LOW-DENSITY PARITY-CHECK CODES FOR DISCRETIZED MIN-SUM DECODING

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ABSTRACT

The performance of low-density parity-check (LDPC) codes transmitted over a memoryless binary-input continuous output additive white Gaussian noise (AWGN) channel and decoded with quantized min-sum decoding is strongly influenced by the decoder's quantization scheme. This paper presents an efficient algorithm that determines the best uniform scalar quantizer for a particular code. To maximize performance, it is necessary to determine degree distributions that best match the characteristics of the quantized min-sum decoder. Toward this end, an iterative optimization framework that jointly optimizes the degree distributions and the quantizer is presented.

1. INTRODUCTION

The problem of designing high-performance irregular low-density parity-check (LDPC) codes under different decoding algorithms and channel models has been studied extensively in the literature (e.g. [1, 2, 3]). Usually, the design objective is to find a code degree distribution that maximizes the decoding threshold for a given rate or, equivalently, to find a degree distribution that maximizes the code rate for a given channel condition (decoding threshold).

The outstanding performance of LDPC codes can be attributed to several factors, including the powerful and efficient probabilistic decoding schemes that they admit. More specifically, since an LDPC code can be represented by a sparse bipartite graph, decoding is performed by passing messages over the edges between check nodes and variable nodes. These messages represent estimates of the variable node values, and may also provide an associated reliability of the estimate. The most powerful message passing algorithm is the sum-product algorithm [4], which computes the posterior distribution of each codeword bit, under the assumption that the graph is cycle free.

A significant benefit associated with LDPC codes is that a wide variety of decoding algorithms exist, each providing a particular performance-complexity tradeoff. The traditional design methodology is to determine threshold-optimized degree distributions under the assumption of a sum-product decoder, but to decode the resulting code with a less complex decoder. A variety of approaches exist that simplify the implementation of the decoder, while attempting to closely approximate the sum-product algorithm. Toward this end, several methods have been suggested to approximate the input-output behaviour of the check node message update equation [5, 6]. These approaches typically use look-up tables and piecewise approximations to the update equation. Furthermore, the messages passed are typically quantized, further simplifying the implementation of the decoder. An alternative approach to the design of low-complexity decoders is suggested in [7, 8], and permits the use of non-uniformly quantized messages, due to a clever implementation of message maps at the variable and check nodes. However, this is an ad-hoc method that is not particularly transparent, and has only been applied to regular LDPC codes. In the following, we focus on the discretized min-sum decoder, due to the fact that it is particularly easy to implement in hardware, and we present code optimization methods for min-sum decoding.

Several issues arise in the design of a practical LDPC coding system that implements discretized min-sum decoding. Foremost among these is the choice of degree distributions. The traditional view is that, since we are attempting to approximate sum-product decoding, we should use threshold-optimized distributions for the sum-product algorithm, such as those produced by `LdpcOpt` [9]. However, it is not clear that these degree distributions remain optimal when the decoder implementation changes. Additionally, the issue of message quantization is not addressed by traditional threshold optimization tools, since the messages are assumed to exist in a continuous domain. In practice, messages passed within the decoder are represented with some small number (say 3 to 6) of bits. The designer must judiciously select the range of values that these quantized messages represent. In the following, we show that significant performance gains occur when one jointly designs the quantizer and degree distributions.

2. BACKGROUND

Following the notations of [1] we define an ensemble of irregular LDPC codes by its variable-degree distribution $\lambda(x) = \sum \lambda_i x^{i-1}$ and its check-degree distribution $\rho(x) = \sum \rho_i x^{i-1}$, where λ_i denotes the fraction of edges incident on variable nodes of degree i and ρ_j denotes the fraction of edges incident on check nodes of degree j. If all the parity constraints are linearly independent, the rate of an irregular LDPC code is related to its degree distribution by

$$R(\lambda, \rho) = 1 - \frac{\sum_i \frac{\rho_i}{i}}{\sum_i \frac{\lambda_i}{i}}. \tag{1}$$

If, due to the random construction of the code, some of the parity check constraints are linearly dependent, the actual rate of the code will be slightly higher.

2.1. Density evolution

There are many decoding algorithms for LDPC codes. If the decoding algorithm and the channel satisfy some symmetry properties [7], the performance of a given code can be studied by density evolution. The inputs to the density evolution algorithm [7] are the probability density function (pdf) of channel log-likelihood ratio (LLR) messages[1], and the pdf of the extrinsic LLR messages from the previous iteration. The output is the pdf of the extrinsic LLR messages at the current iteration. This density will be used as input for finding the message density in the next iteration. The negative tail of the LLR density is the message error rate. Successful decoding implies that the tail vanishes as the number of iteration tends to infinity.

2.2. EXIT charts

An EXIT chart is a design tool that graphically presents the results of density evolution. In the following, we use EXIT charts based on message error rate [10]. EXIT chart analysis based on message error rate allows one to express the EXIT chart of an irregular code as a linear combination of EXIT charts of regular codes (i.e., elementary EXIT charts), which is central to the formulation of the optimization problems herein.

2.3. Decoding algorithms

LDPC codes admit a wide variety of decoding algorithms, each with a particular performance-complexity tradeoff. The most powerful of these is the sum-product algorithm, which computes the posterior distribution of the individual codeword bits, under the assumption that the underlying graph is cycle free. The sum-product algorithm operates by passing messages through the code's graph. Under the usual representation of messages as log likelihood ratios (LLRs), the update equation at a variable node v is

$$m_{v\to c} = m_0 + \sum_{h\in n(v)-c} m_{h\to v},$$

and the update equation at a check node c is

$$m_{c\to v} = 2\tanh^{-1}\left(\prod_{y\in n(c)-v} \tanh\left(\frac{m_{y\to v}}{2}\right)\right),$$

where m_0 is the intrinsic message for variable node v. Immediately, it is apparent that the check node operation will be difficult to implement in practice. However, noting that the $\tanh^{-1}$ of the product of $\tanh$'s is approximated by the minimum of the absolute value of the messages times the product of their signs, the check node update rule can be simplified to

$$m_{c\to v} = \min_{y\in n(c)-v} |m_{y\to v}| \cdot \prod_{y\in n(c)-v} \mathrm{sign}(m_{y\to v}). \qquad (2)$$

The decoding algorithm that replaces the sum-product update equation with (2), while maintaining the same variable update equation, is known as the min-sum algorithm. In the following, we focus exclusively on the min-sum algorithm, due to the relative simplicity of its implementation.

In order to further simplify the implementation of the min-sum decoder, a discretized implementation is required. Following the framework of discretized density evolution for the sum-product algorithm [3], we consider the code design problem for a min-sum decoder whose messages are quantized into n-bit messages.

[1]Under the assumption that the all-zero codeword is transmitted.

3. QUANTIZING DECODER MESSAGES

In order to facilitate a practical implementation, we consider a decoder that implements a discretized min-sum algorithm with uniformly quantized messages. We study the design of codes for transmission over a binary input AWGN channel (with noise variance σ) using binary antipodal signalling ± 1. Additionally, our quantization scheme is symmetrical about the origin, and the message associated with an LLR value of zero is required in the decoder. Therefore, for an n-bit decoder, we have $2^n - 1$ distinct quantization levels. Given that our quantization scheme is uniform, an n-bit quantizer is fully specified by the channel output value associated with the largest positive quantization level.

In the case of regular LDPC codes, a quantization scheme that maximizes the mutual information between the binary channel inputs and the quantized channel output, results in an effective decoder. Such a scheme is satisfying in an information theoretic sense, as it maximizes the capacity of the resulting discrete memoryless channel. However, for $n \geq 3$, the mutual information is relatively insensitive to the quantization scheme. Put another way, there exist many reasonable n-bit quantizers that result in similar mutual informations. Table 1 presents the threshold of a (3,6) regular LDPC code under 4-bit min-sum decoding, as the maximum quantized channel output value varies. Also shown in Table 1 is the mutual information between the binary channel inputs and the quantized channel output. Note that the threshold is maximized for a maximum quantization level at 1.25, while the mutual information is maximized at 1.5. Nevertheless, the mutual information measure introduces minimal losses when used to design the uniform quantizer for a (3,6) regular LDPC code.

Max. Channel Output	Mutual Info.	Threshold ($\sigma^\star$)
0.5	0.5805	0.7911
0.75	0.5983	0.8135
1	0.6066	0.8247
1.25	0.6100	0.8275
1.5	0.6110	0.8252
1.75	0.6109	0.8209
2	0.6103	0.8165

Table 1. Mutual information and threshold of (3,6) regular LDPC code, under 4 bit uniformly quantized min-sum decoding with the specified maximum channel output value. Note that the mutual informations are evaluated at $\sigma = 0.8275$.

While a quantization scheme that maximizes mutual information yields an effective decoder for regular codes, in the irregular case this is no longer true. Recall the intuitive justification for the superior performance of irregular codes. For a given check degree distribution, low degree variable nodes contribute significantly to the code's rate but decrease its threshold, while high degree variable nodes contribute negligibly to the rate but greatly increase the threshold. Therefore, an intelligent choice of variable degree irregularity allows one to achieve a desired code rate while maximizing the threshold. However, note that this implicitly assumes that the decoder is capable of sending messages that represent a wide range of reliabilities (LLR values). In particular, if the range of message reliabilities is not sufficiently large, the increase in threshold associated with high degree variable nodes vanishes.

For an irregular code and a fixed number of quantization bits n, the appropriate quantizer balances the requirements of maintaining high mutual information, while also supplying a sufficiently wide range of message reliabilities. Therefore, to maximize the effective threshold, the quantization scheme must account for the dynamics of the code/decoder pair. Due to the strong relationship between a code's threshold and its performance, it is apparent that the design of the quantizer must depend on the particular code/decoder pair. Therefore, we propose an iterative design process to determine the quantizer and degree distributions.

4. DESIGN METHODOLOGY

We focus on the design of threshold-optimized codes. It is important to note that even though the threshold relates to the asymptotic properties of the code, the benefits of threshold-optimized degree distributions remain apparent at short block-lengths [11]. In the following, we design degree distributions that maximize the decoding threshold under n-bit min-sum decoding. We assume throughout that the check degree distribution is concentrated on a single degree, since it simplifies the implementation of a practical decoder, and because such a restriction has a minimal effect on the achievable performance of LDPC codes [12].

For a particular discretized n-bit min-sum decoder, with an associated quantization scheme Q_c, and a particular code with degree distributions $\lambda(x)$ and $\rho(x)$, it is straightforward to evaluate the threshold of the system by discretized min-sum density evolution. For n in our range of interest, say $3 \leq n \leq 7$, this is not particularly computationally intensive, allowing one to repeat the same procedure for a variety of different Q_c, while maintaining constant values of n, $\lambda(x)$ and $\rho(x)$. Furthermore, given that Q_c is entirely specified by its associated maximum channel output value, a search over a suitable range of such values yields a clear picture of the effects of the quantizer.

Starting with some concentrated check distribution and a reasonable n-bit quantizer, we wish to determine the $\lambda(x)$ that maximizes the threshold while achieving a target rate. Our approach is based on EXIT charts that track the message error rate, but whose elementary EXIT curves are computed by performing n-bit min-sum density evolution. Since the elementary EXIT curves are themselves a function of $\lambda(x)$, an iterative linear programming formulation is required to solve the threshold maximization program.

The search for the best (Q_c,$\lambda(x)$,$\rho(x)$=x^{d_c-1}) triple is simplified by the following empirical observations. For fixed values of d_c and n, the maximum achievable threshold is a unimodal function of the maximum quantized channel output value. Similarly, for a fixed value of n, the maximum achievable threshold (over all uniform n-bit quantizers at each d_c) is a unimodal function of the check degree. Therefore, one can perform an iterative search over the design parameters, thus determining the threshold optimal (Q_c,$\lambda(x)$,$\rho(x)$) triple of the desired rate. More specifically, beginning with a reasonable value for d_c and a fixed n, one must determine the threshold optimal $\lambda(x)$ over a range of maximum quantized channel output values, while exploiting its unimodal relationship with the threshold. This process is repeated over several values of d_c, where the d_c are selected to exploit its unimodal relationship with the threshold.

5. NUMERICAL RESULTS

We present our results for the case of a rate 1/2 code with a 4-bit min-sum decoder, with a constraint that the maximum allowable variable degree be less than or equal to 15. Using the methods of Section 4, the best code has $\lambda(x) = 0.213387x + 0.471904x^2 + 0.023227x^9 + 0.291482x^{14}$, $\rho(x) = x^6$, and the highest quantization level corresponds to a channel output value of 3.75. Recall from Table 1 that the mutual information measure for a code of rate one-half would suggest a channel output value of 1.5, but this decreases the effective threshold of the irregular code drastically.

In order to justify the need to obtain degree distributions specifically designed for a quantized min-sum algorithm, we consider the performance of a threshold-optimized, rate 1/2 code, designed for the sum-product algorithm, with the same constraint on the maximum variable degree [2]. It is important to note that even when we are given a particular set of degree distributions, it remains necessary to use the techniques of Section 4 to determine the best n-bit uniform quantizer. For the present code, the best 4-bit quantizer has a maximum level which corresponds to a channel output value of 4.0.

The simulation results of the preceding codes are presented in Fig. 1, which also includes the results of a (3,6) regular LDPC code. The results are those for a 4 bit uniformly quantized min-sum decoder, with a maximum quantized channel output value optimized for each case. Interestingly, the regular code's performance is far superior to that of the code optimized for the sum-product algorithm, and is only 0.14dB worse than the code designed specifically for a quantized min-sum decoder. This emphasizes the relative performance loss incurred when a uniformly quantized discretized decoder is used to decode an irregular code. However, as the number of quantization bits grows, the merits of irregular code constructions become more apparent, since the decoder more closely approximates the true min-sum algorithm. Furthermore, when n is small, a non-uniformly quantized decoder would allow one to more fully exploit the advantages of irregularity, at the expense of a more difficult decoder implementation. However, non-uniform quantization schemes are not considered in this paper.

6. CONCLUDING REMARKS

We have presented an efficient optimization procedure to determine the (Q_c,$\lambda(x)$,$\rho(x)$) that maximize decoding threshold under n-bit uniformly quantized min-sum decoding. The presented results imply several important facts. First, for an irregular code and some fixed number of quantization bits n, the decoder's quantization scheme must balance the requirements to maintain high mutual information, while providing a sufficiently wide range of message reliabilities. Put another way, high degree variable nodes increase the threshold only when the quantization scheme allows messages with high reliabilities relative to the noisiest channel messages.

Secondly, a threshold optimized degree distribution for a decoder that implements the sum-product algorithm with infinite precision, is no longer optimal when decoded with a quantized min-sum decoder. Recall, that due to the minimal extrinsic information that degree two variable nodes receive, they are the most difficult to decode. Furthermore, in a quantized decoding scheme, the reliability of the strongest messages is necessarily capped. For an n-bit min-sum decoder, degree two variable nodes are therefore even more harmful, relative to their effects in the presence of an infinite precision decoder. Hence, to maximize performance, it is necessary to design degree distributions specifically for the quantized decoder.

7. REFERENCES

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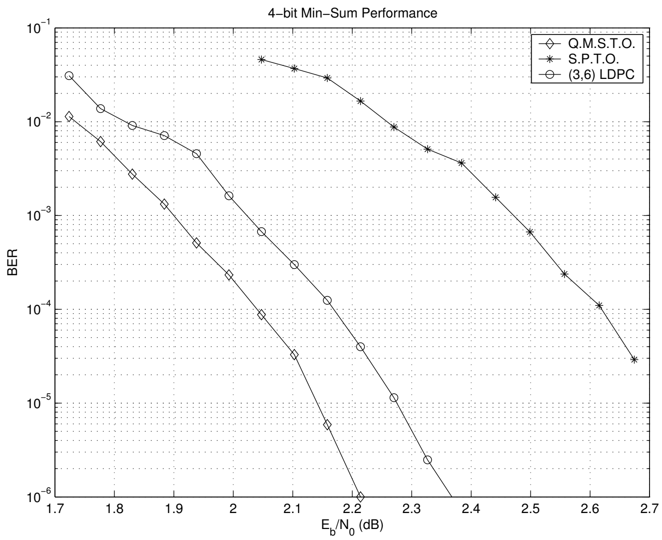

Fig. 1. Comparison between the performance of two irregular codes and a (3,6) regular code for codes of length 4096 and rate one-half. Each code is decoded for 80 decoding iterations with its optimal 4-bit uniformly quantized min-sum decoder. The Q.M.S.T.O. (quantized min-sum threshold optimized) code is given by $\lambda(x) = 0.213387x + 0.471904x^2 + 0.023227x^9 + 0.291482x^{14}$, $\rho(x) = x^6$, while the S.P.T.O. (sum-product threshold optimized) code is from [2].

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