Low-Complexity Near-Optimal Spectrum Balancing for Digital Subscriber Lines

Raymond Lui and Wei Yu
Electrical and Computer Engineering Department
University of Toronto, Toronto, Canada
email: {rwmlui,weiyu}@comm.utoronto.ca

***Abstract*— This paper investigates the multiuser spectrum optimization problem for digital subscriber lines. We propose an iterative and low-complexity spectrum optimization technique that improves upon the recently proposed optimal spectrum balancing (OSB) algorithm. In the optimal spectrum balancing algorithm, the Lagrange multipliers are used to decouple the constrained optimization problem into a series of per-tone unconstrained optimization problems. However, each per-tone problem still has a computational complexity that is exponential in the number of users. This paper proposes an iterative algorithm for the per-tone optimization problem to further reduce the computational complexity of spectrum balancing. The essential idea resembles that of iterative water-filling. In each step of the algorithm, each individual user iteratively optimizes the joint objective function with a fixed set of Lagrange multipliers. The new algorithm has a computational complexity that is polynomial in the number of users. Simulation results show that the new algorithm has a near-optimal performance.**

I. Introduction

Spectrum management is one of the active areas of research in digital subscriber lines (DSL). In a DSL system, multiple copper pairs are bundled together. The electromagnetic coupling between the copper pairs causes crosstalk interference, which has long been identified as the primary source of line impairment in DSL deployments. Current DSL systems use a *static spectrum management* (SSM) approach where a fixed transmit power spectral density is applied for each line regardless of the loop topology or user service requirements. The performance projection under SSM is based on the levels of worst-case crosstalk interference.

Future generation of DSL services are envisioned to utilize *dynamic spectrum management* (DSM). DSM gives each line an ability to adapt to its loop environment and service requirements, and it has the potential to drastically improve the achievable rates and service ranges of current DSL systems. DSM is an active area of research both within the research community and within the standardization bodies [1] [2].

The crosstalk problem is most severe when the channel transfer functions are heavily unbalanced. This is the situation in downstream ADSL systems where a remote optical network unit (ONU) is deployed (ONU is usually located much closer to the customer premise modems served from the central office) and in upstream VDSL systems where some of the upstream transmitters may be much closer to the central office than others. Power back-off methods [2] are traditionally applied in these cases.

Digital Multi-tone (DMT) is the modulation format used in almost all DSL standards. In a DMT system, the frequency spectrum is divided into many parallel subchannels. Power and number of bits can be assigned in each subchannel individually. This gives DSL applications great flexibility in performing spectrum optimization, and spectrum shaping can be done in a tone-by-tone basis. However, because of this flexibility, the number of variables in a spectrum optimization problem is the product of the number of users K and the number of frequency tones N. Further, the objective function in the spectrum optimization problem is non-convex. Thus, a brute-force search-based optimization has a computational complexity that is exponential in KN, which is intractable.

Iterative water-filling [3] is one of the first low-complexity multiuser spectrum optimization techniques that takes advantage of the ability for DSL modems to perform spectral shaping. In this algorithm, each user iteratively maximizes its own achievable rate by performing a single-user water-filling with the crosstalk interference from all other users treated as noise. Since the single-user water-filling process is a convex optimization process and has a complexity of order $O(N\log(N))$, each iteration of the iterative water-filling process has an $O(KN\log(N))$ complexity. However, the iterative water-filling process does not seek to find the global optimum for the entire binder. Instead, each user participates in a non-cooperative game, and the convergence point of the iterative water-filling process corresponds to a competitive equilibrium. Although not optimum, the iterative water-filling algorithm has been shown to significantly outperform SSM schemes.

Recently, an optimal spectrum balancing (OSB) algorithm is proposed in [4] which finds the true global optimal solution to the spectrum optimization problem. The OSB algorithm transforms the spectrum optimization problem into the dual domain by forming the Lagrangian dual of the primal optimization. As further re-interpreted and refined in [5], the class of spectrum optimization problems for DSL has the special property that the primal and the dual optimization problems yield the same solution even when the primal problem is non-convex. As the dual problem has a much lower dimension, the computational complexity of solving the dual problem is much lower. It

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can be shown that the OSB algorithm has a computational complexity that is linear in the number of frequency tones N. As illustrated in [4], the OSB algorithm can provide a significant performance improvement as compared to iterative water-filling.

However, the computational complexity of the OSB algorithm, although linear in N, is still exponential in the number of users K. Implementation experience shows that the complexity of the OSB algorithm becomes unmanageable when the number of users in the binder is larger than two.

The main objective of this paper is to find an appropriate middle ground between iterative water-filling and optimal spectrum balancing. Our goal is to take advantage of both the dual formulation of the optimal spectrum balancing algorithm and the competitive (thus low-complexity) nature of iterative water-filling. Toward this end, this paper proposes an iterative spectrum balancing technique that achieves almost all the gain of optimal spectrum balancing while having a computational complexity that is comparable to iterative water-filling.

The computational methods proposed in this paper have a wider implication beyond that of DSL applications. The DSL spectrum balancing problem is identical to the optimal power and bit loading problem for wireless orthogonal frequency-division multiplex (OFDM) systems [6] [7] [8] [9]. A low-complexity near-optimal solution to the DSL problem is likely to have many practical implications for wireless systems as well.

II. System Model

The achievable data rates in a K-user DMT-based DSL system are computed as follows:

$$R_k = \frac{1}{T}\sum_{n=1}^{N} b_k^n \tag{1}$$

where k is the user's index, n is the tone's index, N is the total number of frequency tones, T is the symbol period. b_k^n denotes the achievable bit rate for user k in tone n, and it is computed as

$$b_k^n = \left\lfloor \log_2\left(1 + \frac{S_k^n}{\sigma_k^n + \sum_{i\neq k}\alpha_{i,k}^n S_i^n}\right)\right\rfloor \tag{2}$$

where S_k^n is the transmit power for user k in tone n, σ_k^n is the normalized channel noise for user k in tone n, and $\alpha_{i,k}^n$ is the normalized crosstalk transfer function from the ith user to the kth user in tone n. Channel noise and crosstalk transfer function are normalized by $\Gamma/|H_k^n|^2$, where Γ is the SNR gap for the system and $|H_k^n|^2$ is the kth user's direct channel transfer function in tone n.

The spectrum optimization problem in a multiuser DSL system is formulated as the maximization of a weighted sum rate of all participating users subject to power constraints

$$\max_{S_1^n,\ldots,S_K^n} \sum_{k=1}^{K} w_k R_k \quad \text{s.t. } P_k \le \mathbf{P_k} \quad \forall k \tag{3}$$

where $\mathbf{P_k}$ is the kth user's power constraint. The weights $w_k \ge 0$ are chosen so that $\sum_{k=1}^{K} w_k = 1$. The total power used by user k is computed as

$$P_k = \Delta f \sum_{n=1}^{N} S_k^n. \tag{4}$$

Here Δf is the frequency width of the DMT tones. The weights $\{w_1, w_2, \ldots, w_K\}$ are the priorities put on the users.

In a two-user system, (3) reduces to

$$\max_{S_1^n,S_2^n} wR_1 + (1-w)R_2 \quad \text{s.t. } P_k \le \mathbf{P_k} \quad \forall k \tag{5}$$

By varying w between 0 and 1, an achievable rate region can be generated.

Throughout the paper, the sidelobe effect between adjacent tones is neglected. This is realistic for frame-synchronous DSL systems implementing a zipper like modulation [10] or where sufficient amount of receiver windowing is included.

III. Optimal Spectrum Balancing

The main idea of the optimal spectrum balancing (OSB) [4] is to solve the constrained optimization problem (3) in the dual domain. Instead of an exhaustive search over all possible bit allocations and over all frequency tones, the optimal spectrum balancing algorithm fixes dual variables $(\lambda_1, \cdots, \lambda_K)$, corresponding to each of the K power constraints, and forms the dual objective function:

$$\begin{aligned}
& g(\lambda_1, \cdots, \lambda_K) \\
=\; & \max_{\{S_1^n,\ldots,S_K^n\}_{n=1}^N} \sum_{k=1}^{K} w_k R_k - \sum_{k=1}^{K} \lambda_k (P_k - \mathbf{P_k}) \\
=\; & \max_{\{S_1^n,\ldots,S_K^n\}_{n=1}^N} \sum_{k=1}^{K} \left(w_k \sum_{n=1}^{N} b_k^n \right) - \sum_{k=1}^{K} \lambda_k \left(\sum_{n=1}^{N} S_k^n - \mathbf{P_k} \right) \\
=\; & \left(\sum_{n=1}^{N} \max_{S_1^n,\ldots,S_K^n} \sum_{k=1}^{K} \left(w_k b_k^n - \lambda_k S_k^n \right) \right) + \sum_{k=1}^{K} \lambda_k \mathbf{P_k}.
\end{aligned} \tag{6}$$

Note that the evaluation of $g(\lambda_1, \cdots, \lambda_K)$ is now decoupled in a tone-by-tone basis. Therefore, if B is the maximum number of bits that can be loaded in each tone, the evaluation of $g(\lambda_1, \cdots, \lambda_K)$ requires $O(NB^K)$ operations. Although still exponential in K, this is nevertheless a significant computational saving as compared to the $O(B^{NK})$ operations needed for an exhaustive search over all frequency tones.

The dual computation algorithm has been further refined in [5]. One of the main insights of [5] is that even though the original problem (3) is a non-convex optimization problem, its duality gap is nevertheless zero. This is true under a mild condition called the time-sharing property (which is always satisfied for a DMT-based system). In particular, the minimal value of $g(\lambda_1, \cdots, \lambda_K)$ over all positive λ's is equal to the optimal solution of (3):

$$\min_{\lambda_1,\cdots,\lambda_K} g(\lambda_1, \cdots, \lambda_K) = \max_{S_1^n,\ldots,S_K^n} \sum_{k=1}^{K} w_k R_k. \tag{7}$$

This crucial observation enables a subgradient search method to be implemented for the spectrum optimization problem. In particular, the number of subgradient steps needed to reach a global optimal solution is a polynomial function of the number of dimensions, which is K.

The above statement is meaningful, however, only if the function $g(\lambda_1, \cdots, \lambda_K)$ can be evaluated efficiently. Unfortunately, as seen in (6), the evaluation of $g(\lambda_1, \cdots, \lambda_K)$ is exponential in K. Therefore, the evaluation of $g(\lambda_1, \cdots, \lambda_K)$ is the computational bottleneck of the optimal spectrum balancing algorithm. Computational experience shows that it is impractical to implement the OSB algorithm if the number of users is larger than two.

IV. Iterative Near-Optimal Spectrum Management

The main contribution of this paper is an efficient algorithm that enables $g(\lambda_1, \cdots, \lambda_K)$ to be approximately evaluated with a complexity that is linear in K. Recall that the evaluation of $g(\lambda_1, \cdots, \lambda_K)$ involves the tone-by-tone optimization of the following function:

$$\max_{S_1^n,\ldots,S_K^n} \sum_{k=1}^{K} (w_k b_k^n - \lambda_k S_k^n) \triangleq \max_{S_1^n,\ldots,S_K^n} h(S_1^n, \cdots, S_K^n) \quad (8)$$

Our main idea is that the optimization of $h(S_1^n, \cdots, S_K^n)$ may be done in an iterative water-filling fashion via coordinate descent. For each fixed set of $(\lambda_1, \cdots, \lambda_K)$, our proposed approach first finds the optimal S_1^n while keeping $S_2^n, \cdots, S_K^n$ fixed, then optimizes S_2^n keeping all other S_k^n fixed, then $S_3^n, \cdots, S_K^n$, then $S_1^n, S_2^n, \cdots$, and so on. Note that when optimizing each S_k^n, only a small finite number of power levels (corresponding to a finite number of integer bits) need to be searched. Further, such an iterative process is guaranteed to converge because each iteration strictly increases the objective function. The convergence point must have integer bit values for all users, and it is guaranteed to be at least a local maximum for $h(S_1^n, \cdots, S_K^n)$.

This new approach is inspired by the iterative water-filling algorithm [3]. However, it differs from iterative water-filling in the following two key aspects. First, unlike the iterative water-filling algorithm where each user maximizes its own rate in each step of the iteration, the above algorithm optimizes an objective function that includes the joint rates of all users. Thus, the new algorithm has the potential to reach the social optimum. Second, the power constraint in the iterative water-filling process is handled in an ad-hoc basis, while the new algorithm proposed in this paper dualizes the power constraint in an optimal fashion. The correct values of the dual variables are then used in a sub-gradient search. We called this approach the Iterative Spectrum Balancing (ISB) algorithm.

It should be noted that the ISB algorithm is a sub-optimal algorithm. Nonetheless, as the simulation results in the next section show, its performance is near-optimal as compared to the optimal spectrum balancing method.

The computational complexity of this new iterative approach is significantly lower than that of OSB algorithm proposed in [4]. In the evaluation of $h(S_1^n, \cdots, S_k^n)$, each iteration has a computational complexity that is linear in K. Let T_1 be the number of iterations needed in the evaluation of each $h(S_1^n, \cdots, S_k^n)$. Let T_2 be the number of subgradient updates needed in the optimal spectrum balancing algorithm. The total computational complexity of ISB is $O(T_1 T_2 BNK)$. Computational experience suggests both T_1 and T_2 polynomial functions of K. This is significant as K is usually large in realistic DSL deployment scenarios. Table I summarizes the computational complexity comparison. (Here, T_3 is the number of iterations needed in iterative water-filling.)

Algorithm	**Computational Complexity**
Exhaustive Search	$O(B^{NK})$
Optimal Spectrum Balancing	$O(T_2 N B^K)$
Iterative Spectrum Balancing	$O(T_1 T_2 BNK)$
Iterative Water-Filling	$O(T_3 KN \log(N))$

TABLE I

Computational Complexity Analysis

V. Simulations

A main objective of this paper is to show that the proposed iterative algorithm has a near-optimal performance as compared to optimal spectrum balancing. This is verified with extensive simulations. In the following simulation, all DSL lines are 26-AWG twisted pairs with a background noise level of -140dBm/Hz. Users are assumed to be symbol synchronized so that the sidelobe interference is not in effect. Also, no spectral masks are enforced.

A. 2-User ADSL Downstream

The first set of simulations examines a 2-user ADSL downstream distributive environment with both users having a loop length of 12k feet and with a crosstalk distance of 3k feet. No other disturbers are assumed to exist in the binder. The loop topology is shown in Fig. 1. Such a distributive environment is expected to benefit significantly from dynamic spectrum management because of its highly unbalanced crosstalk channels. The power constraint for each user is set to 20.4dBm as defined in [11]. For fair comparisons, the number of iterations and initial λ settings are identical for OSB and ISB in simulations.

Fig. 2 shows the achievable rate regions of OSB, ISB, iterative water-filling and SSM algorithms. As can be seen in the figure, the rate regions for OSB and ISB are almost identical to each other. Both outperform iterative water-filling significantly. Interestingly, although the achievable rates of OSB and ISB are identical, the optimal spectra obtained from the two algorithms can be different. Fig. 3 shows the downstream spectra obtained from the two algorithms. The main difference between the spectra of OSB and ISB is in the frequency division multiplexing (FDM) region (frequency beyond 380kHz). Both power spectral densities (PSDs) essentially achieve the same rates because there are many equivalent permutations of frequency tones in the FDM region possible.

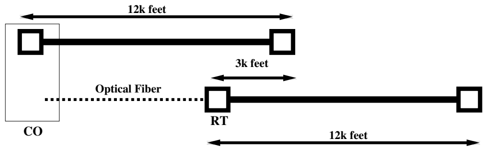

Fig. 1. Loop Topology for Two Downstream ADSL Users

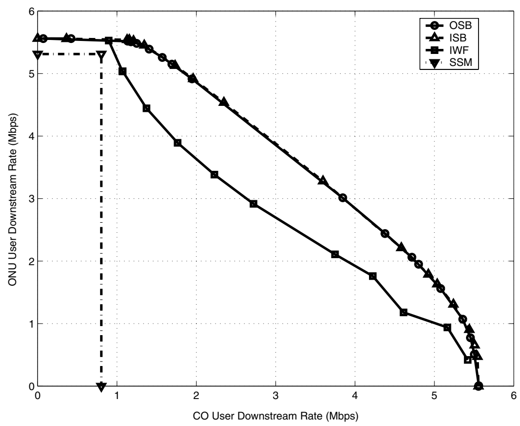

Fig. 2. Rate Region for Two Downstream ADSL Users

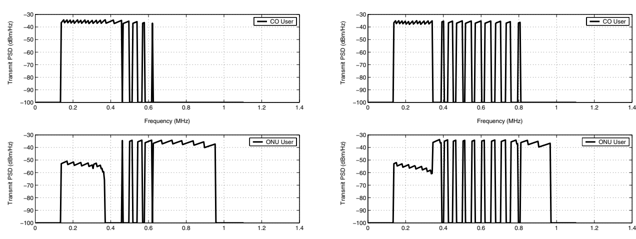

Fig. 3. OSB (left) and ISB (right) power spectral densities for two distributive ADSL users at equal rate. Power spectral densities for both CO-based (top) and RT-based (down) lines are plotted.

B. 5-user VDSL Full Duplex

The current VDSL standard uses a fixed frequency bandplan (i.e. 998) to separate upstream and downstream. This is not optimal because no overlapping of upstream and downstream transmissions is allowed. In this set of simulations, we explore the achievable rate-region and the optimal power allocations with overlap spectra for full duplex transmission in a VDSL environment. The simulation setup consists of 5 users with the same loop length (3k feet long) in the same binder. As the loop characteristics for the five users are identical, this is essentially a two-user scenario between upstream and downstream. Perfect echo cancellation is assumed. The near-end crosstalk (NEXT) is modeled in addition to the far-end crosstalk (FEXT). The downstream transmission has a power constraint of 11.5dBm, and the upstream transmission has a power constraint of 14.5dBm, in accordance to [12].

Fig. 4 shows the achievable rate regions obtained from the OSB and ISB algorithms, As can be seen, the performance of ISB is very close to that of OSB, although ISB is clearly

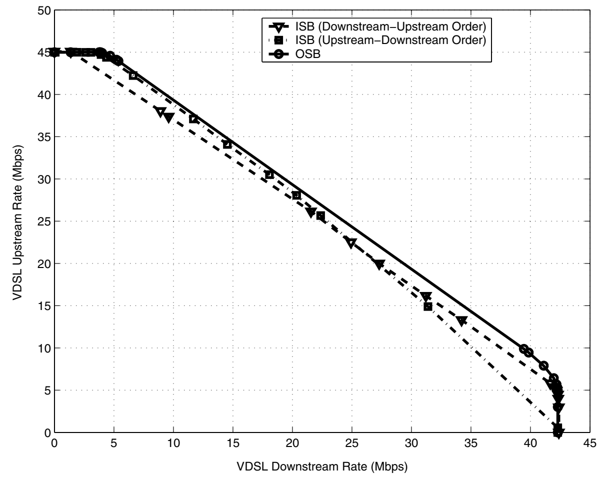

Fig. 4. Rate region for 5-User Full Duplex VDSL

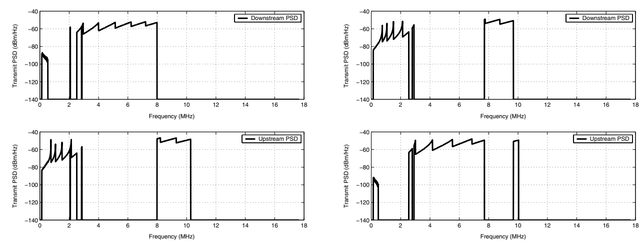

Fig. 5. Downstream (top) and upstream (bottom) power spectral densities for 5-user full duplex VDSL at equal rate. The power spectra depend on the ordering in iteration in ISB. Downstream-upstream order is on the left, and the upstream-downstream order is on the right.

a sub-optimal algorithm. Furthermore, it observed that the solution provided by ISB is not unique. The non-uniqueness of this algorithm is exposed by choosing a different order of users during the iteration procedure in ISB. ISB gives slightly different rate regions for different iteration orders. Interestingly, no particular order has a rate region that is completely superior to the rate regions of all other orders. In addition, as seen in PSD plots ordering affects the power spectral densities as well. Fig. 5 shows the PSD pairs corresponding to the downstream-upstream ordering and the upstream-downstream ordering. As can be seen, a narrow low frequency spectrum is always shared by both directions. In the high frequency range, frequency-division duplex (FDD) separates the upstream and the downstream. FDD is optimal in the high frequency range because of the strong NEXT interference. Interestingly, if the downstream-upstream ordering is used in ISB, the resulting frequency division follows an Up-Down-Up pattern. The situation is completely reversed when the upstream-downstream ordering is used. The upstream-downstream ordering produces a FDD solution that follows Down-Up-Down pattern.

C. 10-user VDSL Full-Duplex

In this final set of simulations, we explore the full duplex transmission of a 10-user VDSL scenario with the topology as in Fig. 6. Again, overlapping spectra is allowed and perfect

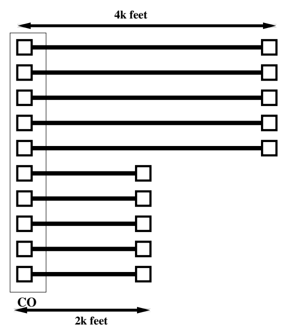

Fig. 6. Loop Topology for 10-User VDSL

echo cancellation is assumed. The OSB algorithm as proposed in [4] is not computationally practical in this case.

Table II compares the performance of the proposed ISB algorithm with that of iterative water-filling (IWF). Iterative water-filling is able to support a minimal data rate of 12.2Mbps, while ISB is able to achieve at least 15.2Mbps. A minimum gain of at least 2.8Mbps is possible.

The power spectral densities obtained from the ISB algorithm are shown in Fig. 7. Interestingly, a small low frequency band is shared by all four transmitters with full duplex operation. In the middle frequency band, frequency-division multiplex (FDD) separates upstream and downstream transmissions of the 2kft and 4kft users. The high frequency band is used exclusively by the 2kft lines. Again, frequency division duplex (FDD) is used there. This type of optimal spectrum usage is non-obvious and is channel and user data rates dependent.

Transmitter	ISB	IWF
4k ft Downstream	15.2Mbps	12.2Mbps
4k ft Upstream	15.2Mbps	12.4Mbps
2k ft Downstream	21.8Mbps	12.7Mbps
2k ft Upstream	25.2Mbps	12.5Mbps

TABLE II

MAXIMUM MINIMUM RATE FOR 10-USER FULL DUPLEX VDSL

VI. CONCLUSION

This paper proposes a low-complexity, iterative and near-optimal spectrum balancing algorithm for digital subscriber line applications. As compared to previous optimal spectrum balancing methods, the new algorithm offers a significant complexity reduction. The complexity is reduced from an exponential complexity in the number of users to a polynomial complexity. The main idea of the algorithm is an iterative evaluation of the Lagrangian function in the optimization step. Simulation results show that the performance of the new algorithm is very close to that of the optimal algorithm. The proposed iterative algorithm is a significant step forward in making optimal spectrum balancing practical.

The iterative algorithm proposed in this paper has a wider implication beyond that of digital subscriber lines. The proposed algorithm can be easily applied to the adaptive bit, power and sub-carrier allocation problems for wireless applications whenever multiuser OFDM is used.

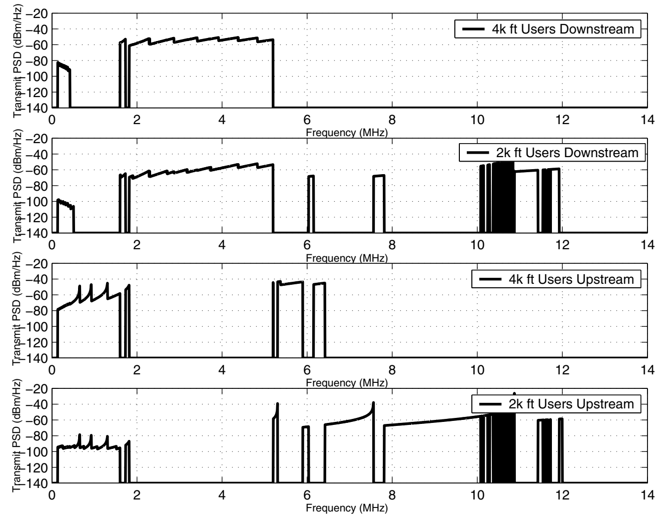

Fig. 7. Power spectral densities for 10-user full duplex VDSL at maximum minimal rate

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