Optimal Power Control in Multiple Access Fading Channels with Multiple Antennas

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Abstract— **This paper characterizes the optimal power control method for maximum sum capacity in a multiple access fading channel with multiple transmitter and receiver antennas when perfect channel side information is available at both the transmitters and the receiver. The profound benefit of multi-antenna diversity is demonstrated by a dimension counting argument. The optimal power allocation strategy in a system with n transmit antennas for each user and m receive antennas is a combination of successive cancelation and a TDMA-like scheme where in each time slot the rank of the transmit signals r_k for all users must satisfy $\sum_k r_k(r_k+1) \leq m(m+1)$. Thus, the total number of users that are allowed to transmit simultaneously is constrained by the number of receiver antennas. Receiver diversity increases the total number of dimensions thus allowing more users to transmit at the same time. By contrast, transmitter diversity allows a single user to occupy multiple dimensions as to benefit its own transmission, thus having the effect of precluding simultaneous transmission by other users.**

I. Introduction

While the use of multiple antennas is increasingly recognized as an effective means to provide both transmitter and receiver diversity for high-rate wireless applications, efforts to quantify their precise benefit have mainly been limited to single-user systems (see [1] and references herein.) This is in part because the single-user fading channel is already fairly involved, with several definitions of capacity and many different assumptions on channel state information possible. It is, however, important to investigate the interaction between the extra dimensions made possible by spatial diversity and the extra dimensions brought into the system by many users. To this end, this paper focuses on the multiple access fading channel with multiple antennas, and explores the optimal power control strategy when perfect and instantaneous channel side information is available at both the transmitters and the receiver. The structure of the optimal power control strategy sheds considerable light on the interaction between spatial diversity and multiuser diversity.

Under the assumption of flat fading and perfect channel side information at both the transmitter and the receiver, Goldsmith and Varaiya [2] showed that the single-user single-antenna ergodic capacity can be achieved with "water-filling-in-time". The idea is that the user should adapt its transmit power to transmit more bits when its channel is good. In a multiuser environment, because of the presence of multiuser interference, the optimal power control strategy needs to consider the fading states of other users as well. With a single transmit antenna for each user and a single receive antenna at the base-station, Knopp and Humblet [3] showed that to maximize the sum ergodic capacity, at any given time instant, only the single user who has the best fading state should transmit. In this situation, a user need to wait for his channel realization to become the best among all users before he could transmit. Recently, Viswanath, Tse and Anantharam [4] showed that such waiting can be completely eliminated if infinitely many antennas are used in the receiver (with a single transmit antenna for each user). Their paper showed that the strategy where each user does "water-filling-in-time" is asymptotically optimal when both the number of users and the number of receive antennas go to infinity. The intuition is that multiple receive antennas provide enough diversity to allow simultaneous transmission by all users. This paper will make this intuition precise by examining the optimal power allocation strategy for the non-asymptotic case. It turns out that when m receiver antennas are used, a maximum of $\frac{1}{2}m(m+1)$ users can be allowed to transmit simultaneously. This observation completes the gap between the single-antenna case and the infinite-antenna case. The extra tool needed in the analysis is the "simultaneous water-filling" idea from convex analysis [5]. Our technique also extends to the analysis of transmitter diversity, and it shows the difference between transmitter and receiver diversity in a multiuser system.

This paper concentrates on flat fading channels. For frequency selective channels that are slow-varying (i.e. with delay spread much smaller than the coherence time), the discrete Fourier transform can be used to partition the channel into independent flat-fading sub-channels, thus eliminating interference among frequency dimensions. The case considered in this paper where spatial dimensions do interfere into each other is the more interesting case, and this is where many subtleties arise.

The rest of the paper is organized as follows. Section II establishes the channel model and formulates the optimization problem. Section III examines the property of the optimal power allocation strategy for a multiuser

This work was supported by the Stanford Graduate Fellowship program.

fading channel when an arbitrary and finite number of transmitter and receiver antennas are used. The separate effects of transmitter diversity and receiver diversity are discussed in Section IV, and conclusions are drawn in Section V.

II. Multi-Antenna Multi-User Fading Channel

A discrete-time multi-antenna multiuser fading channel can be modeled as follows:

$$Y(i) = \sum_{k=1}^{K} H_k(i) X_k(i) + N(i), \tag{1}$$

where i denotes the time index. A total of K users are present in the system. Each user is equipped with n antennas so the input signals $X_k(i)$ are n dimensional vectors. The receiver is equipped with m antennas so the received signal $Y(i)$ is an m dimensional vector. $N(i)$ is the i.i.d. Gaussian noise. The channel seen by user k at time instant i is represented by $H_k(i)$, which is an $m \times n$ matrix. We will assume an i.i.d. fading model where instantaneous channel state information is available to all the transmitters and the receiver. The availability of channel state information is crucial and in practice it must be estimated at the receiver and fedback by a reliable feedback mechanism. For simplicity, the entries of H_k are assumed to be independent and identically distributed Gaussian random variables. This corresponds to a Rayleigh channel model with rich scatterers.

The capacity of a K-user multiple access channel is a K-dimensional convex region whose boundary points characterize the trade-off among data rates for various users. In this paper, the focus will be on the single boundary point that maximizes the K-user rate sum. The characterization of other boundary points involves maximizing a weighted average of all users' data rates, which, while numerically possible [6] [7], does not appear to have a closed-form solution. The single maximum rate-sum point is arguably the most effective single figure of merit for the multiple access channel. It has been the subject of many previous studies [3] [4], and it will be our focus as well.

The capacity for the multiple access channel is achieved with superposition coding and successive decoding. The relevant mutual information corresponding to the sum rate (assuming perfect channel side information at both the transmitters and the receiver) can be expressed in the chain rule as:

$$\begin{aligned} &\mathrm{I}(X_1; Y|H) + \mathrm{I}(X_2; Y|X_1, H) + \cdots + \mathrm{I}(X_K; Y| \\ &X_1, \cdots, X_{K-1}, H) = \mathrm{I}(X_1, X_2, \cdots, X_K; Y|H). \end{aligned} \tag{2}$$

Ergodic capacity is the channel capacity in the traditional Shannon sense. In this case, channel coding is done over a block length sufficiently large to cover all fading states. The mutual information can then be averaged over all fading states, and the ergodic capacity is expressed as an expectation:

$$C_{sum} = \max \mathbf{E}_{H_1, \cdots, H_K} \mathrm{I}(X_1, \cdots, X_K; Y|H_1, \cdots, H_K). \tag{3}$$

Here, the expectation is over the joint channel distribution. The mutual information is now a random variable, which depends on the channel in two ways. First, the explicit computation of $\mathrm{I}(X_1, \cdots, X_K; Y)$ depends on the channel. Secondly, because of the perfect transmitter side information assumption, the input distribution for $(X_1, \cdots, X_K)$ is also a function of $(H_1, \cdots, H_K)$. The maximization is over all such input distributions, which are called power allocation policies.

Gaussian signaling is optimal in the i.i.d. fading multiple access channel, so the optimal transmit signal is a zero-mean Gaussian process. Given the instantaneous channel realization $(H_1, \cdots, H_K)$, each transmitter sets its power spectral density subject to its total power constraint. Let $S_k(H_1, \cdots, H_K)$ be the $m \times m$ signal covariance matrix for user k at the given channel realization, i.e. $S_k(H_1, \cdots, H_K) = \mathbf{E}[X_k X_k^* | H_1, \cdots, H_K]$, where $*$ denotes complex conjugate, and the expectation is over the transmitted codebook. A power allocation policy for user k is a mapping

$$\mathcal{P}_k : (H_1, \cdots, H_K) \mapsto S_k(H_1, \cdots, H_K). \tag{4}$$

The average power constraint P_k for user k is satisfied when

$$\mathbf{E}_{H_1 \cdots, H_K}[\mathrm{tr}(S_k(H_1, \cdots, H_K))] \leq P_k, \tag{5}$$

where the expectation is over the joint channel distribution and "tr" denotes the matrix trace operator. The optimal sum capacity point is the solution to the following optimization problem,

$$\max_{S_1, \cdots, S_K} \mathbf{E}_{H_1, \cdots, H_K} \mathrm{I}(X_1, \cdots, X_K; Y|H_1, \cdots, H_K), \tag{6}$$

subject to the average power (or trace) constraints $(P_1, \cdots, P_K)$ on $(S_1, \cdots, S_K)$.

III. Optimal Power Control Strategy

A. Simultaneous Water-filling

Let ν be the random variable denoting the channel fading state, whose probability density function is $\rho(\nu)$. Denote the channel fading distribution as $H_k(\nu)$, and the power allocation strategy as $S_k(\nu)$. The ergodic sum capacity maximization problem (6) can be posed as follows.

$$\begin{aligned} \max_{S_k(\nu)} \quad & \int_\nu \log \frac{\left|\sum_{k=1}^K H_k(\nu) S_k(\nu) H_k^*(\nu) + Z\right|}{|Z|} d\rho(\nu) && (7) \\ \text{s.t} \quad & \int_\nu \mathrm{tr}(S_k(\nu)) d\rho(\nu) \leq P_k, && (8) \\ & S_k(\nu) \geq 0, && (9) \end{aligned}$$

where $|\cdot|$ denotes the determinant operator, Z denotes the receiver noise power spectrum, and the entropy expression for Gaussian random vectors $H(X) = \log\left((2\pi e)^n |\mathbf{E}[XX^*]|\right)$ is used. Here, $S_k(\nu) \geq 0$ is taken to mean that $S_k(\nu)$ is a positive semi-definite matrix. Equations (8) and (9) need to be satisfied for all $k = 1 \cdots K$ and for all fading states ν.

A key idea in solving the above optimization problem is simultaneous water-filling. At the optimal, each user's power allocation strategy is a single-user "water-filling" against the noise and the combined interference from all other users. This observation is based on the fact that the optimization problem above is convex in the positive semidefinite matrix cone containing $S_k(\nu)$ ([4], [5], [7]). It is then possible to write down its Karush-Kuhn-Tucker (KKT) condition. Associate dual variables λ_k to each power constraint and $U_k(\nu)$ to each positivity constraint, where λ_k is a scalar, and $U_k(\nu)$ are $n \times n$ matrices. We have the following theorem.

Theorem 1: A power control strategy $S_k(\nu)$ maximizes the sum ergodic capacity for a fading multiple access channel (1) with perfect side information at all the transmitters and at the receiver if and only if it satisfies the following at each fading state ν and for each user k:

$$\lambda_k \mathrm{I}_n = H_k^*(\nu) \left(\sum_{j=1}^{K} H_j(\nu) S_j(\nu) H_j^*(\nu) + Z \right)^{-1} H_k(\nu) + U_k(\nu) \tag{10}$$

$$\int_\nu \mathrm{tr}(S_k(\nu)) d\rho(\nu) \leq P_k \tag{11}$$

$$\mathrm{tr}(U_k(\nu) S_k(\nu)) = 0 \tag{12}$$

$$U_k(\nu), S_k(\nu), \lambda_k \geq 0, \tag{13}$$

where I_n is the $n \times n$ identify matrix. Such $S_k(\nu)$ has an interpretation that each user's power allocation is the single-user water-filling allocation against the combined noise and interference from all other users.

Note that the power control strategy is a function of the channel fading state, but the water-filling level λ_k is a function of the fading distribution only, which can be pre-computed. The KKT conditions naturally separate into K groups of single-user water-filling conditions, one corresponding to each user. The only modification from the single user case is that the interference from all other users is now regarded as additional noise. This generalizes an observation made in [5], where the non-fading case is treated. The KKT condition is the key in deriving further properties of optimal power allocation.

B. Single-Antenna Case

When there is only one antenna for each transmitter and for the receiver, [3] showed that the sum-rate maximizing power control strategy is a TDMA-like strategy where a single-user with the highest SNR transmit at every moment. This result will be re-derived here using the simultaneous water-filling interpretation, thus setting the stage for subsequent development where multiple antennas are introduced.

In the single antenna case, the KKT condition simplifies to the following:

$$\lambda_k = \frac{h_k^2(\nu)}{\sum_{j=1}^{K} h_j^2(\nu) s_j(\nu) + \sigma^2} + u_k(\nu) \tag{14}$$

$$\int_\nu \mathrm{tr}(S_k(\nu)) d\rho(\nu) \leq P_k \tag{15}$$

$$u_k(\nu) s_k(\nu) = 0 \tag{16}$$

$$u_k(\nu), s_k(\nu), \lambda_k \geq 0. \tag{17}$$

Without the $u_k(\nu)$ term, equation (14) is the familiar water-filling condition, with water level equal to λ_k^{-1}. The slack variable $u_k(\nu)$ is used to account for the possibility that a fading state may be so bad that no power is allocated for that state. In that case, the positive slack variable $u_k(\nu)$ is used to make up the difference. Note that the slack variable can only be non-zero when $s_k(\nu) = 0$. This is also reflected in the matrix case as (12). For two users k and l to both transmit at a fading state ν, they must both satisfy the single-user water-filling condition:

$$\lambda_k = \frac{h_k^2(\nu)}{\sum_{j=1}^{K} h_j^2(\nu) s_j(\nu) + \sigma^2} \tag{18}$$

$$\lambda_l = \frac{h_l^2(\nu)}{\sum_{j=1}^{K} h_j^2(\nu) s_j(\nu) + \sigma^2} \tag{19}$$

The denominator for the two conditions are the same, so if both users transmit, then

$$\frac{h_k^2(\nu)}{\lambda_k} = \frac{h_l^2(\nu)}{\lambda_l}. \tag{20}$$

In other words, the fading state ν may not be shared by the two users unless the channel gains differ by exactly the factor λ_k/λ_l. Since the channel fading state is assumed to be i.i.d. complex Gaussian distributed, such event has zero probability. Therefore, we have proved the following.

Theorem 2 (Knopp and Humblet [3]) : In a single-antenna multiple access fading channel with i.i.d. Gaussian fading statistics, assuming perfect side information at the transmitters and the receiver, with probability 1, the sum capacity is achieved with a power control strategy that allows only one user to transmit at a time.

This same conclusion was reached earlier by Cheng and Verdu [8] in the context of multiple access channel with intersymbol interference. The power allocation problem for the fading channel is identical to the loading problem for the ISI channel if the fading statistics is assumed to be i.i.d., and if the ISI channel is equipped with guard periods which ensure the orthogonality of subchannels. Cheng and Verdu concluded that FDMA achieves the sum capacity in a multiple access channel with ISI (which corresponds to TDMA for fading channels).

C. Multiple-Antenna Case

The intuition for Knopp and Humblet's result is the following. A single-antenna receiver is limited by the single degree of freedom it has. To achieve the sum capacity, only one user can transmit at a time. With multiple antennas however, multiple dimensions may be available. So, the optimal power control strategy may involve more than one user transmitting at the same time. Nevertheless, the maximum number of simultaneous users should still be related to the number of antennas. This intuition is made precise by the following theorem:

Theorem 3: In a multiple-antenna multiple-access fading channel whose channel matrix entries are i.i.d. Gaussian distributed, the optimal power control strategy with perfect channel side information at all transmitters and at the receiver that achieves the maximum sum ergodic capacity has the following property: with n antennas for each user and m antennas for the receiver, at any time instant, the rank of transmit signal r_k for each users must satisfy $\sum_k r_k(r_k+1) \leq m(m+1)$. In particular, a maximum of $\frac{1}{2}m(m+1)$ users can transmit simultaneously.

Proof: First, consider receiver diversity alone. In this case, each transmitter has one antenna, and the receiver has m antennas, so that the channel matrix $H_k(\nu)$ is an $m\times 1$ vector and the transmitter covariance is just a scalar $s_k(\nu)$, and the slack variable is also a scalar $u_k(\nu)$:

$$\lambda_k = H_k^*(\nu)\left(\sum_{j=1}^K s_j(\nu)H_j(\nu)H_j^*(\nu)+Z\right)^{-1} H_k(\nu) + u_k(\nu). \tag{21}$$

The claim is that at any fading state ν, only a maximum of $\frac{1}{2}m(m+1)$ users can have $s_k(\nu)>0$ and $u_k(\nu)=0$. The rest of the users have $s_k(\nu)=0$ and $u_k(\nu)>0$. As in the single-antenna case, the key is to recognize that the matrix inversion in the expression is common to all users, and each user has an arbitrary channel. Recall that λ_k is determined by the channel fading distribution, so it can be considered fixed. We first ask whether there exists a positive definite symmetric matrix M such that $H_k^* M H_k = \lambda_k$ for more than $\frac{1}{2}m(m+1)$ H_k's. The following Lemma answers this question.

Lemma 1: Fixing positive $\lambda_1,\cdots,\lambda_K$, let $H_1,\cdots,H_K$ be $m\times 1$ random vectors whose entries have an i.i.d. Gaussian distribution. If $K>\frac{1}{2}m(m+1)$, then with probability 1, there does not exists a positive definite symmetric matrix M such that $H_k^* M H_k = \lambda_k, \forall k = 1\cdots K$.

Proof: Let $H_k = (h_{k1}, h_{k2}, \cdots, h_{km})^\tau$. Denote the (i,j) entry of M by m_{ij}. Because of the symmetry, $m_{ij}=m_{ji}$, so there are $\frac{1}{2}m(m+1)$ independent variables in M. To have $H_k^* M H_k = \lambda_k$, we need

$$\sum_{ij} h_{ki}h_{kj}m_{ij} = \lambda_k, \tag{22}$$

for all $k=1\cdots K$. Because H_k are i.i.d. Gaussian, with probability 1, these K equations are linearly independent. So a solution to (22) exists only if $K \leq \frac{1}{2}m(m+1)$. □

Lemma 1 shows that the number of users that can transmit simultaneously is $\frac{1}{2}m(m+1)$ or fewer. It does not guarantee that exactly $\frac{1}{2}m(m+1)$ users will transmit because the existence of a matrix M satisfying (22) does not guarantee that such M can be synthesized by $s_k(\nu)$ as in (21).

Next we turn our attention to transmitter diversity. The water level is now an $n\times n$ identity matrix, and the transmitter power spectrum and the slack variables are both $n\times n$ positive semidefinite matrices. The water-filling condition is:

$$\lambda_k \mathrm{I}_n = H_k^*(\nu)\left(\sum_{j=1}^K H_j(\nu)S_j(\nu)H_j^*(\nu)+Z\right)^{-1} H_k(\nu) + U_k(\nu), \tag{23}$$

for $k=1\cdots K$. Parallel to the previous development, we ask: whether there exists a positive semidefinite matrix M that satisfies $\lambda_k \mathrm{I}_n = H_k^*(\nu) M H_k(\nu) + U_k(\nu)$. The idea is to count the number of independent equations and the number of unknowns. To satisfy the matrix equation, we need to satisfy one equation for each matrix entry. By symmetry, there are $\frac{1}{2}n(n+1)$ independent entries for each k, so there are in total $\frac{K}{2}n(n+1)$ independent equations.

The number of unknown variables is counted as follows. The matrix M introduces $\frac{1}{2}m(m+1)$ degrees of freedom. The number of unknowns introduced by the slack variable $U_k(\nu)$ depends on its rank. An $n\times n$ symmetric matrix has at most $\frac{1}{2}n(n+1)$ degrees of freedom. But, if the matrix is restricted to rank r, the number of degrees of freedom decreases to $\frac{1}{2}n(n+1)-\frac{1}{2}(n-r)(n-r+1)$. To see this, recall that a positive semidefinite symmetric matrix can be represented in its (unique) Cholesky factorization as LL^*. If a $n\times n$ matrix is of rank r, its Cholesky factor is a $n\times r$ triangular matrix, with exactly $\frac{1}{2}n(n+1) - \frac{1}{2}(n-r)(n-r+1)$ independent entries.

Now, the slack variables need to satisfy the complementary slackness condition

$$\mathrm{tr}(U_k(\nu)S_k(\nu)) = 0. \tag{24}$$

So, if the transmit signal $S_k(\nu)$ is of rank $r_k(\nu)$, the rank of $U_k(\nu)$ is at most $n-r_k(\nu)$. Therefore, each $U_k(\nu)$ introduces at most $\frac{1}{2}n(n+1)-\frac{1}{2}r_k(\nu)(r_k(\nu)+1)$ extra degrees of freedom. The total number of unknown variables is then $\frac{1}{2}m(m+1)$ coming from the matrix M plus $\frac{1}{2}n(n+1)-\frac{1}{2}r_k(\nu)(r_k(\nu)+1)$ coming from each of $U_k(\nu)$.

All equations involve the channel realization $H_k(\nu)$, which is a Gaussian random matrix. So, with probability 1, these equations are independent. Thus, for a solution to exist, there need to be at least as many unknown variables as there are equations, so:

$$\frac{m(m+1)}{2} + K\frac{n(n+1)}{2} - \sum_{k=1}^{K} \frac{r_k(\nu)(r_k(\nu)+1)}{2} \geq K\frac{n(n+1)}{2}, \quad (25)$$

from which the condition $\sum_k r_k(r_k+1) \leq m(m+1)$ follows.

At any time instant, a user transmits with positive power if the rank of its transmitted signal is at least 1. Therefore, in a multiple user scenario with m receive antennas, a total of $\frac{1}{2}m(m+1)$ users can transmit at the same time. The power control strategy can be thought of as choosing the "best" set of $\frac{1}{2}m(m+1)$ users, when transmitting together (using power determined by the fading state and their respective water levels), provides the highest sum capacity. This concludes the proof.

Theorem 3 establishes an upper bound on the number of simultaneous users that can transmit simultaneously in a multiple access channel. Although the theorem does not guarantee that the bound is tight, simulation results indicate that the maximum number of simultaneous users is somewhere between m and $\frac{1}{2}m(m+1)$.

IV. Spatial Diversity for Multiple Access

In a multiple access fading channel, the degrees of freedom created by spatial diversity is bounded quadratically. With m receive antennas, the maximum number of degrees of freedom that an optimal multiuser detector can process is $\frac{1}{2}m(m+1)$. These many degrees of freedom are to be divided among the transmitters. Each transmitter, with n transmitting antennas, can potentially use up to $\frac{1}{2}n(n+1)$ degrees of freedom. Therefore, the number of receiver antennas has the effect of allowing more users to transmit simultaneously, while the number of transmitter antennas has the opposite effect. Transmitter antennas have the potential to crowd out receiver dimensions and thus prevent other users from transmitting at the same time. Such crowding-out increases system sum capacity at the expense of delay and fairness.

There is an interesting interplay between spatial diversity and multiuser diversity. In a single-user vector channels, the maximum number of usable dimensions is the rank of the channel matrix. The rank is the number of independent data streams that a vector channel can process. In a multiuser systems however, the maximum number of independent users that the system can accommodate is bounded by the square of the rank. The difference is that in a multiuser environment, no signal coordination among the users is possible, so spatial dimensions for each user overlap. Consequently, more users are cramped into the limited number of dimensions.

This result reduces immediately to the single antenna case considered in [3]. It also extends to the asymptotic result in [4] nicely. As the number of users and the number of receiver antennas both go to infinity while keeping their ratio α fixed, [4] identified a single-user water-filling power allocation strategy which is asymptotically optimal for α either larger than 1 or smaller than 1. With the quadratic growth of maximum number of allowable users, the reason is now clear. By fixing the ratio α, the number of total users in the problem setting of [4] is a linear function of the number of receive antennas, yet the number of degrees of freedom grows quadratically. Since $\frac{1}{2}m(m+1)$ will eventually exceed any linear function of m, as m goes to infinity, all users will be able to transmit.

V. Conclusion

This paper formulated the power control problem for multiple access fading channels when each transmitter and the receiver are equipped with multiple antennas. Under the assumption of i.i.d. fading and perfect and instantaneous transmitter and receiver channel side information, the power control strategy that maximizes the sum ergodic capacity is characterized. It is shown that in a system with n transmitter antennas for each user and m receiver antennas, the optimal power control strategy allows up to $\frac{1}{2}m(m+1)$ degrees of freedom be used at any time, with each user contributing up to $\frac{1}{2}n(n+1)$ degrees of freedom. This property illustrates following: receiver dimensions can be thought of as discrete resources to be distributed among all the users, and transmitter dimensions can be thought of as each user's potential ability to utilize the available dimensions.

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