Minimax Duality of Gaussian Vector Broadcast Channels

Wei Yu and Tian Lan

Electrical and Computer Engineering Department
University of Toronto

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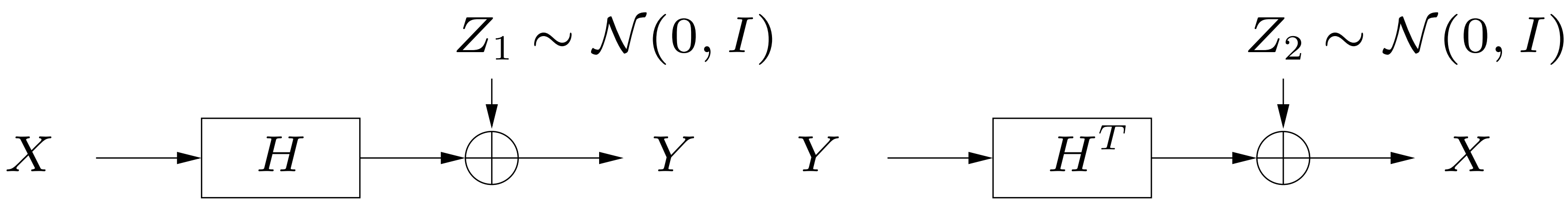
Reciprocity in Gaussian Vector Channels

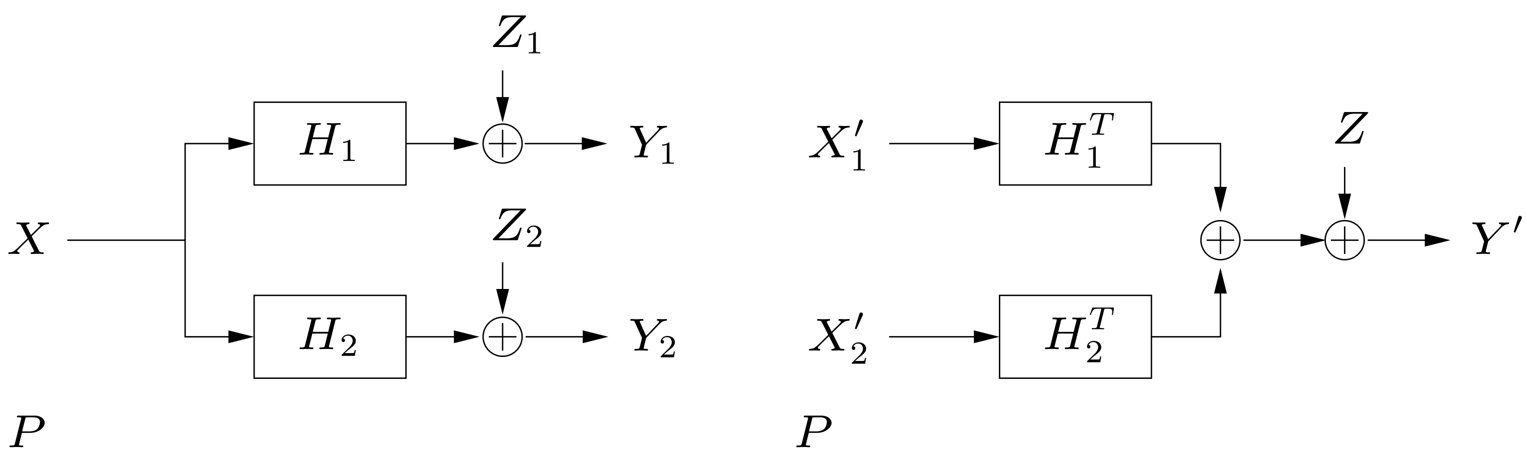


$\mathbf{E}[X^T X] \leq P$ $\qquad$ $\mathbf{E}[Y^T Y] \leq P$

- Capacities of the channels H and H^T are the same
  - under the same power constraint;
  - even if H is not square.

- Proof: H and H^T have the same singular values.

Reciprocity for Multi-User Channels

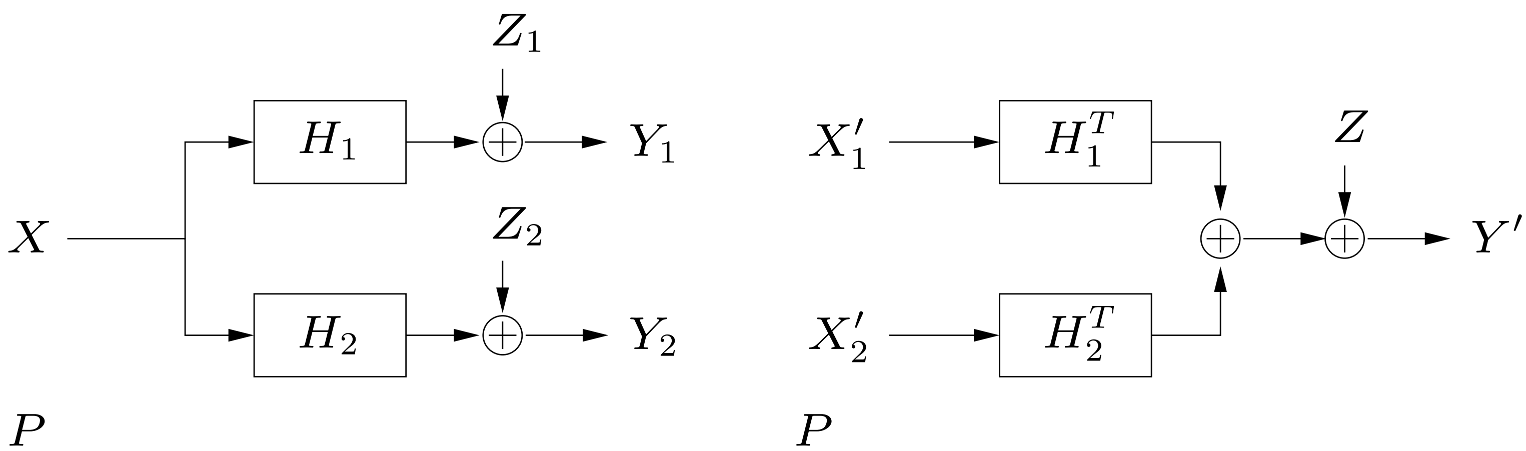


- Duality exists between Gaussian vector MAC and BC Channels (Vishwanath, Jindal, Goldsmith '03, Viswanath, Tse '03).

- Goal of this talk: Generalization and re-interpretation of duality.

Uplink-Downlink Duality

The multiple-access channel and the broadcast channel are duals.



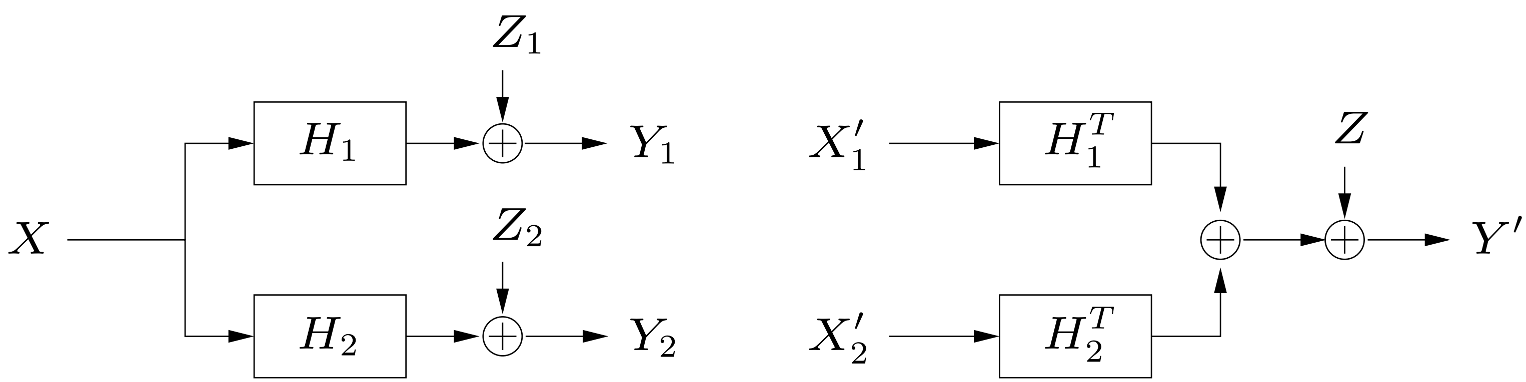
Previous proofs of duality depend critically on the power constraint.

Why is there duality?

Main Results of This Talk

1. Uplink-Downlink Duality is equivalent to Lagrangian Duality

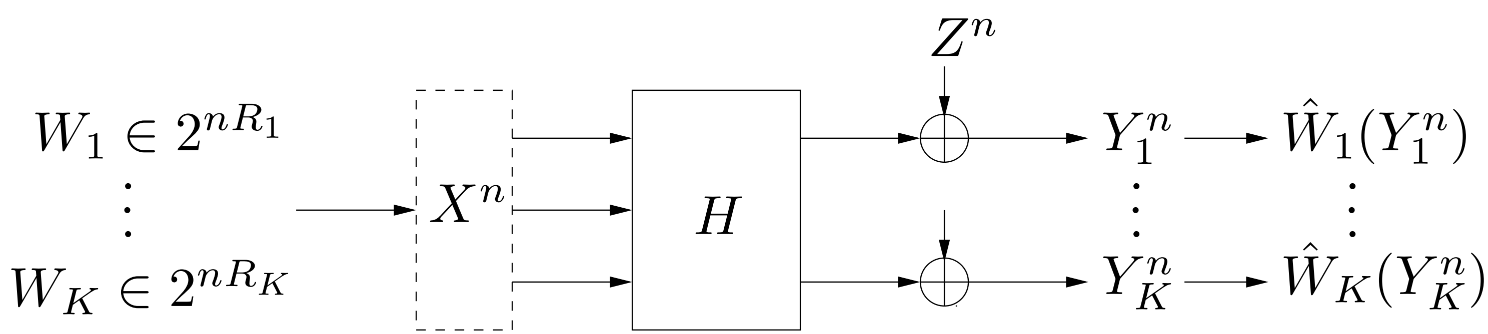
2. Duality generalizes beyond the sum power constraint



Broadcast Channel with Arbitrary input constraints $\Longleftrightarrow$ Multiple Access Channel with Uncertain noise

Multiple Antenna Broadcast Channel

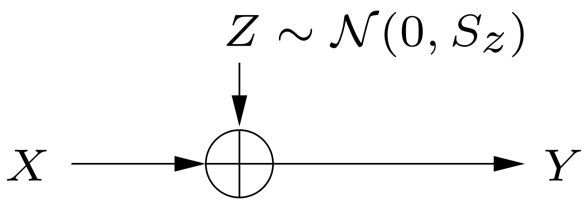
- Non-degraded Gaussian vector broadcast channel:



- Capacity region is solved recently.
  - This talk focuses on sum capacity $C = \max\{R_1 + \cdots + R_K\}$.

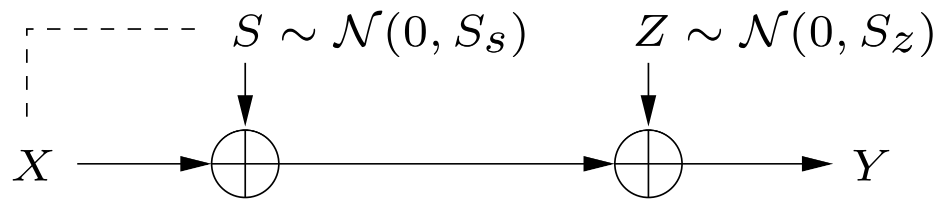
Achievability: Writing on Dirty Paper

Gaussian Channel



$$C = \frac{1}{2} \log \frac{|S_x + S_z|}{|S_z|}$$

... with Transmitter Side Information

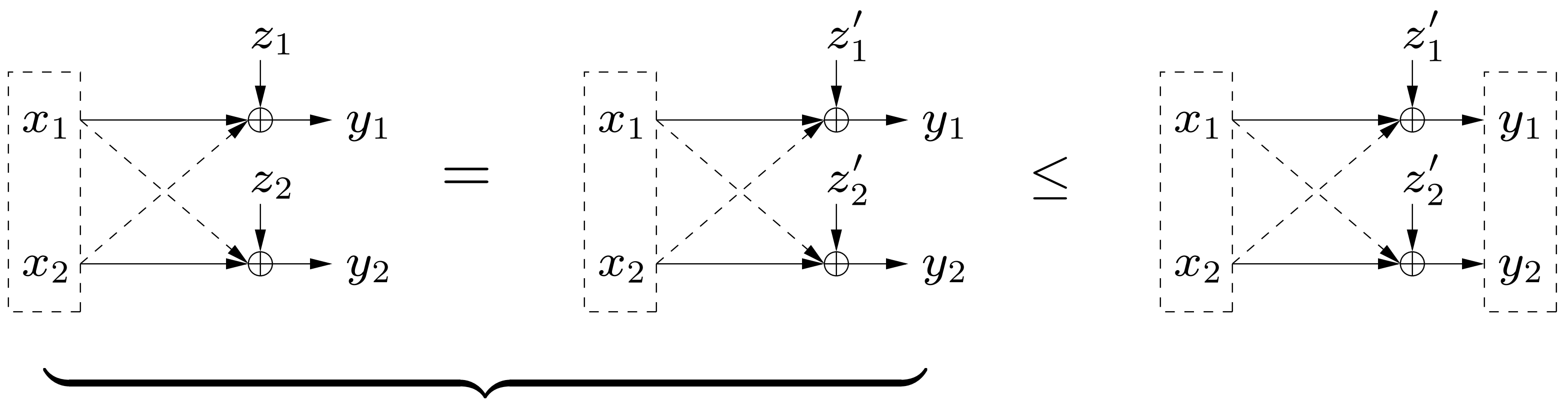


$$C = \frac{1}{2} \log \frac{|S_x + S_z|}{|S_z|}$$

- Capacities are the same if S is known *non-causally* at the transmitter.

Converse: Sato's Outer Bound

- Broadcast capacity does not depend on noise correlation: Sato ('78).



if $\begin{cases} p(z_1) = p(z_1') \\ p(z_2) = p(z_2') \end{cases}$, not necessarily $p(z_1, z_2) = p(z_1', z_2')$.

- So, sum capacity $C \leq \min\limits_{S_z} \max\limits_{S_x} I(\mathbf{X}; \mathbf{Y})$.

Three Achievability Proofs

1. Decision-Feedback Equalization approach (Yu, Cioffi)
   - Worst-noise diagonalizes the feedforward matrix of a DFE.
2. Uplink-Downlink duality approach (Viswanath, Tse)
   - Noise covariance is equivalent to the input constraint in dual channel.
   - Worst-noise decouples the inputs in the dual channel.
3. Convex duality approach (Vishwanath, Jindal, Goldsmith)
   - Channel flipping between multiple access channel and broadcast channel.

This talk: A new derivation of duality.

Gaussian Broadcast Channel Sum Capacity

- Achievability: $C \geq \max_{S_x} \min_{S_z} I(\mathbf{X};\mathbf{Y})$.
- Converse: $C \leq \min_{S_z} \max_{S_x} I(\mathbf{X};\mathbf{Y})$.
- Gaussian vector broadcast channel sum capacity is therefore exactly:

$$C = \max_{S_x} \min_{S_z} \frac{1}{2} \log \frac{|HS_xH^T + S_z|}{|S_z|}$$

.

> Duality can be derived directly from the minimax expression!

Minimax Optimization

- Gaussian vector broadcast channel sum capacity is the solution of

$$\begin{aligned} \max_{S_x} \min_{S_z} \quad & \frac{1}{2} \log \frac{|H S_x H^T + S_z|}{|S_z|} \\ \text{subject to} \quad & \text{tr}(S_x) \leq P \\ & S_z = \begin{bmatrix} I & \star \\ \star & I \end{bmatrix} \\ & S_x, S_z \geq 0 \end{aligned}$$

- The minimax problem is **convex** in S_z, **concave** in S_x.
  - How to solve this minimax problem?

Duality through Minimax

- Two KKT conditions must be satisfied simultaneously:

$$H^T(HS_xH^T + S_z)^{-1}H = \lambda I$$

$$S_z^{-1} - (HS_xH^T + S_z)^{-1} = \begin{bmatrix} \Psi_1 & 0 \\ 0 & \Psi_2 \end{bmatrix}$$

- For the moment, assume that H is invertible.

$$\Rightarrow \quad H^T S_z^{-1} H - \lambda I = H^T \Psi H$$

$$\Rightarrow \quad H(H^T \Psi H + \lambda I)^{-1} H^T = S_z$$

- Further manipulation: $\Rightarrow (\lambda I)^{-1} - (H^T \Psi H + \lambda I)^{-1} = S_x$.

A New Minimax Duality

KKT conditions lead to a new minimax problem:

$$\begin{aligned} \max_{S_x} \min_{S_z} \quad & \frac{1}{2} \log \frac{|H S_x H^T + S_z|}{|S_z|} \\ \text{s.t.} \quad & \mathrm{tr}(S_x) \leq P \\ & S_z = \begin{bmatrix} I & \star \\ \star & I \end{bmatrix} \\ & S_x, S_z \geq 0 \end{aligned}$$

$$\begin{aligned} \min_{\lambda} \max_{\Psi} \quad & \frac{1}{2} \log \frac{|H^T \Psi H + \lambda I|}{|\lambda I|} \\ \text{s.t.} \quad & \mathrm{tr}(\Psi) \leq \lambda P \\ & \Psi \ \text{ is diagonal} \\ & \Psi \geq 0, \lambda \geq 0 \end{aligned}$$

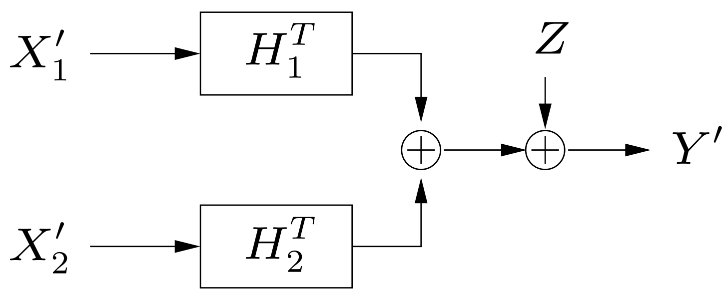
Minimax duality is equivalent to Lagrangian duality.

Construct the Dual Channel

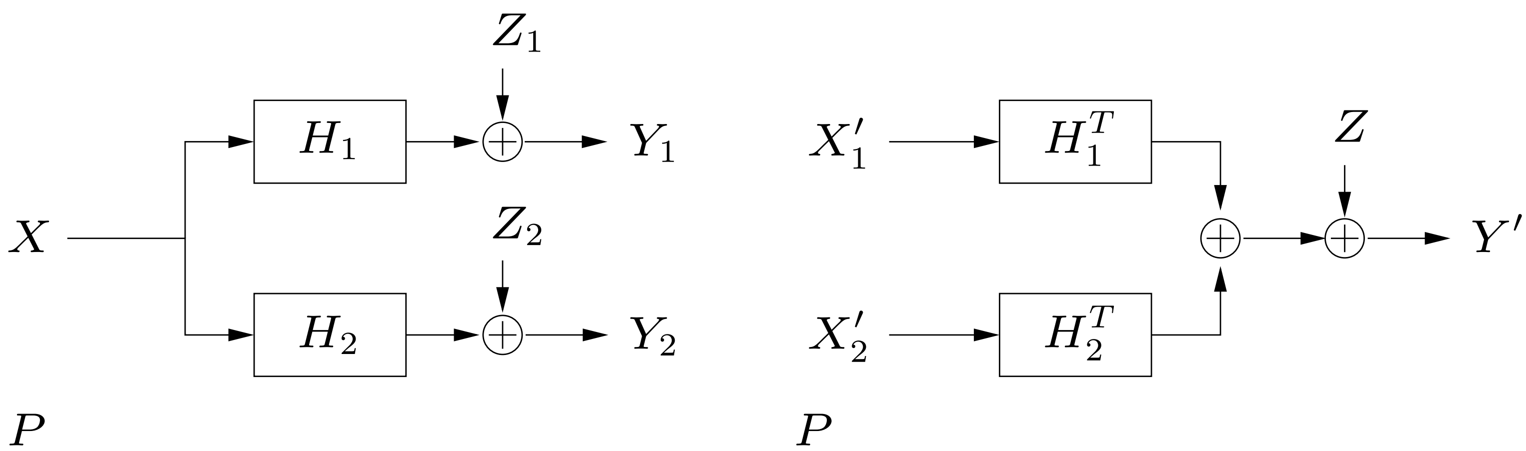
$$\boxed{\text{KKT condition: } H(H^T \Psi H + \lambda I)^{-1} H^T = S_z}$$

- Define the diagonal matrix: $D = \Psi/\lambda$. $\mathrm{trace}(D) = \sum_i \Psi_i/\lambda = P$.
- $S_z = \begin{bmatrix} I & \star \\ \star & I \end{bmatrix}$. Thus, constraint on D: $\mathrm{trace}(D_1) + \mathrm{trace}(D_2) \leq P$.

$$\mathbf{E}[X_1' X_1'^T] = D_1$$
$$\mathbf{E}[X_2' X_2'^T] = D_2$$
$$\mathrm{trace}(D_1) + \mathrm{trace}(D_2) \leq P$$



Uplink-Downlink Duality



Theorem 1. *Under the same power constraint, the Gaussian vector multiple-access channel and the Gaussian vector broadcast channel have the same sum capacity.*

New Proof: By re-defining $D = \Psi/\lambda$, the minimization part of the minimax dual problem disappears.

Generalized Minimax Duality

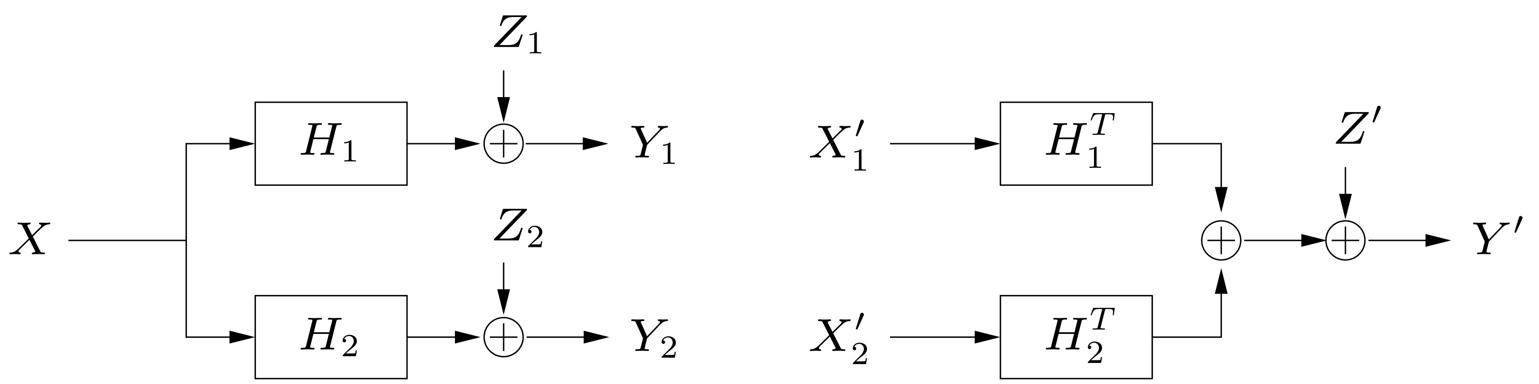
Theorem 1 may be generalized to arbitrary linear constraints:

$$\begin{aligned} \max_{S_x} \min_{S_z} \quad & \frac{1}{2} \log \frac{|H S_x H^T + S_z|}{|S_z|} \\ \text{s.t.} \quad & \operatorname{tr}(S_x Q_x) \leq 1 \\ & \operatorname{tr}(S_z Q_z) \leq 1 \\ & S_x, S_z \geq 0 \end{aligned}$$

$$\begin{aligned} \max_{\Sigma_z} \min_{\Sigma_x} \quad & \frac{1}{2} \log \frac{|H^T \Sigma_z H + \Sigma_x|}{|\Sigma_x|} \\ \text{s.t.} \quad & \operatorname{tr}(\Sigma_z \Psi_z) \leq 1 \\ & \operatorname{tr}(\Sigma_x \Psi_x) \leq 1 \\ & \Sigma_x, \Sigma_z \geq 0 \end{aligned}$$

Relation: $S_x = \lambda_x \Psi_x \quad S_z = \lambda_z \Psi_z \quad \Sigma_x = \lambda_x Q_x \quad \Sigma_z = \lambda_z Q_z$

Generalized Uplink-Downlink Duality



$\mathsf{tr}(S_x Q_1) \leq P$ $\quad S_z \sim \mathcal{N}(0, Q_2)$ $\quad \mathsf{tr}(S_{x'} Q_2) \leq P$ $\quad S_{z'} \sim \mathcal{N}(0, Q_1)$

Q_1: Input constraint in BC and Noise covariance in MAC.
Q_2: Worst noise covariance in BC and Input constraint in MAC.

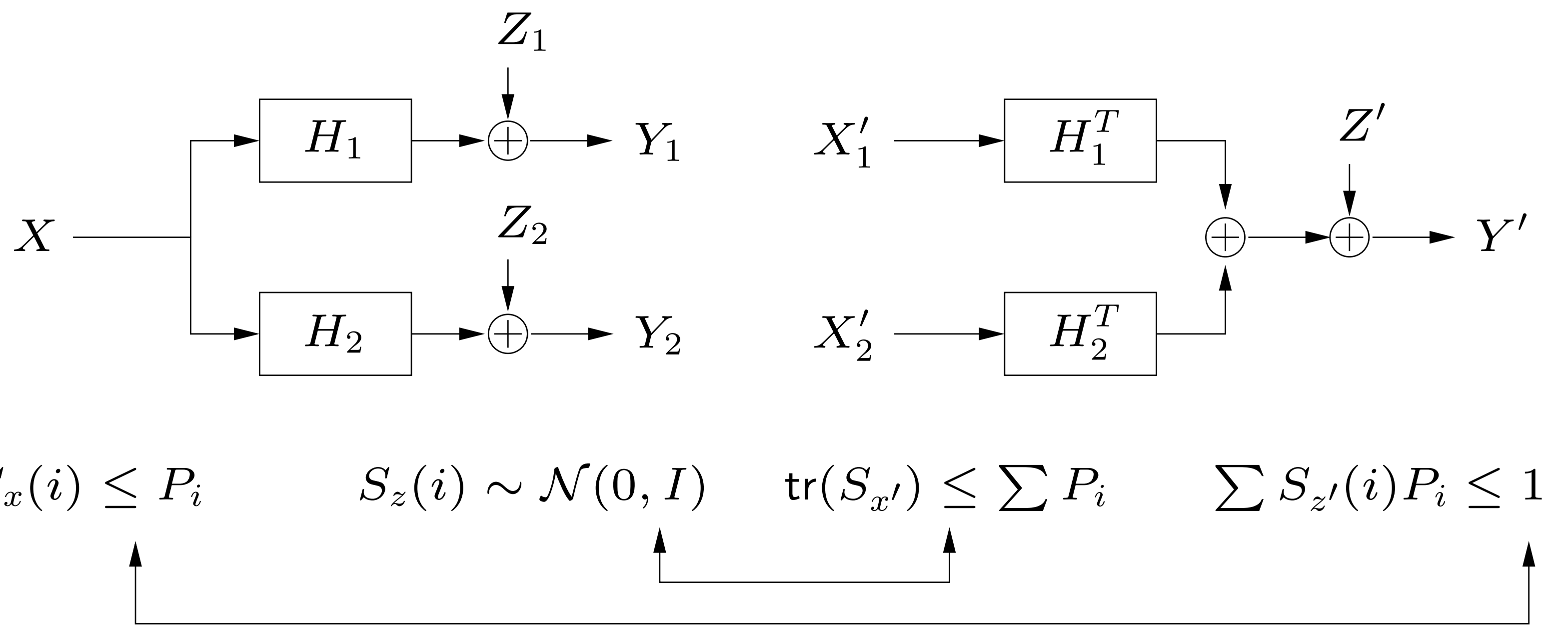
Per-Antenna Power Constrained Broadcast Channel

$$\begin{aligned} \max_{S_x} \min_{S_z} \quad & \frac{1}{2} \log \frac{|H S_x H^T + S_z|}{|S_z|} \\ \text{s.t.} \quad & S_x(i,i) \le P_i \\ & S_z = \begin{bmatrix} I & \star \\ \star & I \end{bmatrix} \\ & S_x, S_z \ge 0 \end{aligned}$$

$$\begin{aligned} \max_{\Psi} \min_{\Lambda} \quad & \frac{1}{2} \log \frac{|H^T \Psi H + \Lambda|}{|\Lambda|} \\ \text{s.t.} \quad & \mathrm{tr}(\Psi) \le \sum_i P_i \\ & \sum_i \Lambda(i,i) P_i \le 1 \\ & \Lambda, \Psi \ge 0, \quad \text{and diagonal} \end{aligned}$$

Theorem 2. *The dual of a Gaussian vector broadcast channel with individual per-antenna power constraint is a multiple-access channel with a diagonal and linearly constrained uncertain noise.*

Per-Antenna Power Constrained Broadcast Channel



In addition, this duality applies not only to the sum capacity but also the entire region.

Summary and Conclusions

- Sum capacity of a Gaussian vector broadcast channel is:

$$C = \max_{S_x} \min_{S_z} \frac{1}{2} \log \frac{|HS_xH^T + S_z|}{|S_z|}$$

- Lagrangian duality leads to uplink-downlink duality.

- The dual of a broadcast channel with individual power constraint is a multiple access channel with unknown noise.

- Duality can be very useful from a computational perspective.