Universal Relaying for the Interference Channel

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***Abstract*—This paper considers a Gaussian relay-interference channel and introduces a generalized hash-and-forward relay strategy, where the relay sends out a bin index of its quantized observation, and the receivers first decode the relay quantization codeword to a list, then use the list to help decode the respective messages from the transmitters. The main advantage of the proposed approach is in a scenario where the relay observes a linear combination of the transmitted signals and broadcasts a common relay message through a digital relay link of fixed rate to help both receivers of the interference channel. We show that when compared to the achievable rates with interference treated as noise, generalized hash-and-forward can provide one bit of rate improvement for every relay bit for both users at the same time in an asymptotic regime where the background noises go down to zero. The proposed approach is universal, in contrast to the compress-and-forward or amplify-and-forward strategies which are not asymptotically optimal for multiple users simultaneously, if at all.**

I. Introduction

Consider a classic Gaussian interference channel augmented by an independent relay node as shown in Fig. 1, where the relay observes a linear combination of the transmitted signals. Suppose further that the relay output is sent to both receivers via a common broadcast digital link of rate R_0 bits per channel use. In this paper, we are interested in designing universal relay strategies that can efficiently help the direct communications of both users in the interference channel at the same time.

This channel model is particularly interesting in the low noise regime, where the noise powers Z_1, Z_2 and Z_r tend to zero. In this case, the relay observation Y_r becomes a deterministic function of both (X_1, Y_1) and (X_2, Y_2). To see this, note that when the noises are zero, we have $Y_1 = h_{11}X_1 + h_{21}X_2$, $Y_2 = h_{22}X_2 + h_{12}X_1$ and $Y_r = g_1X_1 + g_2X_2$, in which case we can write

$$Y_r = \left(g_1 - \frac{h_{11}}{h_{21}}g_2\right)X_1 + \frac{g_2}{h_{21}}Y_1 \tag{1a}$$

$$Y_r = \left(g_2 - \frac{h_{22}}{h_{12}}g_1\right)X_2 + \frac{g_1}{h_{12}}Y_2 \tag{1b}$$

i.e., $Y_r = f_1(X_1, Y_1)$, and $Y_r = f_2(X_2, Y_2)$ for some deterministic functions f_1 and f_2.

For a discrete memoryless relay channel where the relay observation is a deterministic function of the direct channel input and output, Cover and Kim [1] showed a fundamental result that a digital relay link is cut-set-bound achieving, i.e. every relay bit provides one bit of rate improvement for the direct channel. Thus, for the Gaussian relay-interference channel shown in Fig. 1, when the relay observation Y_r is a deterministic function of X_1 and Y_1 (i.e. when the noises are zero), one would expect that a digital link of rate R_0 to be able to improve the direct communication rates between X_1 and Y_1 by exactly R_0, provided that X_2 is treated as noise.

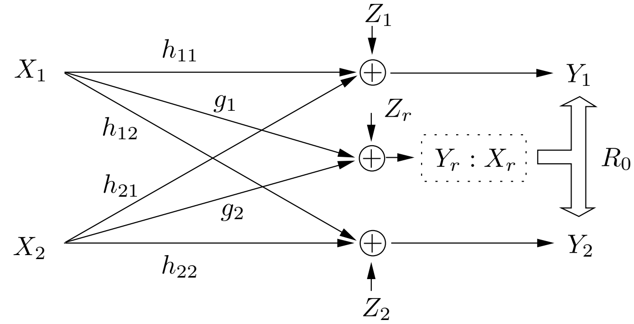

Fig. 1. A two-user Gaussian interference channel with a common broadcast digital relay link of rate R_0 to both receivers.

We now ask the following question: In the relay-interference channel shown in Fig. 1, as the relay observation Y_r is asymptotically a deterministic function of both (X_1, Y_1) and (X_2, Y_2) at the same time, can the *same* relay message improve the asymptotic achievable rates of both users of the interference channel by R_0 at the same time? The main result of this paper is that this is indeed the case. To establish this rigorously, we use a generalized hash-and-forward (GHF) strategy, and show that it is universal and asymptotically optimal for both users of the interference channel at the same time.

The proposed GHF strategy is motivated by the hash-and-forward (HF) strategy originally designed for a discrete memoryless deterministic relay channel in [1]. In fact, [1] proposes an entire range of relay strategies to attain the cut-set-bound achieving performance for the deterministic relay channel: a compress-and-forward (CF) strategy in which the relay uses Wyner-Ziv coding to quantize its observation, or an alternative HF strategy in which the relay simply hashes its observation and forwards a bin index to the receiver, or a combination of both.

The CF strategy, however, is unsuited as a universal relaying strategy, because in general, the two receivers in the interference channel have different side information. Consequently, different Wyner-Ziv codebooks need to be used for the different receivers in order to attain the optimal cut-set-bound achieving performance.

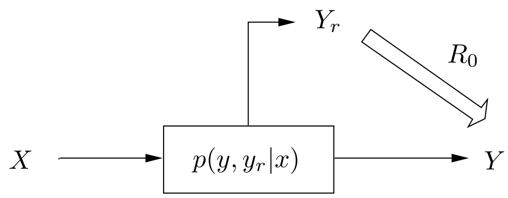

Fig. 2. A single-relay channel with digital relay link of rate R_0

The HF strategy, on the other hand, is universal, as it involves the direct binning of the relay observation. However, the HF strategy as proposed in [1] applies only to the deterministic channel. In [2], an extended hash-and-forward (EHF) strategy is proposed for the general nondeterministic case. This paper generalizes HF in a different direction, and proposes a decoding strategy called GHF. For the single-user relay channel, neither EHF nor GHF provides a higher rate than CF. For the relay-interference channel, however, GHF has the attractive property of being asymptotically optimal for two users at the same time in the low noise limit—something that is not possible with CF or amplify-and-forward (AF) strategies.

The relay-interference channel has been studied extensively in the literature [3]–[6], where interference-forwarding techniques involving binning and forwarding part of the interference to the receiver have emerged as a central theme. This paper shows that in addition to decoding part of the interference then forwarding, or compressing part of the interference then forwarding, it is sometimes advantageous to hash then forward. The latter strategy has the advantage of being universal with respect to more than one receiver.

II. Generalized Hash-and-Forward

Consider a three-terminal single-relay channel comprised of a source, a relay, and a destination node, where the relay can communicate to the destination using a digital link of rate R_0, as shown in Fig. 2. Denote the source signal as X, and the relay and destination observations as Y_r and Y, respectively. When the relay cannot decode the source codeword, a sensible relay strategy is to assist the destination by describing its observation at rate R_0 to the destination. A central question in the design of relay strategy is how such quantization should be performed?

In the classic CF scheme [7, Theorem 6], the relay observation is quantized using a Wyner-Ziv coding technique. In this case, the relay quantizes Y_r using an auxiliary random variable U then sends a bin index at rate R_0 to the destination, so that using side information Y, the destination can uniquely recover U then proceed to decode X with the help of U.

This paper proposes a different strategy in which we choose an auxiliary random variable U and provide its bin index to the destination. But unlike in CF, even with the use of side information Y, the destination cannot determine U uniquely, but only to a list $\mathcal{L}$. Nevertheless, the destination can still search through all source codewords by testing the joint typicality of each source codeword with the list $\mathcal{L}$, then decode a unique X.

For the single-relay channel, the above list decoding strategy gives no higher rate than CF. In other words, the optimal list size in the single-relay case is one. For the relay-interference channel, however, the above strategy opens up the possibility of using a single universal relay encoding function to serve two source-destination pairs at the same time. This universality is our primary motivation for introducing list decoding.

The proposed strategy is called generalized HF, because it generalizes the HF strategy of [1] which is cut-set-bound achieving for for a class of deterministic channels where $Y_r = f(X, Y)$. In this case, we can set $U = Y_r$. The HF strategy is then equivalent to the joint decoding of X and Y_r, because a successful decoding of X automatically produces Y_r when $Y_r = f(X, Y)$.

GHF generalizes HF in that we allow the conditional distribution of U to be of a general form, i.e. as $p(u|y_r)$. This provides considerable flexibility in the choice of U. In particular, for a relay-interference channel in the low noise limit, a single U can be asymptotically optimal for two source-destination pairs at the same time, i.e., every relay bit is worth one bit to two users simultaneously.

Theorem 1 (Achievable rate of GHF): For a memoryless relay channel defined by $p(y, y_r|x)$ and with a digital link of rate R_0 per channel use between the relay and the destination, the rate R is achievable provided that

$$R < I(X; Y) + R_0 - I(U; Y_r|X, Y), \tag{2}$$

for $(X, Y, Y_r, U) \sim p(x)p(y, y_r|x)p(u|y_r)$ such that

$$R_0 \leq I(U; Y_r|Y). \tag{3}$$

Remark 1: CF restricts the auxiliary random variable U to satisfy

$$R_0 \geq I(U; Y_r|Y). \tag{4}$$

This ensures that U is uniquely decoded at the destination. GHF works in the opposite regime, where the destination can only decode U to a list. In this case, the overall achievable rate also takes a penalty. In fact, only by choosing U such that $I(U; Y_r|Y) = R_0$, GHF reduces to CF rate, as shown below:

$$\begin{aligned} R &< I(X; Y) + I(U; Y_r|Y) - I(U; Y_r|X, Y) \\ &= I(X; Y) + h(U|Y) - h(U|X, Y) \\ &= I(X; Y, U) \end{aligned} \tag{5}$$

where the second equality is due to the Markov chair $U - Y_r - (X, Y)$. For any other U satisfying (3), the overall achievable rate would be lower. In fact, as it is clear from (2), the overall achievable rate R decreases as $I(U; Y_r|X, Y)$ increases, or as U more accurately represents Y_r.

However, the point of Theorem 1 is not that it provides a higher overall achievable rate (which it does not). Rather, Theorem 1 provides a computable achievable rate for all U's satisfying (3). This flexibility of being able to choose any of these U's is an advantage that will be apparent when the relay assists multiple destinations simultaneously.

Remark 2: For the class of deterministic relay channels considered in [1], where the relay observation $Y_r = f(X,Y)$ with f being deterministic, we have $I(U;Y_r|X,Y) = 0$. Thus, the rate $I(X;Y) + R_0$ is achievable using GHF whenever the relay quantization scheme satisfies (3). In particular, the GHF strategy reduces to HF if $U = Y_r$, in which case

$$\begin{aligned} R &< \min\{I(X;Y)+R_0, I(X;Y)+I(U;Y_r|Y)\} \\ &= \min\{I(X;Y)+R_0, I(X;Y)+h(Y_r|Y)\} \\ &= \min\{I(X;Y)+R_0, I(X;Y)+I(X;Y_r|Y)\} \\ &= \min\{I(X;Y)+R_0, I(X;Y,Y_r)\} \end{aligned} \tag{6}$$

where the second last equality is due to $Y_r = f(X,Y)$. This recovers the result of [1].

Remark 3: The rate expression for GHF in Theorem 1 is identical to that of EHF in [2]. However, the decoding strategy of GHF differs from that of EHF in that list decoding is performed on U rather than X. The rate expression in Theorem 1 can also be derived using a joint decoding strategy on (U,X) as shown in [8], [9]. All of these strategies share the common feature that unlike CF, U is not decoded exactly first.

Proof of Theorem 1: The source communicates to the destination at rate nR over B consecutive blocks, each of n symbols. For the last block, no message is transmitted. As $B \to \infty$, $nR(B-1)/B$ tends to nR.

Codebook Generation: Randomly and independently generate 2^{nR} codewords $X^n(w)$ of length n indexed by $w \in \{1,\dots,2^{nR}\}$ according to $\prod_{i=1}^n p(x_i)$. Fix a $p(u|y_r)$ such that $I(U;Y_r|Y) \geq R_0$. Randomly and independently generate $2^{n(I(Y_r;U)+\epsilon)}$ codewords $U^n(r)$, $r \in \{1,\dots,2^{n(I(Y_r;U)+\epsilon)}\}$ of length n according to $\prod_{i=1}^n p(u_i)$. We shall also need a random partition of the U^n codewords into bins. Randomly partition the set $\{1,2,\dots,2^{n(I(Y_r:U)+\epsilon)}\}$ into 2^{nR_0} bins $\mathcal{B}_l, l \in \{1,\dots,2^{nR_0}\}$ each of size $2^{n(I(Y_r:U)-R_0+\epsilon)}$.

Encoding: In block i, the source sends $X^n(w_i)$. Having observed $Y_r^n(i-1)$ in block $i-1$, the relay finds a codeword $U^n(t_i)$, $t_i \in \{1,\dots,2^{n(I(Y_r;U)+\epsilon)}\}$, such that $(U^n(t_i), Y_r^n(i-1))$ is ϵ-strongly typical (see [10, Section 13.6] for definition of strong typicality). The relay sends k, the bin index of t_i over the digital channel to the destination in block i, (i.e. $t_i \in \mathcal{B}_k$).

Decoding: In block i, the destination decodes the source message of block $i-1$ in following steps:

1) Upon receiving k, the destination forms an index list $\mathcal{L}$ of possible U^n-codewords by identifying indices $r \in \mathcal{B}_k$ such that $(U^n(r), Y^n(i-1)$ are ϵ-strongly typical.
2) Destination finds a source codeword that is consistent with its own observation Y^n and $\mathcal{L}$ by finding $\hat{w} \in \{1,\dots,2^{nR}\}$ such that the three-tuple $(X^n(\hat{w}), U^n(m), Y^n(i-1))$ is ϵ-strongly typical for some $m \in \mathcal{L}$.

Analysis of Probability of Error: Because of symmetry, we can assume that $X^n(1)$ is sent over all blocks. Since decoding events in different blocks are independent, we can also focus on block i to analyze probability of error, and drop the time indices. The error events are as follows:

E_1: $(X^n(1), Y_r^n, Y^n)$ is not ϵ-strongly typical.
E_2: $\nexists t \in \{1,\dots,2^{n(I(U;Y_r)+\epsilon}\}$ such that $(U^n(t), Y_r^n)$ is ϵ-strongly typical.
E_3: $\nexists r \in \mathcal{B}_k$ such that $(U^n(r), Y^n)$ is ϵ-strongly typical, i.e., $\mathcal{L}$ is empty.
E_4: $\nexists s \in \mathcal{L}$ such that $\big(X^n(1), Y^n, U^n(s)\big) \in A_\epsilon^{*n}$.
E_5: $\exists m, w' : m \in \mathcal{L}, w' \in \{1,\dots,2^R\}$, $w' \neq 1$, such that $(X^n(w'), U^n(m), Y^n) \in A_\epsilon^{*n}$,

where A_ϵ^{*n} denotes the set of ϵ-strongly typical three-tuples.

For n sufficiently large, $P(E_1) \leq \epsilon$ for arbitrarily small $\epsilon > 0$ [10, Lemma 10.6.1]. Following the argument of [10, Section 10.6], $P(E_2 \cap E_1^c) \leq \epsilon$ for sufficiently large n, since the number of U^n codewords is more than $2^{nI(U;Y_r)}$. Similarly, $P(E_3 \cap E_2^c \cap E_1^c) < \epsilon$ for sufficiently large n, provided that

$$I(U;Y_r) - R_0 \geq I(U;Y) \tag{7}$$

or equivalently, since $I(U;Y_r) - I(U;Y) = I(U;Y_r|Y)$ by the Markov chain $U - Y_r - Y$,

$$R_0 \leq I(U;Y_r|Y). \tag{8}$$

Assuming that E_1 does not occur, by the Markov Lemma [10, Lemma 15.8.1], since $(X,Y) - Y_r - U$ forms a Markov chain and $(X^n, Y^n, Y_r^n) \in A_\epsilon^{*n}$, $P(E_4 \cap \bigcap_{j=1}^3 E_j) < \epsilon$ for sufficiently large n.

To bound the probability of E_5, note that for $X^n(m)$ drawn i.i.d. $\sim \prod p(x_i)$ and independent of ϵ-strongly typical pair $(U^n(m), Y^n)$, the probability that $(X^n(m), U^n(m), Y^n) \in A_\epsilon^{*n}$ is less than $2^{-n(I(X;Y,U)-\epsilon)}$ for sufficiently large n and arbitrarily $\epsilon > 0$ [10, Lemma 10.6.2]. Let A be the event that $(X^n(w'), U^n(m), Y^n) \in A_\epsilon^{*n}$ for some $m \in \mathcal{L}$ and $w' \in \{1,\dots,2^{nR}\}, w' \neq 1$, assuming that E_i does not occur for $i = 1, \cdots, 4$. We have

$$\begin{aligned} P\left(\bigcap_{j=1}^5 E_j\right) &= P(A) \\ &= \sum_l P\Big(A \Big| \|\mathcal{L}\| = l\Big) P\Big(\|\mathcal{L}\| = l\Big) \\ &\leq \sum_l P\big(\|\mathcal{L}\| = l\big) \sum_{m \in \mathcal{L}, w'} 2^{-n(I(X;Y,U)-\epsilon)} \\ &= \sum_l P\big(\|\mathcal{L}\| = l\big) \cdot l \cdot 2^{nR} \cdot 2^{-n(I(X;Y,U)-\epsilon)} \\ &= 2^{nR} 2^{-n(I(X;Y,U)-\epsilon)} \mathbb{E}\|\mathcal{L}\|, \end{aligned} \tag{9}$$

where $\|\mathcal{L}\|$ represents the cardinality of $\mathcal{L}$.

Now, the method employed in [7, Lemma 3] can be used to find an upper bound on $\mathbb{E}\|\mathcal{L}\|$. Recall that $\mathcal{L}$ is the list of $U^n(r)$ codewords with $r \in \mathcal{B}_k$ and (U^n, Y) ϵ-strongly typical. Let

$$\psi(r|Y^n) = \begin{cases} 1 & (U^n(r), Y^n) \text{ is } \epsilon\text{-strongly typical}, \\ 0 & \text{otherwise}. \end{cases}$$

Then, $\|\mathcal{L}\|$ can be expressed as:

$$\|\mathcal{L}\| = \sum_{r \in \mathcal{B}_k} \psi(r|Y^n). \tag{10}$$

We have

$$\begin{aligned}\mathbb{E}\|\mathcal{L}\| &= \mathbb{E}\psi(t|Y^n) + \sum_{r\neq t, r\in\mathcal{B}_k} \mathbb{E}\psi(r|Y^n)\\ &= P\Big(\psi(t|Y^n)=1\Big) + \sum_{r\neq t, r\in\mathcal{B}_k} P\Big(\psi(r|Y^n)=1\Big)\\ &\overset{(*)}{\leq} 1 + (2^{\|B_k\|}-1)2^{-n(I(U;Y)-\gamma)}\\ &\leq 1 + 2^{n(I(U;Y_r)-R_0-I(U;Y)+\epsilon+\gamma)}\\ &= 1 + 2^{n(I(U;Y_r|Y)-R_0+\epsilon+\gamma)}, \end{aligned} \tag{11}$$

where $(*)$ follows from [10, Lemma 10.6.2] for sufficiently large n and arbitrarily small $\gamma > 0$.

Combining (9) and (11), along with (8), gives us the following criteria for the probability of error to asymptotically approach zero for large n:

$$R < I(X;Y,U) \tag{12}$$

$$R < I(X;Y,U) - (I(U;Y_r|Y) - R_0) \tag{13}$$

$$R_0 \leq I(U;Y_r|Y). \tag{14}$$

Note that (13) and (14) imply (12), so (12) is redundant. Further, (13) can be simplified as follows

$$\begin{aligned} R &< I(X;Y) + I(X;U|Y) + R_0 - I(U;Y_r|Y)\\ &\overset{(a)}{=} I(X;Y) + I(X;U|Y) + R_0 - h(U|Y) + h(U|Y_r)\\ &\overset{(b)}{=} I(X;Y) + R_0 - \big(h(U|X,Y) - h(U|Y_r,X,Y)\big)\\ &= I(X;Y) + R_0 - I(U;Y_r|X,Y), \end{aligned}$$

where (a) and (b) follow from the Markov chain $U - Y_r - (X,Y)$. This proves (2). ■

III. Universal Relaying for the Gaussian Relay-Interference Channel

The real advantage of GHF lies in a relay-interference channel, where a single relay simultaneously assists more than one receivers. The CF strategy requires the auxiliary random variable U to be decoded uniquely at the receiver at the first step. Because the two receivers in the interference channel may have different side information for Wyner-Ziv decoding, the optimal U for different receivers are in general different. In contrast, GHF provides the possibility of quantizing the relay observation with a single U, which, although not optimal for either receiver at any finite noise level, becomes optimal for both receivers asymptotically as noises go down to zero (i.e. as the channel becomes deterministic).

Consider a Gaussian interference channel as shown in Fig. 1

$$Y_1 = h_{11}X_1 + h_{21}X_2 + Z_1 \tag{15a}$$

$$Y_2 = h_{12}X_1 + h_{22}X_2 + Z_2, \tag{15b}$$

where the relay observes

$$Y_r = g_1X_1 + g_2X_2 + Z_r \tag{16}$$

and has a common broadcast digital relay link of rate R_0 to both Y_1 and Y_2, where the noises Z_1, Z_2, Z_r are i.i.d. Gaussian random variables with zero mean and variance N, and where the powers of X_1 and X_2 are constrained to be less than P_1 and P_2 respectively.

To avoid the case where the relay observation degenerates into one of Y_1 and Y_2, we further assume

$$\frac{g_1}{g_2} \neq \frac{h_{11}}{h_{21}}, \qquad \frac{g_1}{g_2} \neq \frac{h_{12}}{h_{22}}. \tag{17}$$

We compare the rate improvement due to the relay against the baseline rates where independent Gaussian codebooks with $X_1 \sim \mathcal{N}(0,P_1)$, $X_2 \sim \mathcal{N}(0,P_2)$ are used at the transmitters and where interference is always treated as noise:

$$R_1 = I(X_1;Y_1) = \frac{1}{2}\log\left(1 + \frac{h_{11}^2P_1}{h_{21}^2P_2+N}\right) \tag{18a}$$

$$R_2 = I(X_2;Y_2) = \frac{1}{2}\log\left(1 + \frac{h_{11}^2P_1}{h_{21}^2P_2+N}\right) \tag{18b}$$

The main result of this section is that using a common digital relay link of rate R_0, GHF can provide R_0 bits of rate improvement to both users at the same time asymptotically as $N \to 0$, while h_{ij}, g_i and P_i are kept fixed. This is in contrast to CF or AF strategies which do not provide the maximal R_0 bits of rate improvement.

A. Rate Improvement using GHF

We evaluate the achievable rate using GHF by assuming a joint Gaussian auxiliary variable $U = Y_r + \eta$ with $\eta \sim \mathcal{N}(0,Q)$. By Theorem 1, the following rates are achievable using GHF

$$R_1 = I(X_1;Y_1) + R_0 - I(U;Y_r|X_1,Y_1); \tag{19a}$$

$$R_2 = I(X_2;Y_2) + R_0 - I(U;Y_r|X_2,Y_2). \tag{19b}$$

subject to the condition that $p(u|y_r)$ must satisfy

$$R_0 \leq I(U;Y_r|Y_1); \tag{20a}$$

$$R_0 \leq I(U;Y_r|Y_2). \tag{20b}$$

First, we evaluate

$$\begin{aligned} &I(U;Y_r|X_1,Y_1)\\ &\quad= h(U|X_1,Y_1) - h(U|Y_r)\\ &\quad= \frac{1}{2}\log\left(2\pi e(\sigma_{\tilde{U}}^2 - \sigma_{\tilde{U}\tilde{Y}_1}(\sigma_{\tilde{Y}_1}^2)^{-1}\sigma_{\tilde{Y}_1\tilde{U}})\right) - \frac{1}{2}\log(2\pi eQ)\\ &\quad= \frac{1}{2}\log\left(1 + \frac{\left((g_2^2+h_{21}^2)P_2+N\right)N}{(h_{21}^2P_2+N)Q}\right) \end{aligned} \tag{21}$$

where $\tilde{U} = g_2X_2 + Z_r$ and $\tilde{Y}_1 = h_{21}X_2 + Z_1$. Similarly,

$$I(U;Y_r|X_2,Y_2) = \frac{1}{2}\log\left(1 + \frac{\left((g_1^2+h_{12}^2)P_1+N\right)N}{(h_{12}^2P_1+N)Q}\right) \tag{22}$$

We also need to evaluate

$$\begin{aligned} I(U;Y_r|Y_1) &= h(U|Y_1) - h(U|Y_r)\\ &= \frac{1}{2}\log\left(2\pi e(\sigma_U^2 - \sigma_{UY_1}(\sigma_{Y_1}^2)^{-1}\sigma_{Y_1U})\right) - \frac{1}{2}\log(2\pi eQ)\\ &= \frac{1}{2}\log\left(1 + \frac{(g_1h_{21}-g_2h_{11})^2P_1P_2 + c_1N}{(h_{11}^2P_1+h_{21}^2P_2+N)Q}\right) \end{aligned} \tag{23}$$

where $c_1 = (g_1^2 + h_{11}^2)P_1 + (g_2^2 + h_{21}^2)P_2 + N$. Similarly,

$$I(U; Y_r|Y_2) = \frac{1}{2}\log\left(1 + \frac{(g_1h_{22} - g_2h_{12})^2 P_1P_2 + c_2N}{(h_{12}^2P_1 + h_{22}^2P_2 + N)Q}\right) \tag{24}$$

where $c_2 = (g_1^2 + h_{12}^2)P_1 + (g_2^2 + h_{22}^2)P_2 + N$.

For any fixed R_0, we can always find Q that satisfies both constraints in (20). In addition, as long as (17) is satisfied, we can find values for Q that satisfy (20) and do not go to zero as $N \to 0$. In fact, the largest GHF rate is obtained by choosing Q to satisfy the more stringent of (20) with equality. In this case, by (23) and (24), the limiting value of Q as $N \to 0$ is

$$Q = \frac{1}{2^{2R_0} - 1}\min\{a, b\}, \tag{25}$$

where

$$a = \frac{(g_1h_{21} - g_2h_{11})^2 P_1P_2}{(h_{11}^2P_1 + h_{21}^2P_2)}, \tag{26a}$$

$$b = \frac{(g_1h_{22} - g_2h_{12})^2 P_1P_2}{(h_{12}^2P_1 + h_{22}^2P_2)}. \tag{26b}$$

This value for Q is a constant bounded away from zero.

Substituting this Q into (21) and (22), we see that as $N \to 0$, both $I(U; Y_r|X_1, Y_1)$ and $I(U; Y_r|X_2, Y_2)$ vanish, implying that GHF achieves exactly R_0 bits in rate improvement for both users at the same time.

B. Rate Improvement using CF

For a CF strategy to help both receivers in an interference channel at the same time, we need a single U that is uniquely decodable at both receivers. In this case, the achievable rate using CF is

$$R_1 = I(X_1; Y_1, U) \tag{27a}$$

$$R_2 = I(X_2; Y_2, U) \tag{27b}$$

subject to the condition that $p(u|y_r)$ must satisfy

$$R_0 \geq I(U; Y_r|Y_1); \tag{28a}$$

$$R_0 \geq I(U; Y_r|Y_2). \tag{28b}$$

The key observation here is that for a general relay-interference channel, $I(U; Y_r|Y_1) \neq I(U; Y_r|Y_2)$. Thus, we must set U to be such that

$$R_0 = \max\{I(U; Y_r|Y_1), I(U; Y_r|Y_2)\}. \tag{29}$$

Without loss of generality, assume that $I(U; Y_r|Y_1) < I(U; Y_r|Y_2)$. Then for R_2, CF is asymptotically optimal. This is because $R_0 = I(U; Y_r|Y_2)$, so CF and GHF are equivalent as shown in (5), i.e. both provide R_0 bits of rate improvement as $N \to 0$. However, this is not so for R_1. Let

$$R_\Delta = |I(U; Y_r|Y_1) - I(U; Y_r|Y_2)|. \tag{30}$$

Then, the CF rate for R_1 can be computed as

$$\begin{aligned} R_1 &= I(X_1; Y_1, U) \\ &= I(X_1; Y_1) + I(U; Y_r|Y_1) - I(U; Y_r|X_1, Y_1) \\ &= I(X_1; Y_1) + R_0 - R_\Delta - I(U; Y_r|X_1, Y_1) \end{aligned} \tag{31}$$

where the second equality follows from (5). Assume again a Gaussian Wyner-Ziv quantizer $U = Y_r + \eta$ where $\eta \sim \mathcal{N}(0, Q)$. When (17) is satisfied, we can find Q that satisfies (28) and is bounded away from zero, as can be seen from (24). Thus, the $I(U; Y_r|X_1, Y_1)$ term in (31) vanishes as $N \to 0$, as computed in (21). Therefore asymptotically, CF achieves a rate improvement of $R_0 - R_\Delta$. This is not the maximal possible rate improvement unless $R_\Delta = 0$.

We can in fact quantify the asymptotic gap by noting that the asymptotic value of Q can be computed as

$$Q = \frac{1}{2^{2R_0} - 1}\max\{a, b\}, \tag{32}$$

where a and b are as defined in (26). Again, assuming $I(U; Y_r|Y_1) < I(U; Y_r|Y_2)$, then $a < b$. It follows that

$$R_\Delta = \frac{1}{2}\log\left(1 + \frac{b + a}{\frac{1}{2^{2R_0}-1}b + a}\right) \tag{33}$$

asymptotically as $N \to 0$.

To summarize, because CF requires the auxiliary random variable to be decodable uniquely at both receivers, the relay quantizer has to be designed for the receiver with the worse side information. This introduces inefficiency at the other receiver. In contrast, GHF provides flexibility in the choice of auxiliary random variable. It allows the relay quantizer to be designed for the receiver with the better side information. This quantizer also happens to be asymptotically optimal for the other receiver, thus giving us the universality of GHF.

C. Rate Improvement using AF

Finally, consider the AF relay strategy. For fair comparison, we assume that the relay link is now an orthogonal analog broadcast channel from the relay to the two receivers with noises $Z'_1 \sim \mathcal{N}(0, 1)$ and $Z'_2 \sim \mathcal{N}(0, 1)$. The power constraint at the relay P is such that

$$R_0 = \frac{1}{2}\log(1 + P). \tag{34}$$

In AF, the relay simply amplifies its observation Y_r by a factor λ and sends it through the analog broadcast channel. To satisfy the power constraint at the relay, λ is set as

$$\lambda = \sqrt{\frac{P}{\mathbb{E}Y_r^2}} = \sqrt{\frac{2^{2R_0} - 1}{g_1^2P_1 + g_2^2P_2 + N}}. \tag{35}$$

The achievable rate for the first user is then given by

$$\begin{aligned} R_1 &= I(X_1; Y_1, \lambda Y_r + Z'_1) \\ &= \frac{1}{2}\log\frac{\begin{vmatrix} h_{11}^2P_1 + h_{21}^2P_2 + N & \lambda(h_{11}g_1P_1 + h_{21}g_2P_2) \\ \lambda(h_{11}g_1P_1 + h_{21}g_2P_2) & \lambda^2(g_1^2P_1 + g_2^2P_2 + N) + 1 \end{vmatrix}}{\begin{vmatrix} h_{21}^2P_2 + N & \lambda h_{21}g_2P_2 \\ \lambda h_{21}g_2P_2 & \lambda^2(g_2^2P_2 + N) + 1 \end{vmatrix}} \end{aligned}$$

where X_2 is treated as noise. The rate improvement is now given by $R_1 - I(X_1; Y_1)$. A similar expression can be found for R_2. It can be shown numerically that the rate improvement is always strictly less than R_0, even as $N \to 0$. This is because AF does not digitize the relay link, hence is always affected by the noise in it.

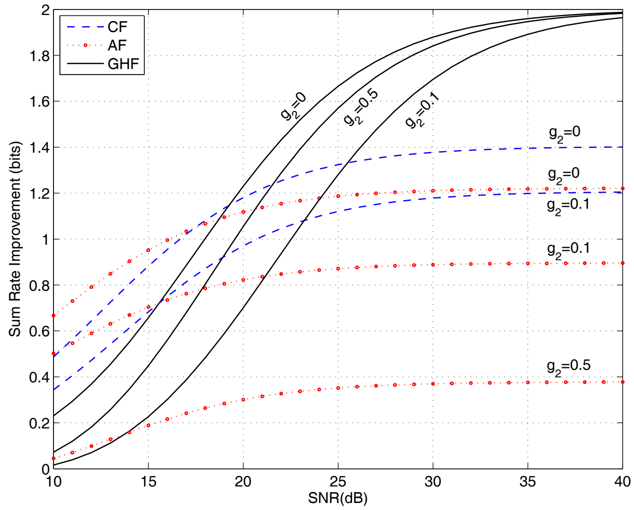

Fig. 3. Sum rate improvement of GHF, CF, and AF as a function of SNR for $R_0 = 1$, $P = 10$, $h_{11} = h_{22} = 1$, $h_{12} = h_{21} = 0.5$, $g_1 = 0.5$, and various values for g_2. The GHF and CF curves are identical for $g_2 = 0.5$.

IV. Numerical Example

Fig. 3 shows a numerical example of the sum rate improvements using GHF, CF, and AF, as a function of the signal-to-noise ratio (SNR), defined as $10\log\left(h_{11}^2 P/N\right)$, for a relay-interference channel with $R_0 = 1$, $P = 10$, $h_{11} = h_{22} = 1$, $h_{12} = h_{21} = 0.5$, $g_1 = 0.5$, and with various values for g_2.

When the channel is symmetric, i.e., $g_2 = g_1 = 0.5$, GHF and CF are identical. Both are asymptotically optimal, i.e., the rate for each user is asymptotically improved by R_0. As asymmetry is introduced, e.g., for $g_2 = 0.1$ and $g_2 = 0$, GHF remains asymptotically optimal, while CF fails to be so.

When $g_2 = 0.1$, we have $Y_r = 0.5X_1 + 0.1X_2 + Z_r$. Thus, Y_r has a higher correlation with $Y_1 = X_1 + 0.5X_2 + Z_1$ as compared to $Y_2 = X_2 + 0.5X_1 + Z_2$. In this case, it is easier for the relay to describe its observation to Y_1 than to Y_2 using Wyner-Ziv coding. But, a CF strategy designed for both Y_1 and Y_2 cannot take advantage of the better correlation at Y_1. It has to accommodate the user with worse side information, as otherwise, the relay quantization codeword would not be decodable at Y_2. In fact, a CF strategy designed for both Y_1 and Y_2 largely helps Y_2. The rate of the first user is improved asymptotically by about 0.2 bits, while the rate of the second user is improved asymptotically by 1 bit. In contrast, by taking advantage of list decoding which does not require unique decoding of the relay quantization codeword, GHF is able to asymptotically improve the rates of both users by 1 bit even when the channel is asymmetric.

An interesting scenario is the case of $g_2 = 0$, where the relay observes X_1 only. Since the second user treats X_1 as noise, the ideal relay strategy for the second user is a CF strategy that describes the interference. The first user, however, would likely benefit more from a decode-and-forward (DF) strategy, since the relay observes X_1 with no interference. Conventional CF and DF strategies clearly cannot be implemented simultaneously by the same relay. The quantize-then-hash strategy of GHF resolves this tension. Asymptotically, it appears like a DF strategy to the first user, while simultaneously appearing like a CF strategy to the second user. As Fig. 3 shows, GHF asymptotically improves the rates of both users by 1 bit.

Fig. 3 also shows the rate improvement due to AF. Asymptotically at high SNR, AF is inferior to both CF and GHF.

V. Concluding Remarks

This paper proposes a relay strategy called generalized hash-and-forward, which combines the features of CF and HF relay strategies. Just as in CF, the relay in GHF quantizes its observation then sends out a bin index. But unlike in CF, the destination in GHF does not decode the relay quantization codeword uniquely first, but only to a list. A joint decoding strategy is then used to recover the source codeword.

A key advantage of GHF is that it provides flexibility in the design of the relay quantizer. This makes GHF particularly suited for the relay-interference channel considered in this paper where a single relay needs to assist two users simultaneously. It is shown that in an asymptotic low-noise regime, when compared to the achievable rates with interference treated as noise, GHF can provides one bit of rate improvement for every relay bit for both users at the same time.

As a final remark, we note that the assumption that interference is always treated as noise is crucial for the above result to hold. Treating interference as noise is, however, not always optimal. This is particularly so when the noise powers approach zero, in which case introducing common message decoding in a Han-Kobayashi strategy [11] can be quite beneficial. Nevertheless, the result of this paper is still useful for the vast number of practical situations, where private messages only are transmitted.

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