achieving the sum capacity is found. It is shown that the sum-rate maximization problem can be solved efficiently using an iterative water-filling algorithm, where each step of the iteration is equivalent to a local maximization of one user's data rate with multiuser interference treated as noise. The iterative water-filling algorithm is shown to converge to the sum capacity from any starting point. The convergence is fast. In particular, it reaches within 1/2 nats per user per output dimension from the sum capacity after just a single iteration. As mentioned before, the vector channel model discussed in this correspondence includes ISI channels and fading channels as special cases. Thus, the iterative water-filling algorithm can also be used to efficiently compute the power allocation across the frequency spectrum for an ISI channel or over time for a fading channel.

Finally, although the iterative water-filling algorithm solves the sum capacity problem efficiently, it does not directly apply to other points in the capacity region. The computations of other capacity points are also convex programming problems. However, how to best exploit the problem structure in these cases is still not clear.

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Maximizing the Spectral Efficiency of Coded CDMA Under Successive Decoding

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***Abstract*—We investigate the spectral efficiency achievable by random synchronous code-division multiple access (CDMA) with quaternary phase-shift keying (QPSK) modulation and binary error-control codes, in the large system limit where the number of users, the spreading factor, and the code block length go to infinity. For given codes, we maximize spectral efficiency assuming a minimum mean-square error (MMSE) successive stripping decoder for the cases of equal rate and equal power users. In both cases, the maximization of spectral efficiency can be formulated as a linear program and admits a simple closed-form solution that can be readily interpreted in terms of power and rate control. We provide examples of the proposed optimization methods based on off-the-shelf low-density parity-check (LDPC) codes and we investigate by simulation the performance of practical systems with finite code block length.**

***Index Terms*—Channel capacity, code-division multiple access (CDMA), low-density parity-check (LDPC) codes, multiuser detection, quaternary phase-shift keying (QPSK) modulation, successive decoding.**

I. Introduction

All points in the capacity region of the scalar Gaussian multiple-access channel are achievable by successive single-user encoding,

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