On Constant Power Water-filling

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Abstract— **This paper derives a rigorous performance bound for the constant-power water-filling algorithm for ISI channels with multicarrier modulation and for i.i.d. fading channels with adaptive modulation. Based on the performance bound, a very-low complexity logarithm-free power allocation algorithm is proposed. Theoretical worst-case analysis and simulation show that the approximate water-filling scheme is close to optimal.**

I. Introduction

When a communication channel is corrupted by severe fading or by strong intersymbol interference, the adaptation of transmit signal to the channel condition can typically bring a large improvement to the transmission rate. Adaptation is possible when the channel state is available to the transmitter, usually by a channel estimation scheme and a reliable feedback mechanism. With perfect channel information, the problem of finding the optimal adaptation strategy has been much studied in the past. If the channel can be partitioned into parallel independent subchannels by assuming i.i.d. fading statistics for the fading channel, or by the discrete Fourier transform for the intersymbol interference channel, the optimal transmit power adaptation is the well-known water-filling procedure. In water-filling, more power is allocated to "better" subchannels with higher signal-to-noise ratio (SNR), so as to maximize the sum of data rates in all subchannels, where in each subchannel the data rate is related to the power allocation by Shannon's Gaussian capacity formula[1] $\frac{1}{2}\log(1+\text{SNR})$. However, because the capacity is a logarithmic function of power, the data rate is usually insensitive to the exact power allocation, except when the signal-to-noise ratio is low. This motivates the search for simpler power allocation schemes that can perform close to the optimal.

Approximate water-filling schemes often greatly simplify transmitter and receiver design, and they have been the subject of considerable study. In the multicarrier context, Chow [1] empirically discovered that as long as the optimal bandwidth is used, a constant-power allocation has a negligible performance loss compared to true water-filling. The same phenomenon is observed in the adaptive modulation setting [2]. There has been several performance bounds on constant-power water-filling reported in the literature. Aslanis [3] compared the worst case difference between a true water-filling and a constant-power water-filling, and derived a bound based on the SNR cut-off value. Schein and Trott [4] derived a different bound also based on SNR. The current work extends the existing results in several directions. First, a worst-case performance bound is derived using a novel approach based on convex analysis, and the bound is valid for SNR. Secondly, it is shown that the new performance bound can be used to design a very-low complexity power allocation algorithm with a bounded worst-case performance. In particular, the algorithm is shown to be at most 0.266 bits/sec/Hz away from capacity on a Rayleigh channel, and it often performs much closer to capacity in practice.

The rest of the paper is organized as follows. Section II formulates the water-filling problem, and derives the new bound. Section III proposes a new low-complexity power adaptation algorithm. Section IV applies the bound to the Rayleigh fading channel. Simulation results are presented in Section V, and conclusions are drawn in Section VI.

II. Sub-optimal Water-filling

A. Problem Formulation

We choose to formulate the problem in the adaptive modulation framework because it is slightly more general than the multicarrier setting. The communication channel is modeled as:

$$Y(i) = \sqrt{\nu(i)} \cdot X(i) + N(i), \tag{1}$$

where i is the discrete time index, $X(i)$ and $Y(i)$ are scalar input and output signals respectively, $N(i)$ is the additive white Gaussian noise, which is independent and identically distributed with a constant variance σ^2, and $\sqrt{\nu(i)}$ is the multiplicative channel fading coefficient. For simplicity, $\nu(i)$, the squared magnitude of the fading coefficient, is assumed to be independent and identically distributed with a probability distribution $\rho(\nu)$. The capacity for this fading channel under an average transmit power constraint when both the transmitter and the receiver have perfect and instantaneous channel side information was characterized by Goldsmith and Variaya [5]. They proposed a water-filling-in-time solution and proved

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[1]In this paper, "log" is used to denote logarithm of base 2; "ln" is used to denote logarithm of base e.

a coding theorem based on finite partitions of channel fading statistics, i.e. ν is restricted to take finite values $\nu_1, \nu_2, \cdots \nu_m$, with probabilities $p_1, p_2, \cdots p_m$. In this case, the maximization problem becomes:

$$\max_{S_k} \quad \sum_{k=1}^{m} p_k \log \left(1 + \frac{S_k \nu_k}{\sigma^2}\right) \tag{2}$$

$$\text{s.t.} \quad \sum_{k=1}^{m} p_k S_k \leq \bar{S}, \tag{3}$$

$$S_k \geq 0, \tag{4}$$

where $\bar{S}$ is the average transmit power constraint, and the maximization is over all power allocation policies S_k based on the instant channel fading state ν_k. Letting $p_k = 1$ reduces the problem to the multicarrier setting. The solution to this optimization problem is the well-known water-filling procedure. Our interest is in finding approximate solutions with provable worst-case performance.

B. Duality Gap

The optimization problem (2) belongs to the class of convex programming problems, where a convex objective function is to be minimized subject to a convex constraint set. A general form of a convex problem is the following:

$$\min_{x} \quad f_0(x) \tag{5}$$

$$\text{s.t.} \quad f_i(x) \leq 0, \tag{6}$$

where $f_i(x)$, $i = 0, 1, \cdots m$ are convex functions. $f_0(x)$ is called the primal objective. The Lagrangian of the optimization problem is defined as:

$$L(x, \lambda) = f_0(x) + \lambda_1 f_1(x) + \cdots \lambda_m f_m(x), \tag{7}$$

where λ_i are positive constants. The dual objective is defined to be $g(\lambda) = \inf_x L(x, \lambda)$. It is easy to see that $g(\lambda)$ is a lower bound on the optimal $f_0(x)$:

$$f_0(x) \geq f_0(x) + \sum_i \lambda_i f_i(x) \tag{8}$$

$$\geq \inf_z \left(f_0(z) + \sum_i \lambda_i f_i(z)\right) \tag{9}$$

$$\geq g(\lambda). \tag{10}$$

So,

$$g(\lambda) \leq \min_x f_0(x). \tag{11}$$

This is the lower bound that we will use to investigate the optimality of approximate water-filling algorithms. The difference between the primal objective $f_0(x)$ and the dual objective $g(\lambda)$ is called the duality-gap. A central result in convex analysis [6] is that when the primal problem is convex, the duality gap reduces to zero at the optimum.

C. Lower Bound

The above general result is now applied to the water-filling problem. First, maximizing the data rate is equivalent to minimizing its negative. The capacity is a concave function of power, so its negative is convex. The constraints are linear, so they are convex as well. Associate dual variable λ to the power constraint, and μ_k to each of the positivity constraints on S_k, the Lagrangian is then:

$$L(S_k, \lambda, \mu_k) = \sum_{k=1}^{m} -p_k \log\left(1 + \frac{S_k \nu_k}{\sigma^2}\right) + \lambda \left[\left(\sum_{k=1}^{m} p_k S_k\right) - \bar{S}\right] + \sum_{k=1}^{m} \mu_k(-S_k). \tag{12}$$

The dual objective $g(\lambda, \mu_k)$ is the infimum of the Lagrangian over primal variables S_k. At the infimum, the partial derivative of the Lagrangian with respect to S_k must be zero:

$$\frac{\partial L}{\partial S_k} = 0 = -p_k \cdot \frac{\nu_k/\sigma^2}{1 + S_k \nu_k/\sigma^2} \cdot \frac{1}{\ln 2} + \lambda \cdot p_k - \mu_k, \tag{13}$$

from which the classical water-filling condition

$$S_k + \frac{\sigma^2}{\nu_k} = \frac{1}{\lambda - \mu_k/p_k} \cdot \frac{1}{\ln 2}. \tag{14}$$

is obtained. This condition, together with the constraints of the original primal problem, the positivity constraints on the dual variables, and the so-called complementary slackness constraints, form the Karush-Kuhn-Tucker (KKT) condition, which is sufficient and necessary in this case.

Substituting the water-filling condition (14) into (12) gives the dual objective:

$$g(\lambda, \mu_k) = -\sum_{k=1}^{m} p_k \log\left(\frac{\nu_k/\sigma^2}{\lambda - \mu_k/p_k} \cdot \frac{1}{\ln 2}\right) - \left(\sum_{k=1}^{m} (\lambda p_k - \mu_k) \cdot \frac{\sigma^2}{\nu_k}\right) - \lambda \bar{S} + \frac{1}{\ln 2} \tag{15}$$

The dual objective is always convex, and it is a lower bound to the primal objective[2] for all positive λ and μ_k. In particular, substituting the dual variables as in (14) gives the following duality gap Γ, which is defined as the difference between the primal and the dual objectives:

$$\Gamma = \sum_{k=1}^{m} p_k \left(\frac{\sigma^2/\nu_k}{S_k + \sigma^2/\nu_k} \cdot \frac{1}{\ln 2}\right) + \lambda \bar{S} - \frac{1}{\ln 2}. \tag{16}$$

[2] $g(\lambda, \mu_k)$ is a lower bound to the minimization problem. So $-g(\lambda, \mu_k)$ is an upper bound to the rate maximization problem. To avoid notational inconvenience, the rest of the paper will be speaking of only the duality gap.

To express the gap exclusively in primal variables S_k, a suitable λ needs to be found. A small λ is desirable because it makes the duality gap small. Since

$$\frac{1}{S_k + \sigma^2/\nu_k} \cdot \frac{1}{\ln 2} = \lambda - \frac{\mu_k}{p_k}, \tag{17}$$

and recall that λ and μ_k need to be non-negative, the smallest non-negative λ is then

$$\lambda = \max_k \left(\frac{1}{S_k + \sigma^2/\nu_k} \cdot \frac{1}{\ln 2} \right) = \frac{1}{\min_k \{S_k + \sigma^2/\nu_k\}} \cdot \frac{1}{\ln 2}. \tag{18}$$

Assume that the approximate water-filling algorithm satisfies the power constraint $\sum_k p_k S_k \leq \bar{S}$ with equality[3], the above gives the following:

$$\Gamma = \frac{1}{\ln 2} \cdot \left[\sum_{k=1}^{m} p_k \left(\frac{S_k}{\min_j \{S_j + \sigma^2/\nu_j\}} - \frac{S_k}{S_k + \sigma^2/\nu_k} \right) \right] \tag{19}$$

The preceeding development is summarized in the following theorem:

Theorem 1: For the optimization problem (2), if $S_k \geq 0$ is a power allocation strategy that satisfies the power constraint with equality, then the achievable data rate using S_k is at most Γ bits/sec per Hz away from the optimal water-filling solution, where Γ is expressed in (19).

This result is a general bound to all approximate water-filling algorithms. For example, it can be used to bound the performance of power allocation strategies with integer-bit constraint[4]. It is clear that if exact water-filling is used, i.e. when $S_k + \sigma^2/\nu_k$ is a constant whenever $S_k > 0$, the gap reduces to zero. Therefore, the cost of not doing water-filling is the decrease in the denominator in the second term. The simplicity of the above expression makes it quite useful in deriving new results, as it shall soon be seen.

D. Constant Power Adaptation

We now turn our attention to the particular class of constant-power adaptation algorithms. As mentioned before, $\log(1 + \mathrm{SNR})$ is more sensitive to SNR when SNR is low. So, it makes sense that the critical task in water-filling should be to ensure that low SNR subchannels are allocated the correct amount of power. In particular, those subchannels that would be allocated zero power in exact water-filling should not receive a positive power in an approximate water-filling algorithm, for otherwise, the power is almost wasted. This intuition allowed Chow [1] to observe that a constant-power allocation strategy, where the transmitter allocates zero power to subchannels that would receiver zero power in exact water-filling, but allocates *constant* power in subchannels that would receive positive power in exact water-filling is often close to the optimal. In this section, this intuition will be made precise using the gap bound derived before.

Consider the following class of constant-power allocation strategies where beyond a cut-off point, ν_0, all subchannels are allocated the same power:

$$S_k = \begin{cases} S_0 & \text{if } \nu_k \geq \nu_0 \\ 0 & \text{if } \nu_k < \nu_0 \end{cases} . \tag{20}$$

Here, the subchannels are assumed to be ordered so that $\nu_k \geq \nu_l$ whenever $k \leq l$. If the same cut-off point ν_0 is used as in exact water-filling, we have,

$$S_0 + \min_{\nu_k \geq \nu_0} \left(\frac{\sigma^2}{\nu_k} \right) \leq \frac{\sigma^2}{\nu_0} \leq \min_{\nu_k < \nu_0} \left(\frac{\sigma^2}{\nu_k} \right). \tag{21}$$

The first inequality is true because in the transmission band (i.e. when $\nu_k \geq \nu_0$) the constant-power allocation is an suboptimal strategy, therefore the minimal sum of power and (normalized) noise is less than the water level (σ^2/ν_0). The second inequality holds because the subchannels are ordered. Equation (21) allows us to replace the $\min_k\{S_k + \sigma^2/\nu_k\}$ term in the gap formula by $S_0 + \min_k\{\sigma^2/\nu_k\}$. In this case, (19) becomes:

$$\begin{aligned} \ln 2 \cdot \Gamma &= \sum_{k=1}^{m^*} p_k \left(\frac{S_0}{S_0 + \min_j\{\sigma^2/\nu_j\}} - \frac{S_0}{S_0 + \sigma^2/\nu_k} \right) \\ &= \sum_{k=1}^{m^*} p_k \frac{S_0 \left(\sigma^2/\nu_k - \min_j\{\sigma^2/\nu_j\}\right)}{\left(S_0 + \sigma^2/\nu_k\right)\left(S_0 + \min_j\{\sigma^2/\nu_j\}\right)} \\ &\leq \sum_{k=1}^{m^*} p_k \left(\frac{\sigma^2/\nu_k}{S_0 + \sigma^2/\nu_k} \right), \end{aligned} \tag{22}$$

where m^* denotes the number of channel states with positive power allocation. Notice that an immediate constant bound can be obtained by replacing $\dfrac{\sigma^2/\nu_k}{S_0 + \sigma^2/\nu_k}$ with 1. In this case, $\Gamma \leq 1/\ln 2 = 1.44$ bits/sec/Hz is an upper bound to the maximum capacity loss for constant-power allocation algorithms. But this is usually too loose to be of practical interest. Instead, we can simplify the notation using the fact that the number of bits allocated in each subchannel is given by $\log\left(1 + S_0\nu_k/\sigma^2\right)$. In this case, Γ can be written in a particularly simple form:

$$\Gamma \leq \frac{1}{\ln 2} \sum_{k=1}^{m^*} p_k 2^{-b_k}, \tag{23}$$

where b_k is the number of bits allocated in each subchannel. Note that b_k are not restricted to integer values in the above bound. Also note that the crucial assumption for the bound to hold is $\min_k\{S_k + \sigma^2/\nu_k\} =$

[3]When the power constraint is not satisfied with equality, $\bar{S}$ must be used in the second sum in (19) instead of $\sum p_k S_k$.

[4]Equation (19) can be used to show that integer-bit restriction costs at most $1/\ln 2$ bits/sec per Hz by noticing an integer bit allocation algorithm essentially doubles $S_k + \sigma^2/\nu_k$ in allocating each additional bit. Unfortunately, this bound is rather loose.

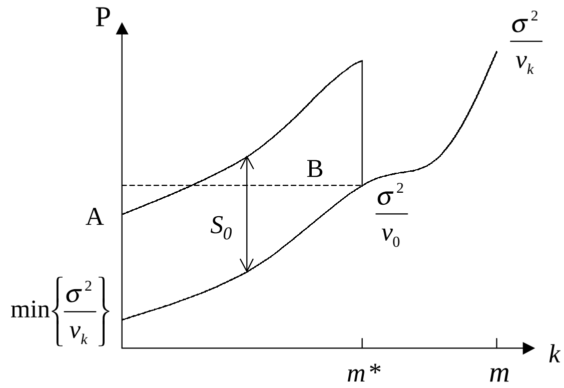

Fig. 1. Constant-Power Allocation

$S_0 + \min_k\{\sigma^2/\nu_k\}$. Having the same cut-off point as in exact water-filling is a sufficient but not necessary. Thus, we have the following theorem.

Theorem 2: For a constant-power allocation strategy of the form (20) that satisfies the power constraint with equality, if $\min_k\{S_k + \sigma^2/\nu_k\} = S_0 + \min_k\{\sigma^2/\nu_k\}$, then it is at most $\left(\sum_{k=1}^{m^*} p_k \cdot 2^{-b_k}/\ln 2\right)$ bits/sec/Hz away from the water-filling optimal, where the sum is over all m^* subchannels that are allocated S_0 amount of power, and b_k is the number of bits allocated in subchannel k, i.e. $b_k = \log(1 + S_0\nu_k/\sigma^2)$.

Fig. 1 illustrates the theorem graphically. As long as level A is lower than level B, the achievable rate is bounded by (23). Note that subchannels with low SNR (and hence low bit allocation) are precisely those contributing most to the bound, thus confirming the intuition that low SNR subchannels are the most sensitive to power mis-allocation.

III. Low Complexity Adaptation

The crucial condition in Theorem 2 is $\min_k\{S_k + \sigma^2/\nu_k\} = S_0 + \min_k\{\sigma^2/\nu_k\}$. This condition says that the bound is valid only if not too few subchannels are used. The condition is trivially satisfied, for example, by putting equal power in all subchannels. In that case, 2^{-b_k} will be nearly 1 for many subchannels, and the duality gap becomes large (although still bounded by the constant 1.44 bits/sec/Hz). Therefore, it is of interest to use as few subchannels as possible without violating the condition so as to simultaneously make the number of terms in the summation small, and make each individual term small (since fewer subchannels implies larger S_0, which in term implies smaller 2^{-b_k}.) This suggests that a simple power allocation strategy which sets the cut-off point to be the largest m^* that satisfies $S_0 + \sigma^2/\nu_1 \le \sigma^2/\nu_{m^*+1}$ is close to the optimal. Graphically, an algorithm that tries to find the smallest m^* so that level A is less than level B has the smallest duality gap. This fact is used to devise the following algorithm:

Algorithm 1: Assume that the channel gain ν_k's are ordered so that $\nu_1 \ge \nu_2 \cdots \ge \nu_m$. Let ν_0 be the cut-off point so that a constant power S_0 is allocated for all $\nu_k \ge \nu_0$. Let m^* be the largest k such that $\nu_k \ge \nu_0$. The following steps find the m^* with the smallest duality gap:
1. Set $m^* = m$.
2. Compute $S_0 = \bar{S}/\sum_{k=1}^{m^*} p_k$.
3. If $\sigma^2/\nu_{m^*+1} \ge S_0 + \sigma^2/\nu_1$, set $m^* = m^* - 1$, repeat step 2. Otherwise, set $m^* = m^* + 1$ and go to the next step.
4. Compute $b_k = \log(1 + S_0\nu_k/\sigma^2)$ for $k = 1, \cdots, m^*$. Then, $R = \sum_{k=1}^{m^*} p_k b_k$ is at most $\left(\sum_{k=1}^{m^*} p_k \cdot 2^{-b_k}/\ln 2\right)$ bits/sec/Hz away from capacity.

Two properties of this algorithm make it attractive. First, unlike most previous low complexity bit-loading methods (e.g. [1]), where the boundary point is found by finding the cut-off point that gives the highest data rate, this algorithm finds the optimal cut-off point without actually computing the data rate achieved in each step, and is therefore free of logarithmic operations. The most expensive operation in this algorithm is the single division in each step, thus making its complexity very low. Secondly, this algorithm has a provable worst-case performance bound as given by Theorem 2. Finally, we note that a binary search of the cut-off point can be used to further improve the algorithm's efficiency.

IV. Rayleigh Channel

The bound developed previously can be explicitly computed if channel fading statistics are known. In particular, for a Rayleigh fading channel, it can be shown that the constant-power adaptation strategy is only a small fraction of one bit away from capacity.

In a wireless channel where a large number of scatterers contribute to the signal at the receiver, application of the central limit theorem leads to a (zero-mean) complex Gaussian model for the channel response. The envelope of the channel response at any time instant has a Rayleigh distribution, whose square magnitude is exponentially distributed, $p_\nu(\nu) = \frac{1}{\Omega} \cdot e^{-\nu/\Omega}$, where Ω, the average channel gain, parameterizes the set of all Rayleigh distributions.

Fixing Ω, the constant-power control strategy is determined by the average power constraint, or alternatively by the cut-off value ν_0. The low complexity power allocation algorithm says that the constant power allocated in each state S_0 should be such that

$$S_0 + \min\left\{\frac{\sigma^2}{\nu}\right\} = \frac{\sigma^2}{\nu_0}. \tag{24}$$

The Rayleigh distribution has a non-zero probability for arbitrarily large amplitudes of ν, so the above reduces to $S_0 = \sigma^2/\nu_0$. Curiously, note that the constant-power allocation algorithm allocates a constant power S_0 to all subchannels that can support at least one bit/second/Hz with S_0.

Now, using the gap bound (22), the spectral efficiency for an optimal constant-power allocation with cut-off ν_0 is bounded within the following constant from capacity:

$$\Gamma(\nu_0) = \frac{1}{\ln 2}\int_{\nu_0}^{-\infty}\left(\frac{\sigma^2/\nu}{\sigma^2/\nu_0+\sigma^2/\nu}\right)\cdot\frac{1}{\Omega}\cdot e^{-\nu/\Omega}d\nu \quad (25)$$

By a change of variable $t = \nu/\Omega$ (and also $t_0 = \nu_0/\Omega$), define

$$f(t_0) = \int_{t_0}^{\infty}\frac{t_0 e^{-t}}{t+t_0}dt, \quad (26)$$

the duality gap can be expressed as

$$\Gamma(\nu_0) = \frac{1}{\ln 2}\cdot f\left(\frac{v_0}{\Omega}\right). \quad (27)$$

The authors are not aware of a closed-form expression for the integral. Numerical evaluation reveals that it has a single maximum occurring at about $t_0 = 0.39$, and the value of the maximum is about 0.1840. The duality gap is largest when the power constraint is such that the cut-off point $\nu_0 = 0.39\Omega$. In this worst case, the average data rate is 1.3631 bits/sec/Hz, and the duality gap is $0.1840/\ln 2 \leq 0.266$ bits/sec/Hz away from capacity. The following theorem summarizes the result.

Theorem 3: For a flat i.i.d. Rayleigh fading channel with perfect side information at the transmitter and the receiver, assuming infinite granularity on the channel state partition, a constant-power adaptation method should allocate S_0 to all subchannels that could support at least one bit with S_0, where S_0 is determined from the power constraint. In this case, the resulting spectral efficiency is at most 0.266 bits/sec/Hz away from capacity.

V. Simulation

Simulation results on the Rayleigh channel are now presented. The average channel gain (Ω) is chosen to be -10dB. In Fig. 2, the average spectral efficiencies of the exact water-filling and the low-complexity constant-power allocation are plotted against the average power constraint together with the duality-gap bound. The average power constraint is normalized by setting noise power σ^2=0dB. The two curves are indistinguishable. For Rayleigh channels, the constant-power allocation method performs even better than the bound suggests, and it has a truly negligible loss compared to the exact water-filling. Note that the constant-power allocation method is designed using the bound. So, while the bound could be loose, the algorithm designed using the bound works very well.

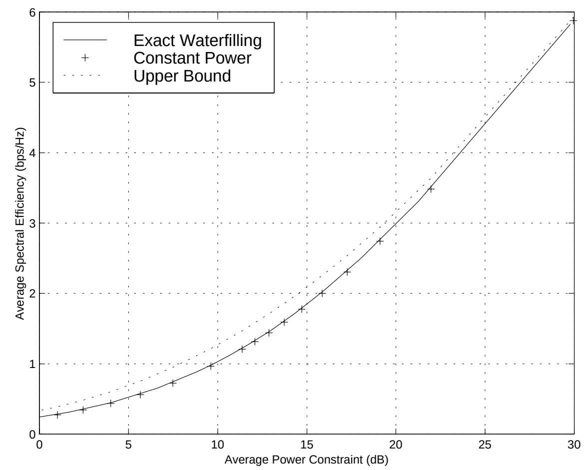

Fig. 2. Spectral efficiency of exact water-filling and constant-power allocation on Rayleigh channel with Ω=-10dB.

VI. Conclusion

Approximate power adaptation algorithms are investigated in this paper. A rigorous performance low bound for sub-optimal power allocation is derived. A very-low complexity constant-power adaptation method is proposed using the bound derived. The low-complexity algorithm has the desirable properties of having a provable worst-case performance and being logarithm-free. The performance bound is applied to Rayleigh fading channels, and it is shown that constant-power adaptive modulation is at most 0.266 bits/sec/Hz away from capacity. Simulation results suggest that the actual gap is even smaller.

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