Grassmannian Beamforming for MIMO Amplify-and-Forward Relaying

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***Abstract*— In this paper, we consider the beamforming codebook design problem for the half-duplex MIMO amplify-and-forward relay channel with Rayleigh fading. The analysis is divided into two steps. First, we present the optimal beamforming scheme with full channel state information (CSI) and derive the optimal source and relay beamforming vectors. Next, we consider the beamforming problem with receiver CSI only and provide a beamforming vector quantization scheme. Based on the statistics of the optimal beamforming vectors, we show that Grassmannian codebooks minimize the upper bound for SNR loss caused by quantization, and therefore these codebooks are appropriate choices for quantizing the optimal beamforming vectors. The efficiency of the Grassmannian codebooks is verified by simulation results.**

***Index Terms*— Amplify-and-forward relaying, Beamforming, Grassmannian codebooks.**

I. Introduction

It is well established that relaying techniques provide considerable advantages over direct transmission, provided that the source and relay cooperate efficiently. The capacity and reliability of the relay channel can be further improved by using multiple antennas at the nodes. The benefits of relaying combined with the advantages of multiple antennas make the multiple-input multiple-output (MIMO) relaying technique a powerful candidate for implementation in the next generation of wireless networks.

In the point-to-point MIMO channel, beamforming schemes can be used to maximize the reliability of the wireless links. In these schemes, known as maximum ratio transmission and receive (MRT-MRC) systems [1], the source maps its symbol to the dominant right singular vector of the channel matrix to maximize the received signal-to-noise ratio (SNR). Clearly, this scheme requires channel state information (CSI) at the source, which is unrealistic in practice. In a practical scenario, the destination must quantize the channel state information and send it back to the source via a rate-limited feedback channel.

Although general-purpose quantizers may be used to quantize each entry of a MIMO channel matrix individually, such a scheme does not preserve the structure of the beamforming vector and would require a large number of feedback bits [2]. A more efficient method is to share a beamforming codebook between the source and the destination, so that the destination can send back the label of the appropriate beamforming vector. These schemes are generally referred to as "limited-feedback" schemes. For Rayleigh fading channels, the optimal beamforming vector has been shown to be uniformly distributed on the unit sphere. Based on this observation, the beamforming codebook design problem has been shown to be equivalent to the Grassmannian line packing problem, which tries to maximize the minimum angle between a fixed number of unit vectors [3,4,5].

In this paper, we consider the problem of beamforming codebook design for half-duplex amplify-and-forward (AF) MIMO relay channels. A general information-theoretic analysis of the MIMO relay channel has been presented in [6]. Although an efficient signaling through the relay channel requires a full-duplex relay with specific processing capabilities (e.g. encoding/decoding), AF schemes are still attractive due to their lower implementation complexity. Moreover, the full-duplex assumption cannot be realized by the current technology, as the input and output signals need to be separated in time or frequency at the relay. For these reasons, this paper focuses on the half-duplex AF relay system.

The half-duplex MIMO AF relay channel has been considered in [7-10], where the authors optimize the source transmission covariance matrix and relay weighting matrix to maximize the instantaneous rate of the channel. Our approach is different from these papers in two major aspects: 1) we focus on the beamforming problem with the objective of received SNR maximization instead of rate maximization; 2) we consider a "limited-feedback" scenario, while the above mentioned papers assume either full CSI or no CSI at the source and/or relay.

The analysis in this paper starts by assuming full CSI at the nodes and deriving the optimal beamforming scheme. It is shown that the optimal relay beamforming vector is the dominant right singular vector of the relay-destination channel matrix, which is uniformly distributed on the unit sphere for Rayleigh fading channel matrix, and therefore Grassmannian codebook should be used for quantizing the relay beamforming vector. For the source beamforming vector, we derive the optimization problem that characterizes the optimal vector. Although this problem does not appear to have an analytic solution, we are able to show that for Rayleigh fading channels, the solution to this problem is also uniformly distributed on the unit sphere, based on which, the appropriateness of the Grassmannian quantizer can be shown analytically. Hence, at both source and relay nodes, the Grassmannian codebooks are proven to be appropriate choices for quantizing the optimal beamforming vectors of the half-duplex AF MIMO relay channel.

It should be noted that, throughput this paper, we assume perfect CSI at the receiver sides of the links, i.e. the relay knows the source-relay channel and the destination knows relay-destination and source-destination channels perfectly. We further assume that the destination knows the source-relay channel (see Section III). The case where such a knowledge is not available at the destination is considered in the full-length version of this paper [12].

The remainder of this paper is organized as follows. In Section II, we present the system model and derive the optimal beamforming scheme with full CSI assumption. Given the limited space available, the proof of optimality can be found in [12]. Section III considers the problem of beamforming codebook design and proves the appropriateness of Grassmannian codebooks. The simulation results are presented in Section IV. Finally, Section V concludes the paper.

Notations: Bold upper case and lower case letters denote matrices and vectors. $\mathbf{I}$ is the identity matrix. $|\cdot|$ and $\|\cdot\|$ denote the absolute value of a scalar and the Euclidean norm of a vector. $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. $(\cdot)^H$ denotes the Hermitian of a matrix. $\mathbb{C}$ denotes the set of complex numbers. The unit sphere of dimension m is defined as $\Omega_m = \{\mathbf{w} \in \mathbb{C}^m | \|\mathbf{w}\| = 1\}$. The chordal distance of any two unit vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ is defined as $d(\mathbf{w}_1, \mathbf{w}_2) = \sqrt{1 - |\mathbf{w}_1^H \mathbf{w}_2|^2}$. The notation $\mathbf{C}(N, \delta)$ denotes a set of N unit vectors with minimum chordal distance of δ. $\mathcal{CN}(0, \mathbf{\Sigma})$ represents a circularly symmetric complex Gaussian distribution with zero mean and covariance matrix $\mathbf{\Sigma}$. Finally, $\mathrm{E}\{\cdot\}$ denotes the expectation operation.

II. MIMO Amplify and Forward Relay Channel with Full CSI

Consider the half-duplex MIMO relay channel model in Fig. 1. The source, relay, and destination are equipped with m, n, and l antennas, respectively. The matrices $\sqrt{P_0}\mathbf{H}_0 \in \mathbb{C}^{l\times m}$, $\sqrt{P_1}\mathbf{H}_1 \in \mathbb{C}^{n\times m}$ and $\sqrt{P_2}\mathbf{H}_2 \in \mathbb{C}^{l\times n}$ model the flat fading channels of the source-destination, source-relay, and relay-destination links, respectively. The coefficients P_0, P_1, and P_2 denote source-destination, source-relay, and relay-destination link SNR's.

In the first time slot, the source uses the beamforming vector $\mathbf{s}$ to map the input symbol x_{in} to its antennas; the relay and destination are in the receiving mode. In the second time slot, the source remains silent and the relay multiples the vector received in the first slot by the weighting matrix $\mathbf{W} \in \mathbb{C}^{n\times n}$ and sends it to destination. The destination uses combining vectors $\mathbf{r}_0$ and $\mathbf{r}_1$ to recover two versions of input symbol that are separated in time:

$$
\begin{aligned}
y_0 &= \sqrt{P_0}\mathbf{r}_0^H \mathbf{H}_0 \mathbf{s} x_{in} + \mathbf{r}_0^H \mathbf{z}_0 \\
y_1 &= \sqrt{P_1 P_2}\mathbf{r}_1^H \mathbf{H}_2 \mathbf{W} \mathbf{H}_1 \mathbf{s} x_{in} + \mathbf{r}_1^H \left(\sqrt{P_2}\mathbf{H}_2 \mathbf{W} \mathbf{z}_1 + \mathbf{z}_2\right),
\end{aligned}
$$

where $\mathbf{z}_0$, $\mathbf{z}_1$, and $\mathbf{z}_2$ are the destination and relay input noise vectors all distributed according to $\mathcal{CN}(0, \mathbf{I})$. By properly

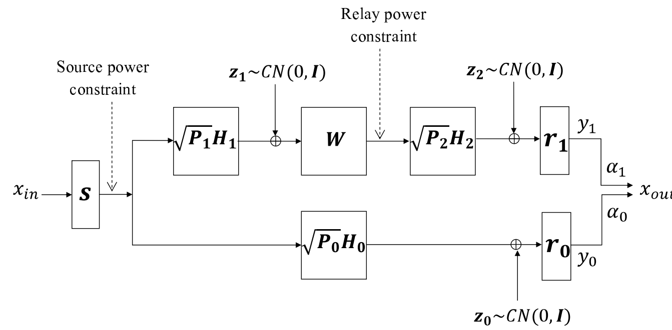

Fig. 1. Half-duplex MIMO AF relay channel model.

choosing the combining coefficients α_0 and α_1, the total received SNR is the summation of the SNR values of the symbols y_1 and y_0:

$$
\gamma = \frac{P_1 P_2 \left|\mathbf{r}_1^H \mathbf{H}_2 \mathbf{W} \mathbf{H}_1 \mathbf{s}\right|^2}{P_2 \left\|\mathbf{W}^H \mathbf{H}_2^H \mathbf{r}_1\right\|^2 + 1} + P_0 \|\mathbf{r}_0^H \mathbf{H}_0 \mathbf{s}\|^2, \tag{1}
$$

where we have assumed $\mathrm{E}\left\{|x_{in}|^2\right\} = 1$ and $\|\mathbf{r}_0\| = \|\mathbf{r}_1\| = 1$ without loss of generality.

We intend to find optimal beamforming vector $\mathbf{s}^\star$, weighting matrix $\mathbf{W}^\star$, and combining vectors $\mathbf{r}_0^\star$, $\mathbf{r}_1^\star$ to maximize the total received SNR subject to the source and relay power constraints, which we assume to be equal to 1. The source power constraint can be satisfied by assuming $\|\mathbf{s}\| = 1$, also the relay power constraint can be expressed as $P_1 \|\mathbf{W}\mathbf{H}_1\mathbf{s}\|^2 + \|\mathbf{W}\|_F^2 = 1$.

Theorem 1: For the problem:

$$
\begin{aligned}
\max_{\mathbf{W}, \mathbf{s}, \mathbf{r}_0, \mathbf{r}_1} \quad & \frac{P_1 P_2 \left|\mathbf{r}_1^H \mathbf{H}_2 \mathbf{W} \mathbf{H}_1 \mathbf{s}\right|^2}{P_2 \left\|\mathbf{W}^H \mathbf{H}_2^H \mathbf{r}_1\right\|^2 + 1} + P_0 \|\mathbf{r}_0^H \mathbf{H}_0 \mathbf{s}\|^2 \\
\text{s.t.} \quad & \begin{cases} \|\mathbf{s}\| = \|\mathbf{r}_0\| = \|\mathbf{r}_1\| = 1 \\ P_1 \|\mathbf{W}\mathbf{H}_1\mathbf{s}\|^2 + \|\mathbf{W}\|_F^2 = 1 \\ \mathbf{W} \in \mathbb{C}^{n\times n},\ \mathbf{s} \in \mathbb{C}^m,\ \mathbf{r}_0, \mathbf{r}_1 \in \mathbb{C}^l. \end{cases}
\end{aligned} \tag{2}
$$

the optimal receive combining vectors $\mathbf{r}_0^\star$ and $\mathbf{r}_1^\star$ are the dominant left singular vectors of $\mathbf{H}_0$ and $\mathbf{H}_2$.
The optimal source beamforming vector is:

$$
\mathbf{s}^\star = \arg\max_{\|\mathbf{s}\|=1} \frac{\|\mathbf{H}_1\mathbf{s}\|^2}{\|\mathbf{H}_1\mathbf{s}\|^2 + \lambda} + \mu \|\mathbf{H}_0 \mathbf{s}\|^2, \tag{3}
$$

where $\lambda = \frac{1+P_2\phi^2}{P_1}$, $\mu = \frac{P_0}{P_2\phi^2}$, and ϕ is largest singular value of $\mathbf{H}_2$.
The optimal relay weighting matrix is:

$$
\mathbf{W}^\star = \sigma \mathbf{v} \mathbf{u}^H, \tag{4}
$$

where $\mathbf{u} = \mathbf{H}_1\mathbf{s}^\star / \|\mathbf{H}_1\mathbf{s}^\star\|$, $\mathbf{v} = \mathbf{H}_2^H \mathbf{r}_1^\star / \left\|\mathbf{H}_2^H \mathbf{r}_1^\star\right\|$, and $\sigma = \left(1 + P_1 \|\mathbf{H}_1\mathbf{s}^\star\|^2\right)^{-\frac{1}{2}}$.

Proof: See [12]. ∎

Note that the optimal source beamforming vector $\mathbf{s}^\star$ in Theorem 1 is given in the form of a maximization problem.

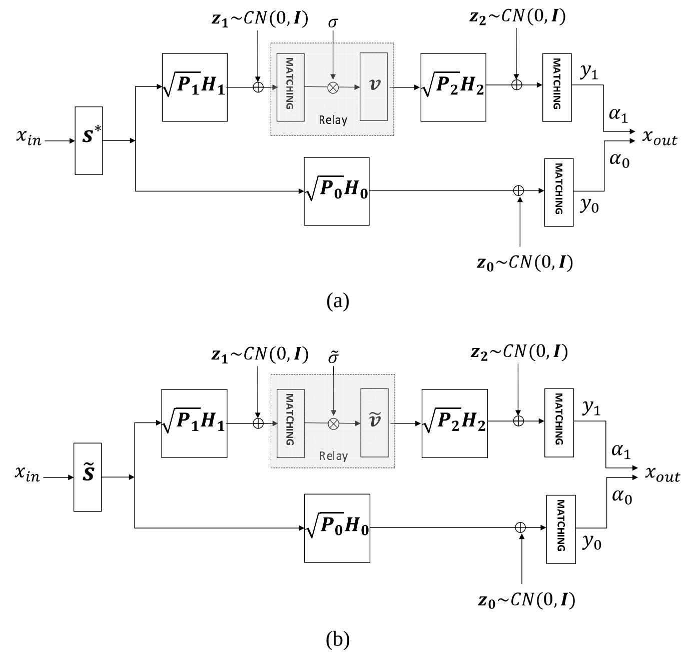

Fig. 2. (a) Optimal beamforming scheme for MIMO AF relay channel with full CSI; (b) MIMO AF relaying with quantized beamforming vectors $\tilde{\mathbf{s}} \in \mathbf{C}_1(N_1, \delta_1)$, $\tilde{\mathbf{v}} \in \mathbf{C}_2(N_2, \delta_2)$.

The objective function of this problem, i.e. (3), may have multiple local maxima and moreover, the global maximum is not unique[1]. This problem does not appear to have an analytic solution. In Section IV, we use a numerical approach to find the local optimum of this maximization problem.

The structure of the optimal beamforming scheme given by Theorem 1 is shown in Fig. 2a. As the rank-one structure of the optimal weighting matrix suggests, the relay first matches to the equivalent source-relay channel $\mathbf{H}_1\mathbf{s}^\star$ and scales the resulting scalar by σ to meet its unit power constraint. Then the relay uses $\mathbf{v}$, the dominant right singular vector of relay-destination channel, for beamforming in the second time slot[2]. The destination, on the other hand, matches to the equivalent source-destination and relay-destination channels $\mathbf{H}_0\mathbf{s}^\star$ and $\mathbf{H}_2\mathbf{v}$ in the first and second time slots.

The resulting maximum received SNR value can be computed by substituting the optimal values given in Theorem 1 in equation (2):

$$\gamma^\star = \frac{\gamma_1^\star \gamma_2^\star}{1 + \gamma_1^\star + \gamma_2^\star} + \gamma_0^\star, \tag{5}$$

where $\gamma_0^\star = P_0\|\mathbf{H}_0\mathbf{s}^\star\|^2$, $\gamma_1^\star = P_1\|\mathbf{H}_1\mathbf{s}^\star\|^2$, and $\gamma_2^\star = P_2\phi^2$.

The aim of this paper is to design quantized beamforming scheme based on the optimal beamforming scheme with full CSI, where the source and relay beamforming vectors belong to certain codebooks with finite cardinalities. To reveal the structure of these codebooks, we need to know the distribution of the optimal beamforming vectors.

[1]If $\mathbf{s}$ is a global maximum point, so is $e^{j\theta}\mathbf{s}$, for any $\theta \in \mathbb{R}$.

[2]Since $\mathbf{r}_1^\star$ is the dominant left singular vector of $\mathbf{H}_2$, the vector $\mathbf{v}{=}\mathbf{H}_2^H\mathbf{r}_1^\star/\|\mathbf{H}_2^H\mathbf{r}_1^\star\|$ is the dominant right singular vector of $\mathbf{H}_2$.

The optimal relay beamforming vector $\mathbf{v}$ is the dominant right singular vector of $\mathbf{H}_2$. For Rayleigh fading channel matrix $\mathbf{H}_2$, the singular vectors have been shown to be uniformly distributed on the unit sphere of the corresponding dimensions (see [4]). Hence, the optimum relay beamforming vector has a uniform distribution on Ω_n.

For source beamforming vector $\mathbf{s}^\star$, although we do not have a closed form expression, we are still able to identify the distribution for Rayleigh fading channels.

Theorem 2: For independent Rayleigh fading channel matrices $\mathbf{H}_0$ and $\mathbf{H}_1$, the optimal source beamforming vector $\mathbf{s}^\star$ that maximizes the total received SNR (or equivalently the objective function in (3)) is uniformly distributed on Ω_m.

Proof: See the Appendix. ■

Note that if we had only one channel from the source to the destination, the optimal source beamforming vector would be the dominant right singular vector of source-destination channel and therefore uniformly distributed on Ω_m. Interestingly, Theorem 2 states that the optimal source beamforming vector is still uniformly distributed on the unit sphere, when there are two independent parallel channels from source to destination. This is basically due to the independence of $\mathbf{H}_0$ and $\mathbf{H}_1$, and the specific properties of the Rayleigh fading channel matrices.

The distributions of optimal source and relay beamforming vectors are used in the next section to derive the structure of the appropriate beamforming vector quantization codebooks.

III. MIMO Amplify and Forward Relay Channel with Limited CSI Feedback

In this section, we assume that CSI is only available at the receiver sides of the links, i.e. relay knows $\mathbf{H}_1$, and destination knows $\mathbf{H}_0$ and $\mathbf{H}_2$.

We further assume that the destination knows $\mathbf{H}_1$, as the relay has a dedicated forward link to destination and can inform destination of $\mathbf{H}_1$. For example, relay may append the received training symbols of the source-relay channel to relay-destination training symbols and forward it to the destination. The destination first estimates $\mathbf{H}_2$ from relay-destination channel training symbols, and having $\mathbf{H}_2$, estimates $\mathbf{H}_1$ from the forwarded source-relay channel training symbols [13].

Based on the CSI assumption above, the quantized beamforming vectors are determined as follows. The source beamforming vector $\tilde{\mathbf{s}}$ is chosen from a codebook $\mathbf{C}_1(N_1, \delta_1) \subset \Omega_m$ shared between the source and destination. The relay beamforming vector $\tilde{\mathbf{v}}$ is chosen from a codebook $\mathbf{C}_2(N_2, \delta_2) \subset \Omega_n$ shared between the relay and destination.

The quantized beamforming scheme is shown in Fig 2b. In the first time slot, the source uses $\tilde{\mathbf{s}} \in \mathbf{C}_1$ for beamforming, and relay and destination match to $\mathbf{H}_1\tilde{\mathbf{s}}$ and $\mathbf{H}_0\tilde{\mathbf{s}}$. In the second time slot, the relay scales its symbol to meet its power constraint and uses $\tilde{\mathbf{v}} \in \mathbf{C}_2$ for beamforming, and destination matches to $\mathbf{H}_2\tilde{\mathbf{v}}$. The source-destination, source-relay, relay-destination, and total received SNR values are given by:

$$\gamma = \frac{\gamma_1\gamma_2}{1+\gamma_1+\gamma_2} + \gamma_0, \tag{6}$$
$$\gamma_0 = P_0\|\mathbf{H}_0\tilde{\mathbf{s}}\|^2, \quad \gamma_1 = P_1\|\mathbf{H}_1\tilde{\mathbf{s}}\|^2, \quad \gamma_2 = P_2\|\mathbf{H}_2\tilde{\mathbf{v}}\|^2.$$

The beamforming vectors $\tilde{\mathbf{s}}$ and $\tilde{\mathbf{v}}$ should be chosen such that the total received SNR is maximized. Clearly, $\tilde{\mathbf{v}}$ should be chosen to maximize γ_2, since it only contributes to γ through the term γ_2.

$$\tilde{\mathbf{v}} = \arg\max_{\mathbf{w}\in\mathbf{C}_2} P_2\|\mathbf{H}_2\mathbf{w}\|^2. \tag{7}$$

The corresponding relay-destination received SNR is:

$$\tilde{\gamma}_2 = \max_{\mathbf{w}\in\mathbf{C}_2} P_2\|\mathbf{H}_2\mathbf{w}\|^2. \tag{8}$$

By substituting $\tilde{\gamma}_2$ in (6), the source beamforming vector should be chosen as follows:

$$\tilde{\mathbf{s}} = \arg\max_{\mathbf{w}\in\mathbf{C}_1} \frac{\|\mathbf{H}_1\mathbf{w}\|^2}{\|\mathbf{H}_1\mathbf{w}\|^2+\tilde{\lambda}} + \tilde{\mu}\|\mathbf{H}_0\mathbf{w}\|^2, \tag{9}$$

where $\tilde{\lambda} = \frac{1+\tilde{\gamma}_2}{P_1}$ and $\tilde{\mu} = \frac{P_0}{\tilde{\gamma}_2}$.

The maximum total received SNR of the quantized scheme $\tilde{\gamma}$ can be computed by substituting (7) and (9) in (6).

Using the expressions for optimal and quantized beamforming vectors in Theorem 1 and equations (7) and (9), and the distributions of optimal vectors (Theorem2), we can prove the following upper bound on the average total received SNR loss caused by quantization[3]:

$$\begin{aligned} &\mathrm{E}\{\gamma^\star\} - \mathrm{E}\{\tilde{\gamma}\} \\ &\quad \le 2\left(mlP_0 + mnP_1\right)\left(1-N_1\left(\frac{\delta_1}{2}\right)^{2(m-1)}\left(1-\frac{\delta_1}{2}\right)\right) \\ &\quad\quad + 2nlP_2\left(1-N_2\left(\frac{\delta_2}{2}\right)^{2(n-1)}\left(1-\frac{\delta_2}{2}\right)\right). \end{aligned} \tag{10}$$

For any number of antennas $m, n > 1$, the above SNR loss upper bound is decreasing in δ_1, δ_2 the minimum distances of the codebooks $\mathbf{C}_1$ and $\mathbf{C}_2$. Therefore, to minimize the SNR loss upper bound, the minimum distances of the codebooks should be maximized and this justifies the use of Grassmannian codebooks, $\mathbf{C}_1$ and $\mathbf{C}_2$, for quantizing the optimal source and relay beamforming vectors.

The next section provides the simulation results that show the performance of the Grassmannian codebooks.

IV. Simulation Results

The general setup for the simulations is as follows. The input symbols belong to a BPSK constellation with unit power. The entries of the channel matrices, which model the i.i.d. Rayleigh fading channels, are generated independently according to $\mathcal{CN}(0,1)$. To model quasi-static fading channels, the simulation time is divided into $20,000$ coherence intervals, each consisting of 200 symbols. The channels are assumed to be constant over each coherence interval and independent from one interval to the other. The simulation results compare the bit error rates of different quantized and unquantized schemes. All the stations are equipped with three antennas ($m = n = l = 3$).

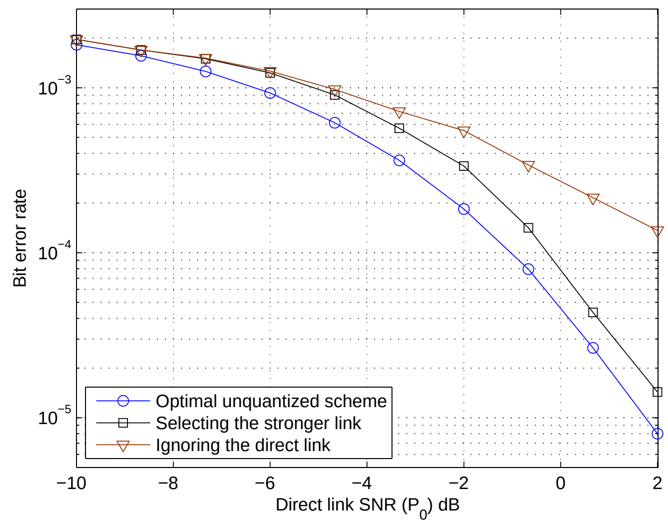

Fig. 3. Comparison of the optimal unquantized beamforming scheme with other unquantized schemes. The source-relay and the relay-destination link SNR's are fixed at $P_1 = P_2 = 2$dB.

Fig. 3 compares the optimal (unquantized) beamforming scheme (Fig. 2a) with other unquantized schemes. The source-relay and the relay-destination link SNR's are fixed at $P_1 = P_2 = 2$dB and the BER values are recorded for different values of the direct link SNR P_0. For the optimal scheme, we use the gradient descent method to find a locally optimal source beamforming vector using (3). For this purpose, the constraint $\|\mathbf{s}\| = 1$ is removed by substituting $\mathbf{s} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$ with $\mathbf{u} \in \mathbb{C}^m$. The gradient of the objective function is computed with respect to $\mathbf{u}$ and used in the gradient descent iterations. Multiple random initial values of $\mathbf{u}$ are used to increase the chances of finding the global optimum point.

The curve marked by $\triangledown$ shows the performance of a scheme that ignores the direct link in determining the source beamforming vector. For this scheme, the source beamforming vector is always set to be the dominant right singular vector of the source-relay channel. As expected, the performance of this scheme diverges from the optimal scheme as the direct link gets stronger. The next curve, marked by $\square$, shows the performance of a scheme that considers only the stronger of the source-destination and the source-relay-destination links for determining the source beamforming vector. In this scheme, the source switches between the dominant right singular vectors of the source-relay and source-destination links depending on their received SNR values.

In the next two simulation setups, we study the performance of the quantized schemes. As discussed in Section III, the

[3]The proof of this upper bound is rather involved and is not presented here (see [12] for the proof). The main challenge is that, unlike point-to-point MIMO channel in [4], the optimal source beamforming vector $\mathbf{s}^\star$ cannot be expressed in a closed form here (see equation (3)).

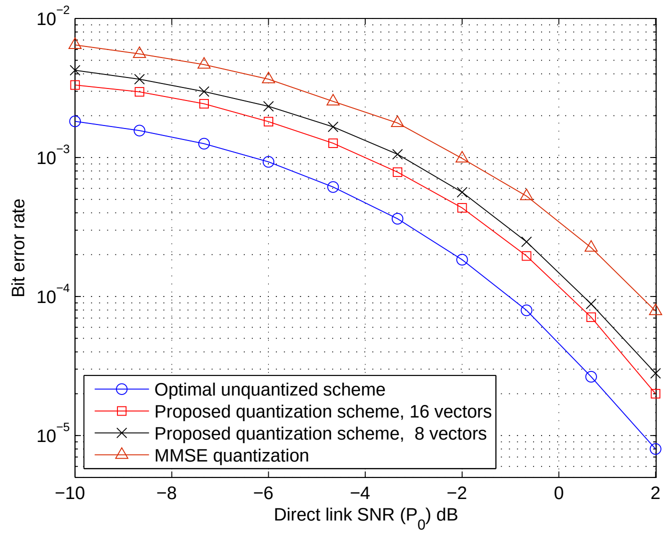

Fig. 4. Comparison of the proposed quantization scheme with MMSE quantizer. The source-relay and the relay-destination link SNR's are fixed at $P_1 = P_2 = 2$dB.

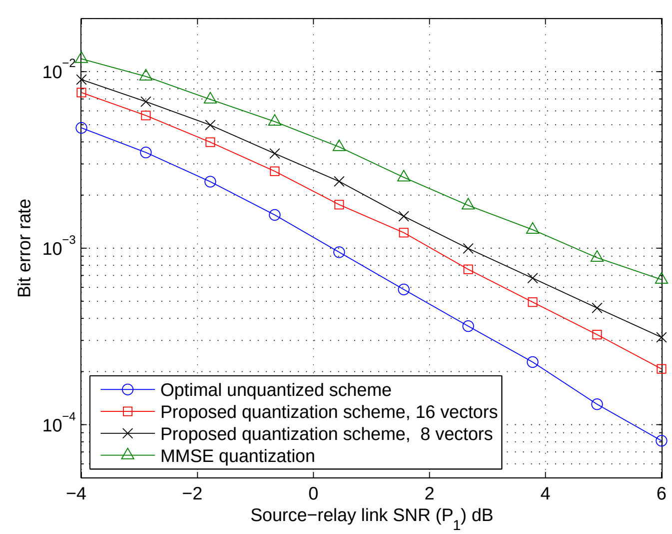

Fig. 5. Comparison of the proposed quantization scheme with MMSE quantizer. The direct link and the relay-destination link SNR's are fixed at $P_0 = -4$dB and $P_2 = 2$dB.

scheme consists of two codebooks $\mathbf{C}_1$ and $\mathbf{C}_2$ of sizes N_1 and N_2. The codebook $\mathbf{C}_1$ determines the source beamforming vector in the first time slot. The codebook $\mathbf{C}_2$ is used to determine the relay beamforming vector in the second time slot. The quantization scheme requires $\log_2(N_1)$ feedback bits for sending the label of $\tilde{\mathbf{s}}$ to the source, and $\log_2(N_2)$ feedback bits to send the label of $\tilde{\mathbf{v}}$ to the relay. Therefore the total number of feedback bits is $\log_2(N_1 N_2)$.

Fig. 4 shows the performance of the proposed quantization scheme with Grassmannian codebooks of sizes $N_1 = N_2 = N_3 = 8, 16$. The source-relay and the relay-destination link SNR's are fixed at $P_1 = P_2 = 2$dB and the BER values have been recorded for different values of the direct link SNR P_0. The Grassmannian codebooks are adopted from [11].

Fig. 4 also shows the performance of the MMSE quantizer. In this scheme, the destination and relay quantize $\mathbf{H}_0$ and $\mathbf{H}_1$ entry-by-entry and send the quantization bits to the source; the source determines its beamforming vector from (3) based on quantized $\mathbf{H}_0$ and $\mathbf{H}_1$. The destination also quantizes $\mathbf{H}_2$ and sends it to relay; the relay determines its beamforming vector by finding the dominant right singular vector of the quantized $\mathbf{H}_2$. If we use two bits to quantize each of the complex channel entries, this scheme requires a total of $2(mn + ml + ln)$ feedback bits. Table I compares the total required number of feedback bits for Grassmannian and MMSE quantizers.

Fig. 5 compares the performance of the same schemes of Fig. 4 in a different scenario. For this figure, the direct link and the relay-destination link SNR's are fixed at $P_0 = -4$dB and $P_2 = 2$dB. The BER values have been recorded for different values of the source-relay link SNR P_1.

As the simulation results in Figs. 4 and 5 verify, the proposed quantization scheme shows better performance compared with the naive MMSE quantizer with much fewer

TABLE I

COMPARISON OF THE TOTAL NUMBER OF FEEDBACK BITS FOR $m = n = l = 3$ ANTENNAS AND CODEBOOK SIZES OF $N = N_1 = N_2 = 8$ AND 16.

Scheme	$N = 8$	$N = 16$
Proposed quantization	6	8
MMSE	54	

feedback bits.

V. CONCLUSION

This paper derives the optimal beamforming scheme for half-duplex MIMO amplify-and-forward relay channel with full CSI. Based on the distributions of the optimal source and relay beamforming vectors, we prove the efficiency of the Grassmannian quantization codebooks. The results were verified by comparing the performance of different unquantized and quantized schemes under different simulation scenarios.

APPENDIX
PROOF OF THEOREM 2

In this appendix, we show that there exists a solution $\mathbf{s}^\star$ to the problem (3) that is uniformly distributed on the unit sphere in $\mathbb{C}^m$, where m is the number of source antennas.

The problem (3) is repeated here:

$$\mathbf{s}^\star = \arg\max_{\|\mathbf{s}\|=1} \frac{\|\mathbf{H}_1\mathbf{s}\|^2}{\|\mathbf{H}_1\mathbf{s}\|^2 + \lambda} + \mu\|\mathbf{H}_0\mathbf{s}\|^2, \qquad (11)$$

Consider $\mathbf{H}_0 = \mathbf{U}_0\boldsymbol{\Sigma}_0\mathbf{V}_0^H$ and $\mathbf{H}_1 = \mathbf{U}_1\boldsymbol{\Sigma}_1\mathbf{V}_1^H$ as the singular value decompositions of $\mathbf{H}_0$ and $\mathbf{H}_1$. Clearly:

$$\|\mathbf{H}_0\mathbf{s}\| = \left\|\boldsymbol{\Sigma}_0\mathbf{V}_0^H\mathbf{s}\right\|$$

and

$$\|\mathbf{H}_1\mathbf{s}\| = \left\|\boldsymbol{\Sigma}_1\mathbf{V}_1^H\mathbf{s}\right\|,$$

since $\mathbf{U}_0$ and $\mathbf{U}_1$ are unitary matrices.

It is easy to check that

$$\mathbf{s}^{\star} = \mathbf{V}_0 \eta(\mathbf{\Sigma}_0, \mathbf{\Sigma}_1, \mathbf{V}_1^H \mathbf{V}_0)$$

is a solution to (11), where the function $\eta(\cdot,\cdot,\cdot)$ is defined to be a solution to the following problem:

$$\eta(\mathbf{\Sigma}_0,\mathbf{\Sigma}_1, \mathbf{V}_1^H \mathbf{V}_0) \stackrel{def}{=} \arg \max_{\|\mathbf{t}\|=1} \frac{\left\|\mathbf{\Sigma}_1 \mathbf{V}_1^H \mathbf{V}_0 \mathbf{t}\right\|^2}{\left\|\mathbf{\Sigma}_1 \mathbf{V}_1^H \mathbf{V}_0 \mathbf{t}\right\|^2 + \lambda} + \mu \left\|\mathbf{\Sigma}_0 \mathbf{t}\right\|^2. \quad (12)$$

If we fix $\mathbf{\Sigma}_0$ and $\mathbf{\Sigma}_1$, the solution $\mathbf{s}^{\star}$, identified above, can be expressed as a function of $\mathbf{V}_0$ and $\mathbf{V}_1$:

$$\mathbf{s}^{\star} = \zeta_{\mathbf{\Sigma}_0,\mathbf{\Sigma}_1}(\mathbf{V}_0, \mathbf{V}_1) \stackrel{def}{=} \mathbf{V}_0 \eta(\mathbf{\Sigma}_0, \mathbf{\Sigma}_1, \mathbf{V}_1^H \mathbf{V}_0). \quad (13)$$

Now, for any unitary matrix $\mathbf{Q}$, we have the following from (13):

$$\zeta_{\mathbf{\Sigma}_0,\mathbf{\Sigma}_1}(\mathbf{Q}\mathbf{V}_0, \mathbf{Q}\mathbf{V}_1) = \mathbf{Q}\zeta_{\mathbf{\Sigma}_0,\mathbf{\Sigma}_1}(\mathbf{V}_0, \mathbf{V}_1) = \mathbf{Q}\mathbf{s}^{\star}.$$

For a Rayleigh fading channel matrix $\mathbf{H}_0$, we know the the random matrix $\mathbf{V}_0$ is independent of $\mathbf{\Sigma}_0$ and its distribution does not change by pre-multiplication by a unitary matrix $\mathbf{Q}$. The same argument holds for $\mathbf{H}_1$, $\mathbf{V}_1$ and $\mathbf{\Sigma}_1$. Therefore, conditioned on $\mathbf{\Sigma}_0$ and $\mathbf{\Sigma}_1$, the matrix $\mathbf{Q}\mathbf{V}_0$ has the same distribution as $\mathbf{V}_0$, and similarly $\mathbf{Q}\mathbf{V}_1$ has the same distribution as $\mathbf{V}_1$.

Since the source-destination and source-relay channels are assumed to be independent, $\mathbf{V}_0$ and $\mathbf{V}_1$ are also independent, and therefore the joint distribution of $(\mathbf{V}_0, \mathbf{V}_1)$ is the same as the joint distribution of $(\mathbf{Q}\mathbf{V}_0, \mathbf{Q}\mathbf{V}_1)$. Hence, any arbitrary function of these pairs will have the same distribution. By applying this to the function $\zeta_{\mathbf{\Sigma}_0,\mathbf{\Sigma}_1}(\cdot,\cdot)$, we conclude that $\mathbf{s}^{\star} = \zeta_{\mathbf{\Sigma}_0,\mathbf{\Sigma}_1}(\mathbf{V}_0, \mathbf{V}_1)$ and $\mathbf{Q}\mathbf{s}^{\star} = \zeta_{\mathbf{\Sigma}_0,\mathbf{\Sigma}_1}(\mathbf{Q}\mathbf{V}_0, \mathbf{Q}\mathbf{V}_1)$ have the same distribution. Since this is true for any unitary matrix $\mathbf{Q}$, we conclude that $\mathbf{s}^{\star}$ is uniformly distributed on the complex unit sphere, conditioned on $\mathbf{\Sigma}_0$ and $\mathbf{\Sigma}_1$. Finally, we note that if the conditional distribution of $\mathbf{s}^{\star}$ is uniform, its unconditional distribution is also uniform.

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