Constant-Power Waterfilling: Performance Bound and Low-Complexity Implementation

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***Abstract*—In this letter, we investigate the performance of constant-power waterfilling algorithms for the intersymbol interference channel and for the independent identically distributed fading channel where a constant power level is used across a properly chosen subset of subchannels. A rigorous performance analysis that upper bounds the maximum difference between the achievable rate under constant-power waterfilling and that under true waterfilling is given. In particular, it is shown that for the Rayleigh fading channel, the spectral efficiency loss due to constant-power waterfilling is at most 0.266 b/s/Hz. Furthermore, the performance bound allows a very-low-complexity, logarithm-free, power-adaptation algorithm to be developed. Theoretical worst-case analysis and simulation show that the approximate waterfilling scheme is very close to the optimum.**

***Index Terms*—Bit loading, duality gap, waterfilling.**

I. Introduction

When a communication channel is corrupted by severe fading or by strong intersymbol interference (ISI), the adaptation of the transmit signal to the channel condition can typically bring a large improvement to the transmission rate. Adaptation is possible when the channel state is available to the transmitter, usually by a channel-estimation scheme and a reliable feedback mechanism. With perfect channel information, the problem of finding the optimal adaptation strategy has been much studied in the past. If the channel can be partitioned into parallel independent subchannels, for example, when the fading statistics for the fading channel is independent and identically distributed (i.i.d.) or by a discrete Fourier transform (DFT) in the case of an ISI channel, the optimal transmit power adaptation scheme is the well-known waterfilling procedure. In a waterfilling power spectrum, more power is allocated to better subchannels with higher signal-to-noise ratios (SNRs), so that the sum of data rates in all subchannels is maximized, where the data rate in each subchannel is related to the power allocation by Shannon's Gaussian capacity formula[1] $(1/2)\log(1+\mathrm{SNR})$. However, because capacity is a logarithmic function of power, the data rate is usually insensitive to the exact power allocation, except when the SNR is low. This motivates the search for simpler power-allocation schemes that can perform close to the optimum.

Approximate waterfilling schemes often greatly simplify transmitter and receiver design, and they have been the subject of considerable study. In the multicarrier context, Chow [1] empirically discovered that as long as a correct frequency band is used, a constant power allocation has a negligible performance loss compared with true waterfilling. The same phenomenon is observed in the adaptive modulation setting [2]. There have been several performance bounds on constant-power waterfilling reported in the literature. Aslanis [3] compared the worst-case difference between a true waterfilling and a constant-power waterfilling, and derived a bound based on the SNR cutoff value. Schein and Trott [4] derived a different bound, also based on SNR. This letter extends the existing results in several directions. First, a worst-case performance bound is derived using an approach based on convex analysis. The upper bound derived is valid for any arbitrary SNR. Second, it is shown that the new performance bound can be used to design a low-complexity power-allocation algorithm which is free of logarithm operations, and which has a bounded worst-case performance. In particular, the algorithm is shown to be at most 0.266 bs/s/Hz away from capacity on a Rayleigh fading channel and often performs much closer to capacity in practice.

In this letter, the primary focus is on power adaptation. The bit allocation is allowed to vary, and is not restricted to integer values. This approach is justifiable with the use of channel coding. In this case, the Shannon capacity for channels with perfect transmitter and receiver side information can be achieved with a concatenation of a standard random Gaussian codebook and a power-adaptation device [5]. In a related work [6], schemes with both constant power and constant bit allocation are investigated.

The rest of the letter is organized as follows. In Section II, the waterfilling problem is formulated and the new upper bound is derived. In Section III, a new low-complexity power-adaptation algorithm is proposed, and its performance analyzed. In Section IV, the performance bound is applied explicitly to the Rayleigh fading channel. Simulation results for both wireless and wireline applications are presented in Section V. Conclusions are drawn in Section VI.

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[1] In this letter, "$\log$" is used to denote logarithm of base 2; "$\ln$" is used to denote logarithm of base e.

II. Constant-Power Waterfilling

A. Problem Formulation

We choose to formulate the problem in the adaptive modulation framework because it is slightly more general than the multicarrier setting. The communication channel is modeled as

$$Y(i) = \sqrt{\nu(i)} \cdot X(i) + N(i) \tag{1}$$

where i is the discrete time index, $X(i)$ and $Y(i)$ are scalar input and output signals, respectively, $N(i)$ is the additive white Gaussian noise (AWGN), which is i.i.d. with a constant variance σ^2, and $\sqrt{\nu(i)}$ is the multiplicative channel fading coefficient. For simplicity, $\nu(i)$, the squared magnitude of the fading coefficient, is assumed to be i.i.d. with a probability distribution $\rho(\nu)$. The capacity of this fading channel under an average transmit power constraint when both the transmitter and the receiver have perfect and instantaneous channel side information was characterized by Goldsmith and Variaya [7]. They proposed a waterfilling-in-time solution and proved a coding theorem based on a finite partition of channel fading statistics, i.e., ν is restricted to take finite values $\nu_1, \nu_2, \ldots, \nu_m$, with probabilities $p_1, p_2, \ldots, p_m$. In this case, the maximization problem becomes

$$\text{maximize} \quad \sum_{k=1}^{m} p_k \log\left(1 + \frac{S_k \nu_k}{\sigma^2}\right) \tag{2}$$

$$\text{subject to} \quad \sum_{k=1}^{m} p_k S_k \le \bar{S} \tag{3}$$

$$S_k \ge 0 \tag{4}$$

where $\bar{S}$ is the average transmit power constraint, and the maximization is over all power-allocation policies S_k based on the instant channel fading state ν_k. Putting $p_k = 1$ reduces the problem to the multicarrier setting. This optimization problem has a well-known waterfilling solution. Our interest is in finding approximate solutions with provable worst-case performance.

Note that Shannon's Gaussian channel capacity formula is used here, and a capacity-achieving Gaussian codebook is assumed. In reality, where practical codes and modulation methods are used, the achievable rate can be computed by the same formula with the noise variance σ^2 increased by a constant factor "SNR gap," which denotes the amount of extra coding gain needed to achieve Shannon capacity [8], [2]. (SNR gap is called SNR_{norm} in [9].) Without loss of generality, the SNR gap is assumed to be 0 dB for the rest of the letter, unless otherwise stated.

B. Duality Gap

The optimization problem (2) belongs to the class of convex optimization problems, where a convex objective function is to be minimized subject to a convex constraint set. A general form of a convex optimization problem is the following:

$$\text{minimize} \quad f_0(x) \tag{5}$$

$$\text{subject to} \quad f_i(x) \le 0 \tag{6}$$

where $f_i(x)$, $i = 0, 1, \ldots, m$ are convex functions. $f_0(x)$ is called the primal objective. The Lagrangian of the optimization problem is defined as

$$L(x, \lambda) = f_0(x) + \lambda_1 f_1(x) + \cdots + \lambda_m f_m(x) \tag{7}$$

where λ_i are nonnegative. The dual objective is defined to be $g(\lambda) = \inf_x L(x, \lambda)$. It is easy to see that $g(\lambda)$ is a lower bound on the optimal $f_0(x)$

$$f_0(x) \ge f_0(x) + \sum_i \lambda_i f_i(x) \tag{8}$$

$$\ge \inf_z \left(f_0(z) + \sum_i \lambda_i f_i(z) \right) \tag{9}$$

$$\ge g(\lambda). \tag{10}$$

So

$$g(\lambda) \le \min_x f_0(x). \tag{11}$$

This is the lower bound that we will use to investigate the optimality of approximate waterfilling algorithms. The difference between the primal objective $f_0(x)$ and the dual objective $g(\lambda)$ is called the duality gap. A central result in convex analysis [10] is that when the primal problem is convex, the duality gap reduces to zero at the optimum under some general conditions known as constraint qualifications (which are satisfied for the problem considered in this letter). In other words, the optimal value of the primal objective may be obtained by maximizing the dual objective $g(\lambda)$ over nonnegative dual variables λ_i. Thus, for convex problems, the lower bound is tight.

C. Lower Bound

The above general result is now applied to the waterfilling problem. First, maximizing the data rate is equivalent to minimizing its negative. The capacity is a concave function of power, so its negative is convex. The constraints are linear, so they are convex, as well. Associate dual variable λ with the power constraint, and μ_k with each of the positivity constraints on S_k, the Lagrangian is then

$$L(S_k, \lambda, \mu_k) = \sum_{k=1}^{m} -p_k \log\left(1 + \frac{S_k \nu_k}{\sigma^2}\right) + \lambda \left[\left(\sum_{k=1}^{m} p_k S_k \right) - \bar{S} \right] + \sum_{k=1}^{m} \mu_k(-S_k). \tag{12}$$

The dual objective function $g(\lambda, \mu_k)$ is the infimum of the Lagrangian over primal variables S_k. At the infimum, the partial derivative of the Lagrangian with respect to S_k must be zero

$$\frac{\partial L}{\partial S_k} = 0 = -p_k \cdot \frac{\frac{\nu_k}{\sigma^2}}{1 + \frac{S_k \nu_k}{\sigma^2}} \cdot \frac{1}{\ln 2} + \lambda \cdot p_k - \mu_k \tag{13}$$

from which the classical waterfilling condition is obtained

$$S_k + \frac{\sigma^2}{\nu_k} = \frac{1}{\lambda - \frac{\mu_k}{p_k}} \cdot \frac{1}{\ln 2}. \tag{14}$$

This condition, together with the constraints of the original primal problem, the positivity constraints on the dual variables, and the complementary slackness constraints, form the Karush–Kuhn–Tucker (KKT) condition, which is sufficient and necessary in this case. More specifically, the complementary slackness condition states that the constraint for the original primal problem is satisfied with equality if and only if the dual variable associated with the inequality is strictly greater than zero. In the waterfilling problem, this translates to the condition that S_k is greater than zero if and only if μ_k is zero. Thus, when a positive power is allocated in a subchannel, (i.e., $S_k > 0$, $\mu_k = 0$), the sum of the signal power S_k and the normalized noise power σ^2/ν_k in each subchannel must be a constant, otherwise (i.e., when $S_k = 0$ and $\mu_k > 0$), the normalized noise power must exceed the water level. The waterfilling condition gives the following optimal adaptation strategy:

$$S_k = \begin{cases} \frac{\sigma^2}{\nu_0} - \frac{\sigma^2}{\nu_k}, & \text{if } \nu_k \geq \nu_0 \\ 0, & \text{if } \nu_k < \nu_0 \end{cases} \tag{15}$$

where the cutoff point ν_0 is determined by the average power constraint and the fading distribution.

Substituting the waterfilling condition (14) into (12) gives the dual objective

$$g(\lambda, \mu_k) = -\sum_{k=1}^{m} p_k \log\left(\frac{\frac{\nu_k}{\sigma^2}}{\lambda - \frac{\mu_k}{p_k}} \cdot \frac{1}{\ln 2}\right) - \left(\sum_{k=1}^{m} (\lambda p_k - \mu_k) \cdot \frac{\sigma^2}{\nu_k}\right) - \lambda \bar{S} + \frac{1}{\ln 2}. \tag{16}$$

The dual objective is always convex, and it is a lower bound to the primal objective[2] for all nonnegative λ and μ_k. The lower bound is, in fact, tight when λ and μ_k achieve the optimum of the dual program. Finding the tightest λ and μ_k is equivalent to solving the original optimization problem, which is complicated. However, if instead, the dual variables associated with the primal variables are chosen via (14), then a simple bound emerges. In this case, the duality gap, defined as the difference between the primal objective and the dual objective, and denoted as Γ, has the following form:

$$\Gamma = \sum_{k=1}^{m} p_k \left(\frac{\frac{\sigma^2}{\nu_k}}{S_k + \frac{\sigma^2}{\nu_k}} \cdot \frac{1}{\ln 2}\right) + \lambda \bar{S} - \frac{1}{\ln 2}. \tag{17}$$

To express the gap exclusively in primal variables S_k, a suitable λ needs to be found. A small λ is desirable, because it makes the duality gap small. Since

$$\frac{1}{S_k + \frac{\sigma^2}{\nu_k}} \cdot \frac{1}{\ln 2} = \lambda - \frac{\mu_k}{p_k} \tag{18}$$

and recall that λ and μ_k need to be nonnegative, the smallest nonnegative λ is then

$$\lambda = \max_k \left(\frac{1}{S_k + \frac{\sigma^2}{\nu_k}} \cdot \frac{1}{\ln 2}\right) = \frac{1}{\min_k \left\{S_k + \frac{\sigma^2}{\nu_k}\right\}} \cdot \frac{1}{\ln 2}. \tag{19}$$

[2] $g(\lambda, \mu_k)$ is a lower bound to the minimization problem. So $-g(\lambda, \mu_k)$ is an upper bound to the rate-maximization problem. To avoid notational inconvenience, the rest of the letter will use the term "duality gap" only.

Assuming that the approximate waterfilling algorithm satisfies the power constraint $\sum_k p_k S_k \leq \bar{S}$ with equality,[3] the above gives the following:

$$\Gamma = \frac{1}{\ln 2} \cdot \left[\sum_{k=1}^{m} p_k \left(\frac{S_k}{\min_j \left\{S_j + \frac{\sigma^2}{\nu_j}\right\}} - \frac{S_k}{S_k + \frac{\sigma^2}{\nu_k}}\right)\right]. \tag{20}$$

The preceding development is summarized in the following theorem.

Theorem 1: For the optimization problem (2), if $S_k \geq 0$ is a power-allocation strategy that satisfies the power constraint with equality, then the achievable data rate using S_k is at most Γ b/s/Hz away from the optimal waterfilling solution, where Γ is expressed in (20).

This bound applies to all approximate waterfilling algorithms in general. For example, it can be used to bound the performance of power-allocation strategies with an integer-bit constraint.[4] It is clear that if exact waterfilling is used, i.e., when $S_k + \sigma^2/\nu_k$ is a constant whenever $S_k > 0$, the gap reduces to zero. Therefore, the cost of not doing waterfilling is in the decrease of the denominator in the second term. The simplicity of the above expression makes it quite useful in deriving new results, as it shall soon be seen.

D. Constant Power Adaptation

We now turn our attention to the particular class of constant-power adaptation algorithms. As mentioned before, $\log(1 + \text{SNR})$ is more sensitive to SNR when SNR is low. So, it makes sense that the critical task in waterfilling should be to ensure that low SNR subchannels are allocated the correct amount of power. In particular, those subchannels that would be allocated zero power in exact waterfilling should not receive a positive power in an approximate waterfilling algorithm, for otherwise, the power is mostly wasted. This intuition allowed Chow [1] to observe that a constant-power allocation strategy, where the transmitter allocates zero power to subchannels that would receiver zero power in exact waterfilling, but allocates *constant* power in subchannels that would receive positive power in exact waterfilling, is often close to the optimal. In this section, this intuition will be made precise using the gap bound derived before.

Consider the following class of constant-power allocation strategies, where beyond a cutoff point ν_0, all subchannels are allocated the same power

$$S_k = \begin{cases} S_0, & \text{if } \nu_k \geq \nu_0 \\ 0, & \text{if } \nu_k < \nu_0. \end{cases} \tag{21}$$

Here, the subchannels are assumed to be ordered so that $\nu_k \geq \nu_l$ whenever $k \leq l$. If the same cutoff point ν_0 is used as in exact waterfilling, we have

$$S_0 + \min_{\nu_k \geq \nu_0} \left(\frac{\sigma^2}{\nu_k}\right) \leq \frac{\sigma^2}{\nu_0}. \tag{22}$$

[3] When the power constraint is not satisfied with equality, $\bar{S}$ must be used in the second sum in (20) instead of $\sum p_k S_k$.

[4] Equation (20) can be used to show that integer-bit restriction costs at most $1/\ln 2$ b/s/Hz by noticing that an integer bit-allocation algorithm essentially doubles $S_k + \sigma^2/\nu_k$ in allocating each additional bit. Unfortunately, this bound is rather loose.

The inequality holds because in the transmission band (i.e., when $\nu_k \geq \nu_0$), the constant-power allocation is a suboptimal strategy, therefore, the minimal sum of power and (normalized) noise is less than the water level, which is σ^2/ν_0. Equation (22) ensures that

$$\min_k \left\{ S_k + \frac{\sigma^2}{\nu_k} \right\} = S_0 + \min_k \left\{ \frac{\sigma^2}{\nu_k} \right\}. \tag{23}$$

In this case, (20) becomes

$$\begin{aligned} \ln 2 \cdot \Gamma &= \sum_{k=1}^{m^*} p_k \left(\frac{S_0}{S_0 + \min_j \left\{ \frac{\sigma^2}{\nu_j} \right\}} - \frac{S_0}{S_0 + \frac{\sigma^2}{\nu_k}} \right) \\ &= \sum_{k=1}^{m^*} p_k \frac{S_0 \left(\frac{\sigma^2}{\nu_k} - \min_j \left\{ \frac{\sigma^2}{\nu_j} \right\} \right)}{\left(S_0 + \frac{\sigma^2}{\nu_k} \right) \left(S_0 + \min_j \left\{ \frac{\sigma^2}{\nu_j} \right\} \right)} \\ &\leq \sum_{k=1}^{m^*} p_k \left(\frac{\frac{\sigma^2}{\nu_k}}{S_0 + \frac{\sigma^2}{\nu_k}} \right) \end{aligned} \tag{24}$$

where m^* denotes the number of channel states with positive power allocation. Note that an immediate constant bound can be obtained by replacing $(\sigma^2/\nu_k)/(S_0 + \sigma^2/\nu_k)$ with 1. In this case, $\Gamma \leq 1/\ln 2 = 1.44$ b/s/Hz is an upper bound to the maximum capacity loss for constant-power allocation algorithms. But this bound is usually too loose to be of practical interest. Instead, we can simplify the notation using the fact that the number of bits allocated in each subchannel is given by $\log(1 + S_0\nu_k/\sigma^2)$. In this case, Γ can be written in a particularly simple form

$$\Gamma \leq \frac{1}{\ln 2} \sum_{k=1}^{m^*} p_k 2^{-b_k} \tag{25}$$

where b_k is the number of bits allocated in each subchannel. Note that b_k are not restricted to integer values in the above bound. Also note that the crucial assumption for the bound to hold is $\min_k\{S_k + \sigma^2/\nu_k\} = S_0 + \min_k\{\sigma^2/\nu_k\}$. Having the same cutoff point as in exact waterfilling is sufficient, but not necessary. Thus, we have the following theorem.

Theorem 2: For a constant-power allocation strategy of the form (21) that satisfies the power constraint with equality, if $\min_k\{S_k + \sigma^2/\nu_k\} = S_0 + \min_k\{\sigma^2/\nu_k\}$, then it is at most $(\sum_{k=1}^{m^*} p_k \cdot 2^{-b_k}/\ln 2)$ b/s/Hz away from the waterfilling optimal, where the sum is over all m^* subchannels that are allocated S_0 amount of power, and b_k is the number of bits allocated in subchannel k, i.e., $b_k = \log(1 + S_0\nu_k/\sigma^2)$.

Fig. 1 illustrates the theorem graphically. As long as the level A in Fig. 1 is lower than the level B, the achievable rate is bounded by (25). Note that subchannels with low SNRs (and hence, low bit allocations) are precisely those contributing most to the bound, thus confirming the intuition that low SNR subchannels are the most sensitive to power misallocation.

III. Low-Complexity Adaptation

The crucial condition in *Theorem 2* is $\min_k\{S_k + \sigma^2/\nu_k\} = S_0 + \min_k\{\sigma^2/\nu_k\}$. This condition states that the bound is valid only if not too few subchannels are used. The condition is trivially satisfied, for example, by putting equal power in all subchannels. In that case, 2^{-b_k} will be nearly 1 for many subchannels, and the duality gap becomes large (although still bounded by the constant 1.44 b/s/Hz.) It is therefore of interest to use as few subchannels as possible without violating the condition, so as to simultaneously make the number of terms in the summation small, and make each individual term small (since fewer subchannels implies larger S_0, which, in turn, implies smaller 2^{-b_k}). This suggests that a simple power-allocation strategy which sets the cutoff point to be the largest m^* that satisfies $S_0 + \sigma^2/\nu_1 \leq \sigma^2/\nu_{m^*+1}$ is close to the optimal. Graphically, an algorithm that tries to find the smallest m^* so that level A is less than level B has the smallest duality gap. This fact is used to devise the following algorithm.

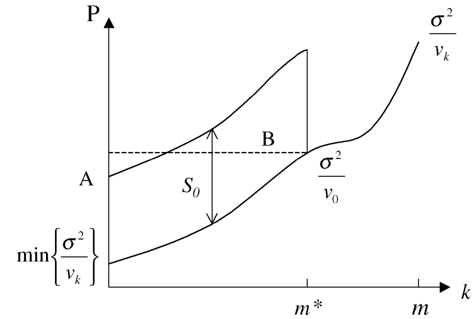

Fig. 1. Constant-power waterfilling.

Algorithm 1: Assume that the channel gain ν_k's are ordered so that $\nu_1 \geq \nu_2 \cdots \geq \nu_m$. Let ν_0 be the cutoff point, so that a constant power S_0 is allocated for all $\nu_k \geq \nu_0$. Let m^* be the largest k, such that $\nu_k \geq \nu_0$. The following steps find the m^* with the smallest duality gap.

1) Set $m^* = m - 1$.
2) Compute $S_0 = \bar{S}/\sum_{k=1}^{m^*} p_k$.
3) If $\sigma^2/\nu_{m^*+1} \geq S_0 + \sigma^2/\nu_1$, set $m^* = m^* - 1$, and repeat step 2. Otherwise, set $m^* = m^* + 1$ and go to the next step.
4) Compute $b_k = \log(1 + S_0\nu_k/\sigma^2)$ for $k = 1, \ldots, m^*$. Set $R = \sum_{k=1}^{m^*} p_k b_k$.

Theorem 3: The low-complexity constant-power algorithm produces a rate R that is at most $(\sum_{k=1}^{m^*} p_k \cdot 2^{-b_k}/\ln 2)$ b/s/Hz from capacity.

The proof of the theorem follows directly from the development in Section II. Two properties of this algorithm make it attractive. First, unlike most previous low-complexity bit-loading methods (e.g., [1]), where the boundary point is found by finding the cutoff point that gives the highest data rate, this algorithm finds the optimal cutoff point without actually computing the data rate achieved in each step, and is therefore free of logarithmic operations. The most expensive operation in this algorithm is the single division in each step, thus making its complexity very low. Second, this algorithm has a provable worst-case performance bound, as given by *Theorem 3*. Note that the algorithm is designed to minimize the duality-gap bound, which is not the same as maximizing the actual data rate. There are examples where the two criteria give quite different cutoff points.

The presentation of the above algorithm has been simplified using a linear search. In practice, a binary search for m^* can easily be implemented, thus further increasing its efficiency. However, the asymptotic algorithmic efficiency is bounded by the sorting of the subchannels, which is $O(m \log(m))$.

The simplicity of the algorithm also points to the possibility of easy adaptive implementation when the channel distribution is not known in advance. Since the power allocation is parameterized by the single cutoff point ν_0, an adaptive algorithm can set an initial cutoff point, then adjust ν_0 based on the resulting power consumption. Hence, it is possible to approximately waterfill without estimating the exact channel distribution in advance, and therefore, without the sorting operation.

IV. Rayleigh Channel

The bound developed previously can be explicitly computed if channel-fading statistics are known. In particular, for a Rayleigh fading channel, it can be shown that the constant-power adaptation strategy is only a small fraction of one bit away from capacity.

In a wireless channel where a large number of scatterers contribute to the signal at the receiver, application of the central limit theorem leads to a (zero-mean) complex Gaussian model for the channel response. The envelope of the channel response at a given time instant has a Rayleigh distribution, i.e., the square magnitude of the channel gain is exponentially distributed: $p_\nu(\nu) = 1/\Omega \cdot e^{-\nu/\Omega}$, where Ω, the average channel gain, parameterizes all Rayleigh distributions.

Fixing Ω, the constant-power control strategy is determined by the average power constraint, or alternatively, by the cutoff value ν_0. The low-complexity power-allocation algorithm states that the constant power allocated in each state S_0 should be such that

$$S_0 + \min\left\{\frac{\sigma^2}{\nu}\right\} = \frac{\sigma^2}{\nu_0}. \qquad (26)$$

The Rayleigh distribution has a nonzero probability for arbitrarily large amplitudes of ν, so the above reduces to $S_0 = \sigma^2/\nu_0$. Interestingly, the constant-power allocation algorithm allocates a constant power S_0 to all subchannels that can support at least 1 b/s/Hz with S_0.

Now, using the gap bound (24), the spectral efficiency for an optimal constant-power allocation with cutoff ν_0 is bounded within the following constant from capacity:

$$\Gamma(\nu_0) = \frac{1}{\ln 2}\int_{\nu_0}^{-\infty}\left(\frac{\frac{\sigma^2}{\nu}}{\frac{\sigma^2}{\nu_0}+\frac{\sigma^2}{\nu}}\right)\cdot\frac{1}{\Omega}\cdot e^{-\nu/\Omega}d\nu. \qquad (27)$$

By a change of variable $t = \nu/\Omega$ (and also $t_0 = \nu_0/\Omega$), define

$$f(t_0) = \int_{t_0}^{\infty}\frac{t_0 e^{-t}}{t+t_0}dt \qquad (28)$$

the duality gap can be expressed as

$$\Gamma(\nu_0) = \frac{1}{\ln 2}\cdot f\left(\frac{\nu_0}{\Omega}\right). \qquad (29)$$

The authors are not aware of a closed-form expression for the integral (28). Numerical evaluation reveals that it has a single maximum occurring at about $t_0 = 0.39$, and the value of the maximum is about 0.1840. The duality gap is largest when the power constraint is such that the cutoff point $\nu_0 = 0.39\Omega$. In this worst case, the average data rate is 1.3631 b/s/Hz, and the duality gap is $0.1840/\ln 2 \leq 0.266$ b/s/Hz away from capacity. The following theorem summarizes the result.

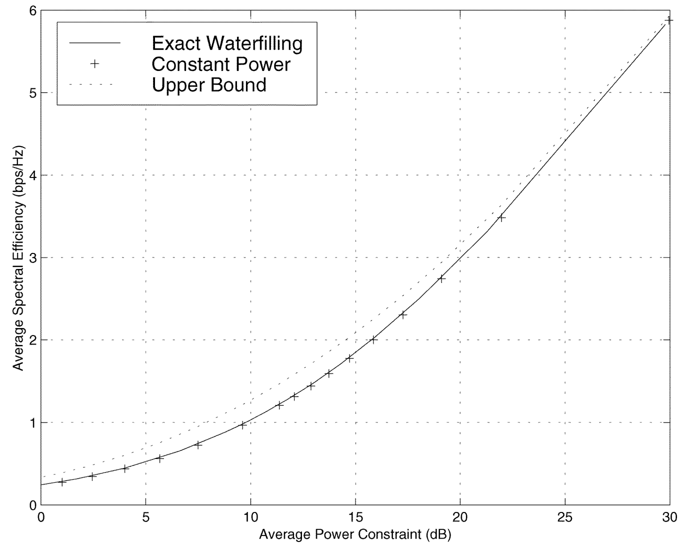

Fig. 2. Spectral efficiencies of exact waterfilling and constant-power allocation on a Rayleigh channel.

Theorem 4: For a flat i.i.d. Rayleigh fading channel with perfect side information at the transmitter and the receiver, assuming infinite granularity on the channel state, a constant-power adaptation method should allocate S_0 to all subchannels that could support at least one bit, where S_0 is determined from the power constraint. In this case, the resulting spectral efficiency is at most 0.266 b/s/Hz away from capacity.

V. Simulation Results

A. Wireless Rayleigh Channels

The performance of the power-adaptation algorithm is simulated on a Rayleigh channel. The low-complexity constant-power adaptation is used. The power transmission level is determined by the instantaneous channel gain. The average channel gain (Ω) is chosen to be -10 dB. In Fig. 2, the average spectral efficiencies of the exact waterfilling and the low-complexity constant-power allocation are plotted against the average power constraint, together with the duality-gap bound. The average power constraint shown in the figure is the normalized value with noise power spectral density level set to $\sigma^2 = 0$ dB. It is seen from Fig. 2 that the true waterfilling and the low-complexity algorithm give indistinguishable results. The capacity upper bound for the low-complexity power allocation as computed by (29) is also plotted in Fig. 2. As it can be seen, the true capacity lies well within the upper bound. However, while the capacity bound may be loose, the constant-power allocation method designed using the performance bound nevertheless works very well.

B. Digital Subscriber Line Channels

The performance of the proposed low-complexity constant-power waterfilling algorithm is also simulated for the very-high-speed digital subscriber line (VDSL) application.

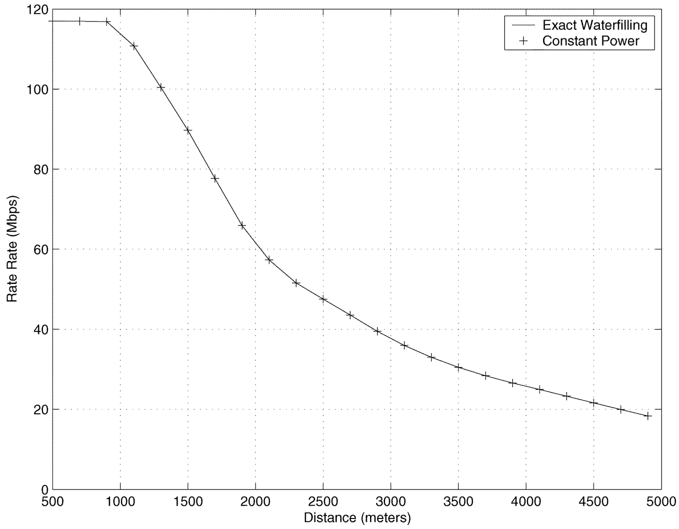

Fig. 3. Achievable rates for VDSL lines at various line lengths.

VDSL 26-gauge transmission lines are simulated at various distances. VDSL twisted-pair cables are severe ISI channels. VDSL transmission can potentially use 4069 tones, with each tone occupying 4.3125 kHz bandwidth. A mix of alien crosstalk signals are also included. A moderate combined SNR gap and margin of 12 dB is assumed. The data rates achievable with the low-complexity constant-power waterfilling algorithm and that with true waterfilling are plotted as a function of line length in Fig. 3. At distances shown in Fig. 3, between 700–3000 tones are typically used. Clearly, constant-power waterfilling has negligible rate loss, compared with true waterfilling.

VI. Conclusions

In this letter, we investigate low-complexity power-adaptation algorithms for both the wireless fading channel and the wireline ISI channel. Our main contribution is a rigorous performance bound for the constant-power waterfilling algorithm based on the duality-gap analysis in convex optimization. Furthermore, the duality-gap analysis allows a very-low-complexity constant-power adaptation method to be developed. The low-complexity algorithm has the desirable properties of having a provable worst-case performance and being logarithm-free. The performance bound is applied to Rayleigh fading channels. It is shown that constant-power adaptive modulation is at most 0.266 b/s/Hz away from capacity. Simulation results suggest that the actual gap is even smaller.

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