COMPETITION AND COOPERATION IN MULTI-USER COMMUNICATION ENVIRONMENTS

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To my parents

Abstract

A communication environment with multiple transmitters and multiple receivers is inherently a competitive environment. The aim of this thesis is to illustrate the role of competition and the value of cooperation in a multi-user communication environment from an information-theoretical perspective. Various scenarios will be treated, including the multiple access channel where receivers cooperate, the broadcast channel where transmitters cooperate, and the interference channel where neither transmitters nor receivers cooperate.

There are three main results in this thesis: First, it is shown that in a Gaussian multiple access channel with multiple transmit and receive antennas, the optimum transmit strategy that maximizes the sum capacity can be found by an iterative water-filling procedure, where each user competitively maximizes its own rate while treating interference from other users as noise. Thus, a competitive optimum in a Gaussian multiple access channel is also a global optimum. Second, it is shown that in a Gaussian broadcast channel with multiple transmit and receive antennas, under a certain non-singularity condition, the sum-capacity can be achieved using a decision-feedback precoder. Further, the sum capacity can be interpreted as a saddle-point of a mutual information game, where the transmitter chooses a transmit strategy to maximize the mutual information, and "nature" chooses a fictitious noise correlation to minimize the mutual information. Thus, the sum capacity of a Gaussian broadcast channel corresponds to a competitive equilibrium. Third, it is shown that in a Gaussian interference channel, although a competitive optimum is not necessarily the global optimum, it leads to a desirable operating point. This suggests a distributed dynamic spectrum management scheme for digital subscriber line (DSL) applications.

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Chapter 1

Introduction

The modern theory of communications started in 1948 with Claude E. Shannon's classic paper "A mathematical theory of communication" [1]. In this ground breaking work, Shannon proved the following fundamental result: reliable communication between a transmitter and a receiver is possible if and only if the rate of communication is below a certain quantity called channel capacity. Further, Shannon explicitly showed how channel capacity can be computed from the statistical properties of the communication environment. This result, known as the noisy channel coding theorem, revealed the fundamental performance limit in a communication channel and laid down a foundation for the modern science of digital communications. For the next fifty years, a principal goal of communication engineers has been to devise practical methods to approach this ultimate limit.

Shannon's original work focused on communication scenarios between a single transmitter and receiver pair. This communication model is referred to as a single-user channel for which the capacity is now well-established. Practical communication systems, on the other hand, often involve multiple transmitters and receivers sharing the same transmission medium. For these multi-user channels, although the definition of channel capacity can be easily generalized, the characterization of capacity is much more difficult than for single-user channels. In fact, Shannon himself started the study of multi-user information theory by introducing a two-way communication scenario with a transmitter and a receiver at each end of a channel [2]. The capacity of this seemingly simple channel is still an open problem to this date.

A multi-user communication environment differs from a single-user environment in several crucial aspects. First of all, as multiple transmitters and receivers share the same communication medium, they cause mutual interference into each other. Interference is typically detrimental to the system performance, so a multi-user environment is inherently a competitive environment where users compete for network resources. On the other hand, the interaction among the users also creates opportunities for the multiple transmitters and receivers to cooperate. There is an interesting interplay between competition and cooperation in multi-user channels. The purpose of this thesis is to capture some of such interplay by investigating three specific multi-user channels with varying degrees of cooperation. The goal is to answer the following two questions: In a communication situation with multiple transmitters and multiple receivers, what is the role of competition? What is the value of cooperation?

Compared to a single-user communication environment, multi-user communication is substantially more complicated for several reasons. First, a multi-user environment is usually interference limited. Multi-user transmission techniques need to exploit the structure of the interference. The optimal way to do so, however, is still poorly understood except in the simplest cases. Second, multiple transmitters and receivers in a multi-user environment can potentially cooperate. However, it is not always clear what the best form of cooperation is in a given situation. Third, the standard machinery for proving the converse in the single-user channel capacity theorem is often inadequate for multi-user channels. The upper bound for multi-user channel capacity is typically more difficult to derive than for the single-user case. Further, a multi-user communication environment can involve feedback. The role of feedback has so far resisted rigorous information-theoretical treatment in most cases. For these and many other reasons, multi-user information theory is largely incomplete, and multi-user channel capacity problems represent some of the most challenging open problems in information theory.

In spite of the lack of a complete understanding in multi-user communication theory, the recent explosion in information technology has created a large demand for practical communication solutions that accommodate many users. For example, the mobile wireless system is inherently a multi-user system as signals from all mobile devices share the same air interface. A good understanding of multi-user communication is critical in wireless system design. As another example, in broadband access networks such as digital subscriber lines (DSL) and Ethernet, the network topology typically involves a central office transmitting and receiving signals from multiple users at the same time. Signals from multiple users interfere with each other, so the network is best modeled as a multi-user system. In fact,

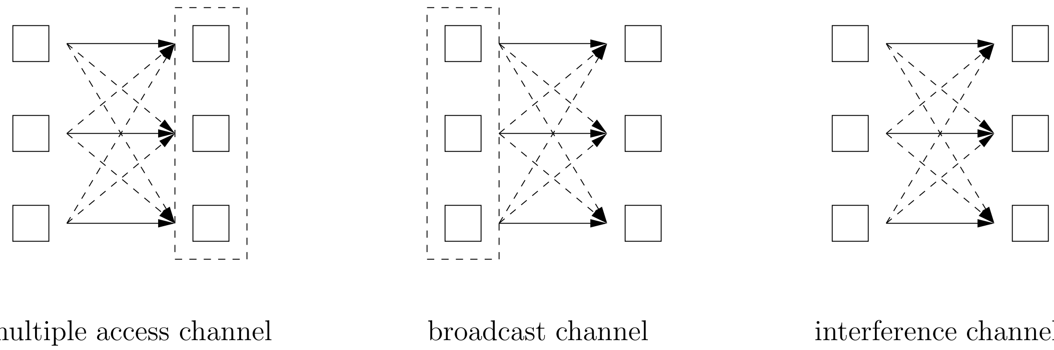


Figure 1.1: Multi-user communication models

a multi-user system design for DSL has the potential to bring substantial performance improvement to the network.

This thesis deals with the design of optimal transmission strategies for multi-user communication situations. An information-theoretical approach is taken. The focus is on Gaussian channels with multiple transmitters and multiple receivers and with varying degrees of cooperation among the transmitters and the receivers. Three specific channel models are treated, and they are illustrated in Figure 1.1. The first channel model is called *the multiple access channel*, where transmitters do not cooperate and receivers cooperate. In this scenario, each transmitter represents a different user sending independent information, and the joint receiver must decode information from all users. The second channel model is called *the broadcast channel*, where transmitters cooperate and receivers do not cooperate. In this scenario, the joint transmitter sends independent information to multiple receivers at the same time, and each receiver is interested in decoding its own message. The third channel model is called *the interference channel*, where neither transmitters nor receivers cooperate. In this scenario, each transmitter-receiver pair attempts to communicate in the presence of interference from all other users. These three scenarios are prototypical examples of multi-user channels, and they capture a variety of practical situations. A comparative study of the three cases also illustrates the role of competition and the value of cooperation in multi-user situations. In the rest of this chapter, practical motivations for the channel models are described in more detail, precise statements of the problems are presented, and the main results of the thesis are outlined.

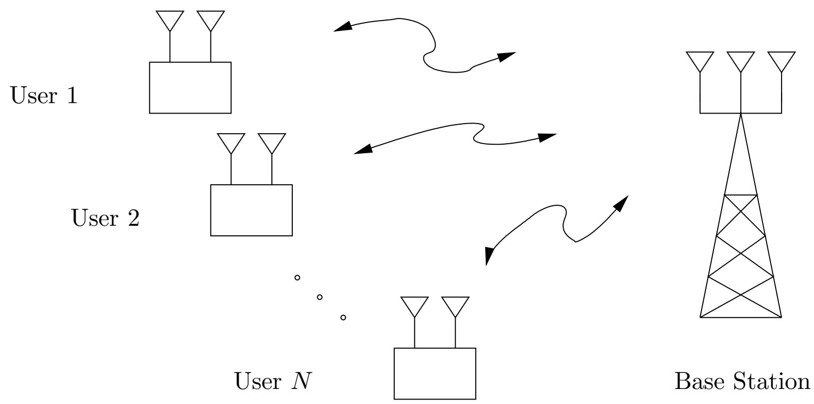


Figure 1.2: Wireless cellular network

1.1 Motivation

Practical communication systems are often multi-user in nature. This is mainly because system designs are often constrained by physical resources such as bandwidth, and a system that allows multiple users to share resources is often the most economical. This is the case for broadcast systems such as terrestrial television, cable television and satellite systems where all receivers share the same spectrum, and also for computer networks such as the original Ethernet where many computers are connected to the same cable. The multi-user communication situations studied in this thesis are motivated by these examples. Two particular systems of interest are the wireless system and digital subscriber lines.

1.1.1 Wireless Systems

Wireless communication systems are often designed as cellular systems. Wireless applications typically require an arbitrary pair of mobile devices to be able to communicate with each other. The most economical way to ensure such “anywhere” connectivity is to deploy base-stations that collectively provide coverage to the entire geographical area of interest, and to let mobile devices communicate with the nearest base-station wherever they are. The coverage area of each base-station is referred to as a cell. At any given time, a single base-station needs to serve potentially many mobile devices. So, the communication environment between a base-station and mobile devices is a multi-user environment.

Consider a general system model, where both the base-station and the mobile devices are equipped with multiple antennas. Multiple antennas create spatial dimensions, and they are effective means to enhance wireless system performance. In a multiple-antenna system, the transmit and receive signals are vector valued, and the communication channel between the base-station and the mobiles may be modeled as a matrix. Note that antennas at the base-station can always cooperate in the sense that antennas may jointly encode and decode information. However, at the mobile side, only antennas that belong to the same mobile may cooperate. Figure 1.2 illustrates a typical multi-antenna wireless environment.

Two-way communication takes place between the base-station and the mobiles. The direction from the mobiles to the base-station is referred to as the uplink direction. It can be modeled as a multiple access channel where transmitters belonging to different mobiles do not cooperate, but receivers cooperate. The direction from the base-station to the mobiles is referred to as the downlink direction. It can be modeled as a broadcast channel where receivers belonging to different mobiles do not cooperate, but transmitters cooperate. Thus, for wireless systems, both the multiple access channel model and the broadcast channel model are applicable.

1.1.2 Digital Subscriber Lines

Digital subscriber line (DSL) technology brings high-speed data service to home via ordinary telephone copper twisted-pairs [3]. The DSL environment is traditionally thought of as a single-user environment, as the communication between each pair of transmit and receive modems takes place on a dedicated link. However, telephone lines from different customer sites are bundled together on the way to the central office. Multiple lines within a bundle create electromagnetic interference into each other. Such electromagnetic coupling is called crosstalk, and it is often the dominant noise source in a line. For this reason, the DSL environment is more accurately modeled as a multi-user environment.

Figure 1.3 illustrates a typical DSL environment. There are two types of crosstalk. Near-end crosstalk (NEXT) refers to the interference emitted by transmitters located on the same side as the receiver. Far-end crosstalk (FEXT) refers to the interference emitted by transmitters located on the opposite end of the line. Note that cooperation is not possible at the remote terminals, as they are located in different geographical locations. Cooperation at the central office is potentially possible, and if it is done, the upstream direction can be modeled as a multiple access channel and the downstream direction can be modeled as a

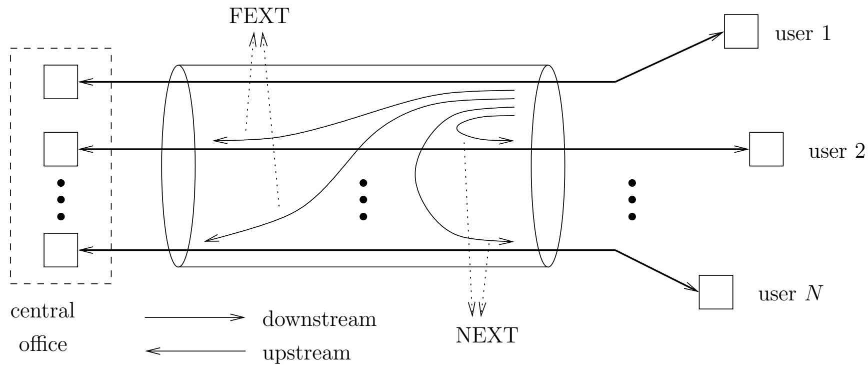


Figure 1.3: Digital subscriber lines

broadcast channel. However, cooperation at the central office is possible in practice only if all lines are owned by the same service provider, which might not be the case, especially in a unbundled environment where different service providers compete commercially in the local access market. In this scenario, the DSL environment must be modeled as an interference channel where neither transmitters nor receivers cooperate. Thus, for DSL applications, all three multi-user channel models considered in this thesis are applicable.

1.2 Overview of Thesis

The objective of this thesis is to characterize the channel capacity, optimal spectrum and optimal coding techniques for a vector Gaussian channel with varying degrees of cooperation among the transmitters and the receivers. As a first step, it is instructive to consider the case where both transmitters and receivers cooperate. This single-user Gaussian vector channel can be modeled as

$$\mathbf{y} = H\mathbf{x} + \mathbf{z}, \tag{1.1}$$

where $\mathbf{x}$ and $\mathbf{y}$ are vector-valued signals, H is a matrix channel, and $\mathbf{z}$ is the vector Gaussian noise. The input signal must satisfy a power constraint: $\mathbf{E}[\mathbf{X}^T\mathbf{X}] \leq P$. Figure 1.4 illustrates a Gaussian vector channel. The information-theoretical capacity of the channel is defined as follows: Let a $(n, 2^{nR})$ codebook consist of an encoding function $\mathbf{X}^n(W)$, where $W \in$

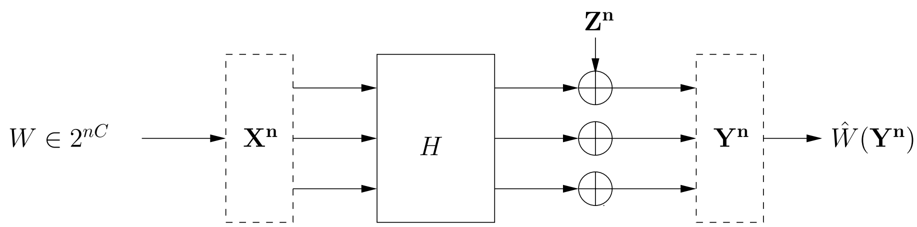


Figure 1.4: Gaussian vector channel

$\{1, \cdots, 2^{nR}\}$, and a decoding function $\hat{W}(\mathbf{Y}^n)$. The probability of error P_e^n is defined as the probability: $\Pr\{W \neq \hat{W}\}$, averaged over the entire codebook. A rate R is achievable if there exists a sequence of $(n, 2^{nR})$ codebooks for which $P_e^n \to 0$ as $n \to \infty$. The channel capacity C is defined to be the supremum of all achievable rates.

Shannon's noisy channel coding theorem states that the capacity of a discrete memoryless channel is the maximum mutual information between the input terminals and the output terminals, maximized over all possible input distributions. For the vector Gaussian channel, this implies that $C = \max I(\mathbf{X}; \mathbf{Y})$, where the maximization is over all input distributions that satisfy the power constraint [4]. It is not difficult to show that the maximum mutual information is achieved with Gaussian inputs, and in this case the mutual information can be evaluated as:

$$I(\mathbf{X}; \mathbf{Y}) = \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|}, \tag{1.2}$$

where $|\cdot|$ denotes matrix determinant, and S_{xx} and S_{zz} denote the covariance matrices of the input $\mathbf{X}$ and the noise $\mathbf{Z}$, respectively. The power constraint becomes a constraint on the input covariance matrix: $\text{trace}(S_{xx}) \leq P$.

The mutual information maximization problem has a well-known solution based on the singular-value decomposition of H called water-filling [5]. Assuming that S_{zz} is the identity matrix, the optimum S_{xx} must have its eigenvectors equal to the right singular-vectors of H, and its eigenvalues follow a water-filling power allocation on the singular-values of H. To achieve the vector channel capacity, coordination is necessary among the transmit terminals of $\mathbf{x}$ and among the receive terminals of $\mathbf{y}$. Transmitter coordination is necessary because the capacity-achieving transmit covariance matrix is not necessarily diagonal. The optimal transmit signals from different transmit terminals may need to be correlated, and producing such a correlated signal requires coordination at the transmitter. Receiver coordination is necessary because an optimal detector is required to process signals from all receive terminals jointly. With full coordination, it is possible to choose a transmit filter to match the right singular-vectors of H and to choose a receive filter to match the left singular-vectors of H, so that the vector Gaussian channel is diagonalized [5]. Such diagonalization decomposes the Gaussian vector channel into a series of independent Gaussian scalar sub-channels so that Gaussian scalar codes can be used on each sub-channel to collectively achieve the vector channel capacity.

When coordination is possible only among the receive terminals, but not among the transmit terminals, the vector channel becomes a multiple access channel. The Gaussian vector multiple access channel is the focus of Chapter 2 of this thesis. In a multiple access channel, the maximum sum capacity can still be computed in terms of the maximum mutual information $I(\mathbf{X};\mathbf{Y})$. But, because the different transmit terminals of $\mathbf{x}$ are required to be uncorrelated, the water-filling covariance, which is optimum for a coordinated vector channel, can no longer be used. The main result of Chapter 2 is an extension of single-user water-filling to the multiple access case. The proposed optimization routine is iterative in nature, and it has a game-theory interpretation. In a Gaussian multiple access channel, the sum capacity is achieved with each user choosing an input distribution to maximize its own rate while treating interference from all other users as noise. Thus, in a Gaussian multiple access channel, a competitive optimum is also a global optimum.

When coordination is possible only among the transmit terminals, but not among the receive terminals, the Gaussian vector channel becomes a broadcast channel. The Gaussian vector broadcast channel is the focus of Chapter 3 of this thesis. Unlike the multiple access channel, the capacity region for a broadcast channel is still not known in general [6]. The main difficulty is that a broadcast channel distributes information across several receive terminals, and without joint processing of the received signals, a data rate equal to $I(\mathbf{X};\mathbf{Y})$ cannot be supported. The main result of Chapter 3 is a characterization of the sum capacity for a Gaussian vector broadcast channel under a certain non-singularity condition. It is shown that the sum capacity is achieved using a precoding scheme for Gaussian channels with additive side information non-causally known at the transmitter. Further, the optimal precoding structure corresponds to a decision-feedback equalizer that decomposes the broadcast channel into a series of single-user channels with interference pre-subtracted at the transmitter. In fact, the sum capacity is a saddle-point of a Gaussian mutual information game, where a signal player chooses a transmit covariance matrix to maximize the mutual information, and a noise player chooses a fictitious noise correlation to minimize the mutual information. Thus, in a Gaussian broadcast channel, the sum capacity corresponds to a competitive equilibrium.

When coordination is possible neither among the transmit terminals nor among the receive terminals, the Gaussian vector channel becomes an interference channel. The Gaussian interference channel is the focus of Chapter 4 of this thesis. The capacity region for the Gaussian interference channel is still poorly understood. Instead of dealing with Shannon capacity, Chapter 4 of this thesis looks at practical digital subscriber line channels and proposes a dynamic power allocation scheme for this interference network. The proposed power allocation scheme is based on a game-theory formulation of the problem. The main result is that under certain conditions, a competitive equilibrium exists and is unique in a two-user Gaussian interference channel game. Further, a competitive equilibrium corresponds to a power allocation that gives better performance than the current static power allocation approach. Thus, although competitive equilibrium is not optimal in an interference channel, it corresponds to a desirable operating point for DSL applications.

Chapter 5 summarizes the main points of the thesis. Competition and cooperation are key features in multi-user communication. A game-theoretical perspective can give useful insights into the capacity and optimal transmission problems for multi-user channels.

1.3 Notations

The notation used in this thesis is as follows: Lower case letters are used to denote scalar signals, e.g. x, y. Upper case letters are used to denote scalar random variables, e.g. X, Y, or matrices, e.g. H, where the distinction should be clear from the context. Bold face letters are used to denote vector signals, e.g. $\mathbf{x}$, $\mathbf{y}$, or vector random variables, e.g. $\mathbf{X}$, $\mathbf{Y}$. For a random vector $\mathbf{X}$, $p(\mathbf{x})$ denotes its probability law, $\mathbf{E}[\mathbf{X}]$ denotes its expectation, and S_{xx} denotes its covariance matrix. For matrices, $\cdot^T$ denotes the transpose operation, $|\cdot|$ denotes the determinant operation, and $\mathrm{tr}(\cdot)$ denotes the trace operation. The operator ≥ 0 is used to denote that a matrix is positive semi-definite. An n-dimensional identity matrix is denoted as either $I_{n\times n}$ or I.

Chapter 2

Multiple Access Channel

A communication situation where multiple uncoordinated transmitters send independent information to a common receiver is referred to as a multiple access channel. Figure 2.1 illustrates a two-user multiple access channel, where X_1 and X_2 are uncoordinated transmitters encoding independent messages W_1 and W_2, respectively, and the receiver is responsible for decoding both messages at the same time. A $(n, 2^{nR_1}, 2^{nR_2})$ codebook for a multiple access channel consists of encoding functions $X_1^n(W_1)$, $X_2^n(W_2)$, where $W_1 \in \{1, \cdots, 2^{nR_1}\}$ and $W_2 \in \{1, \cdots, 2^{nR_2}\}$, and decoding functions $\hat{W}_1(Y^n)$, $\hat{W}_2(Y^n)$. An error occurs when $W_1 \neq \hat{W}_1$ or $W_2 \neq \hat{W}_2$. A rate pair (R_1, R_2) is achievable if there exists a sequence of $(n, 2^{nR_1}, 2^{nR_2})$ codebooks for which the average probability of error $P_e^n \to 0$ as $n \to \infty$. The capacity region of a multiple access channel is the union of all achievable rate pairs.

The capacity region for the multiple access channel has the following well-known single-letter characterization [7] [8]. For a discrete-time memoryless multiple access channel characterized by the channel transition probability $p(y|x_1, x_2)$, assuming a fixed input distribution $p_1(x_1)p_2(x_2)$, the capacity region is a pentagon:

$$\begin{aligned} R_1 &\leq I(X_1; Y|X_2); \\ R_2 &\leq I(X_2; Y|X_1); \\ R_1 + R_2 &\leq I(X_1, X_2; Y), \end{aligned} \tag{2.1}$$

where the mutual information expressions are computed with respect to the joint distribution $p(y|x_1, x_2)p_1(x_1)p_2(x_2)$. When the input distribution is not fixed, but constrained in certain ways, the capacity region is the convex hull of the union of all capacity pentagons

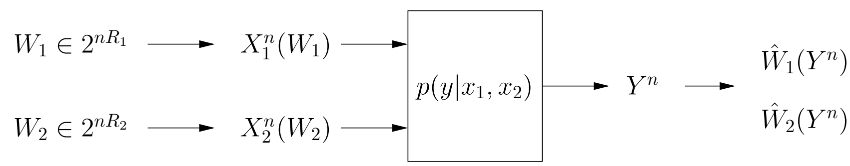


Figure 2.1: Multiple access channel

whose corresponding input distributions satisfy the input constraint after the convex hull operation [9] [10]. Since the input signals in a multiple access channel are independent, the input distribution must take a product form $p_1(x_1)p_2(x_2)$. This product constraint is non-convex, so the problem of finding the optimal input distribution for a multiple access channel is in general non-trivial [11]. (This is in contrast to the single-user case where convex programming techniques are applicable [12] [13].) The aim of this chapter is to solve the input optimization problem for a particular type of multiple access channel: the Gaussian vector multiple access channel.

A Gaussian multiple access channel refers to a multiple access channel where the law of the channel transition probability $p(y|x_1, x_2)$ is Gaussian. When a Gaussian multiple access channel is memoryless and when x_1 and x_2 are scalar, the input optimization problem has a simple solution. Let the power constraints on x_1 and x_2 be P_1 and P_2, respectively. Gaussian independent distributions $X_1 \sim \mathcal{N}(0, P_1)$ and $X_2 \sim \mathcal{N}(0, P_2)$ are optimal for every boundary point of the capacity region. In fact, for scalar Gaussian channels, the union and the convex hull operations are superfluous, and the capacity region is just a simple pentagon, known as the Cover-Wyner region [4]. However, the input optimization problem becomes non-trivial when the Gaussian multiple access channel has vector inputs. In this case, different points in the capacity region may correspond to different input distributions, and a characterization of the capacity region involves an optimization over vector random variables. The main contribution of this chapter is an efficient numerical algorithm for this input optimization problem.

The input optimization problem for the vector Gaussian multiple access channel has been studied in the literature for several special cases. The capacity region of a Gaussian multiple access channel with intersymbol interference (ISI) was characterized by Cheng and Verdú [14]. For the multiple access channel with ISI, the input optimization problem can be formulated as a problem of optimal power allocation over frequencies. An analogous problem for i.i.d. fading channels was studied by Knopp and Humblet [15] and Tse and Hanly [16] where the optimal power allocation over time was characterized. Both the ISI channel and the scalar i.i.d. fading channel can be thought of as special cases of the vector multiple access channel considered here. In both cases, all individual channels in the multiple access channel can be simultaneously decomposed into independent sub-channels. For the ISI channel, a cyclic prefix can be appended to the input sequence so that the channel can be diagonalized in the frequency domain by a discrete Fourier transform. For the i.i.d. fading channel, the independence among the sub-channels in time is explicitly assumed. In both cases, the optimal signaling direction is just the direction of the simultaneous diagonalization, and the input optimization problem is reduced to the power allocation problem among the sub-channels.

The situation is considerably more complicated if simultaneous diagonalization is not possible. This more general setting corresponds to a multiple access situation where both the transmitters and the receiver are equipped with multiple antennas. In the spatial domain, the channel gain between a transmit antenna and a receive antenna can be arbitrary, so the channel matrix can have an arbitrary structure. It is in general not possible to simultaneously decompose an arbitrary set of matrix channels into parallel independent sub-channels. Unlike the ISI channels where the time-invariance property gives a special Toeplitz structure to the channel matrix, a multi-antenna channel does not follow spatial-invariance. Consequently, the equivalence of a cyclic prefix does not exist in the spatial domain, and the transmitter optimization problem becomes a combination of choosing the optimal signaling directions for each user and allocating a correct amount of power in each signaling direction. Such joint optimization is non-trivial, as the optimal solution needs to find a compromise between maximizing each user's data rate and minimizing its interference into other users. In this regard, only asymptotic results have been reported so far [17]. A similar situation exists for CDMA systems, where the matrix channel is determined by the spreading sequences. Recent results in this area have been obtained in [18] [19] [20].

The main result of this chapter is that the joint optimization of signaling power and signaling directions for a multiple access channel can be performed by a generalization of single-user input optimization. In a single-user vector channel, the optimal signaling directions are the eigen-modes of the channel matrix, and the optimal power allocation is the so-called water-filling allocation. For a vector multiple access channel, although each

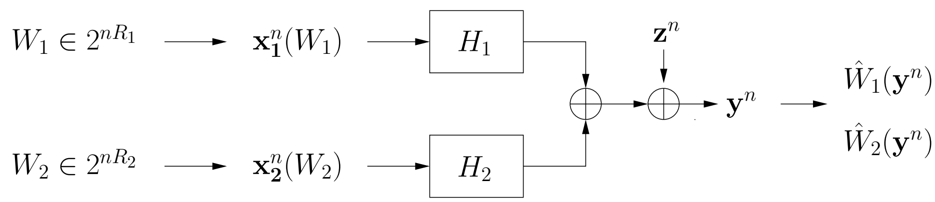


Figure 2.2: Gaussian vector multiple access channel

user has a different channel and experiences a different interference structure, it is possible to apply single-user water-filling iteratively to reach a compromise among the signaling strategies for different users. As it is shown later, this iterative water-filling procedure always converges, and it converges to the sum capacity of a vector multiple access channel.

Recently, the author learned that a similar iterative procedure was proposed independently by Médard [21] for single-antenna multi-path fading channels. Although the contexts are different, the idea and the principles are essentially the same.

The rest of this chapter is organized as follows: Section 2.1 formulates the input optimization problem for the Gaussian vector multiple access channel in a convex programming framework. Section 2.2 focuses on the rate-sum point and derives an iterative water-filling algorithm. Section 2.3 contains concluding remarks.

2.1 Gaussian Vector Multiple Access Channel

A memoryless two-user Gaussian vector multiple access channel is shown in Figure 2.2:

$$\mathbf{y} = H_1\mathbf{x}_1 + H_2\mathbf{x}_2 + \mathbf{z}, \tag{2.2}$$

where $\mathbf{x}_1$, $\mathbf{x}_2$ are input vector signals under power constraints P_1 and P_2 respectively, $\mathbf{y}$ is the output vector signal, $\mathbf{z}$ is the additive Gaussian noise vector whose covariance matrix is denoted as S_{zz}, and H_1, H_2 are channel matrices. The aim of this section is to formulate the input optimization problem for this multiple access channel. The development here is restricted to the two-user case for simplicity. The results can be easily generalized to cases with more than two users.

Following the development in [9] and [10], define the directly achievable region of a

Gaussian vector multiple access channel with power constraints q_1 and q_2 as:

$$\mathcal{A}(q_1, q_2) = \bigcup_{p_1(\mathbf{x}_1)p_2(\mathbf{x}_2)} \left\{ (R_1, R_2) : \begin{array}{c} R_1 \leq I(\mathbf{X}_1; \mathbf{Y}|\mathbf{X}_2); \\ R_2 \leq I(\mathbf{X}_2; \mathbf{Y}|\mathbf{X}_1); \\ R_1 + R_2 \leq I(\mathbf{X}_1, \mathbf{X}_2; \mathbf{Y}). \end{array} \right\} \tag{2.3}$$

where the union is taken over all independent input distributions $p_1(\mathbf{x}_1)p_2(\mathbf{x}_2)$ that satisfy the power constraints $\mathrm{tr}(\mathbf{E}[\mathbf{X}_1\mathbf{X}_1^T]) \leq q_1$ and $\mathrm{tr}(\mathbf{E}[\mathbf{X}_2\mathbf{X}_2^T]) \leq q_2$.

For a fixed input distribution $p_1(\mathbf{x}_1)p_2(\mathbf{x}_2)$, let S_1 and S_2 be the covariance matrices of $\mathbf{X}_1$ and $\mathbf{X}_2$ under the respective marginals:

$$S_1 = \mathbf{E}[\mathbf{X}_1\mathbf{X}_1^T], \tag{2.4}$$

$$S_2 = \mathbf{E}[\mathbf{X}_2\mathbf{X}_2^T]. \tag{2.5}$$

The mutual information expressions in (2.3) can be bounded as follows:

$$\begin{aligned} I(\mathbf{X}_1; \mathbf{Y}|\mathbf{X}_2) &= h(\mathbf{Y}|\mathbf{X}_2) - h(\mathbf{Y}|\mathbf{X}_1, \mathbf{X}_2) \\ &= h(H_1\mathbf{X}_1 + \mathbf{Z}) - h(\mathbf{Z}) \\ &\leq \frac{1}{2}\log\frac{|H_1S_1H_1^T + S_{zz}|}{|S_{zz}|}, && (2.6) \\ I(\mathbf{X}_2; \mathbf{Y}|\mathbf{X}_1) &\leq \frac{1}{2}\log\frac{|H_2S_2H_2^T + S_{zz}|}{|S_{zz}|}, && (2.7) \\ I(\mathbf{X}_1, \mathbf{X}_2; \mathbf{Y}) &\leq \frac{1}{2}\log\frac{|H_1S_1H_1^T + H_2S_2H_2^T + S_{zz}|}{|S_{zz}|}, && (2.8) \end{aligned}$$

where the inequalities follow from the fact that the maximum-entropy distribution under a covariance constraint is a Gaussian distribution. It is then easy to see that Gaussian distributions $\mathbf{X}_1 \sim \mathcal{N}(0, S_1)$ and $\mathbf{X}_2 \sim \mathcal{N}(0, S_2)$ simultaneously maximize all three mutual information bounds.

Assuming Gaussian inputs, the directly achievable region can now be expressed as:

$$\mathcal{A}(q_1, q_2) = \bigcup_{\substack{\mathrm{tr}(S_1) \leq q_1, \\ \mathrm{tr}(S_2) \leq q_2, \\ S_1, S_2 \geq 0}} \mathcal{B}(S_1, S_2), \tag{2.9}$$

where

$$\mathcal{B}(S_1, S_2) = \left\{ (R_1, R_2) : \begin{array}{c} R_1 \leq C_1(S_1); \\ R_2 \leq C_2(S_2); \\ R_1 + R_2 \leq C_{12}(S_1, S_2). \end{array} \right\}, \tag{2.10}$$

and

$$C_1(S_1) = \frac{1}{2} \log \frac{|H_1 S_1 H_1^T + S_{zz}|}{|S_{zz}|}, \tag{2.11}$$

$$C_2(S_2) = \frac{1}{2} \log \frac{|H_2 S_2 H_2^T + S_{zz}|}{|S_{zz}|}, \tag{2.12}$$

$$C_{12}(S_1, S_2) = \frac{1}{2} \log \frac{|H_2 S_2 H_2^T + H_1 S_1 H_1^T + S_{zz}|}{|S_{zz}|}. \tag{2.13}$$

The directly achievable region for a multiple access channel is the rate region achievable with only stationary inputs. The directly achievable region is the capacity region if the input terminals do not have the ability to synchronize with each other [22]. For multiple access channels where input terminals can coordinate their timing, the time-sharing or convex combinations of directly achievable rate pairs are also achievable. The convex hull operation can potentially enlarge the region, and the convex hull must be taken over the constraint sets as well as the rate regions themselves. More precisely, as characterized in [9] and [10], the capacity region can be expressed as:

$$\mathcal{C}(P_1, P_2) = \text{closure} \left\{ (R_1, R_2) : \begin{array}{l} ((R_1, R_2), (P_1, P_2)) \in \\ \text{convex} \bigcup\limits_{q_1, q_2 \geq 0} (\mathcal{A}(q_1, q_2), (q_1, q_2)) \end{array} \right\}. \tag{2.14}$$

However, it turns out that for Gaussian vector multiple access channels, the capacity region is just the directly achievable region, and the convex hull operation is not necessary.

Lemma 2.1 $\log|M|$ *is concave in the space of positive semi-definite matrices* M*.*

Proof: See [23, p.466], [24, p.48], or [25]. □

Theorem 2.1 *For a Gaussian vector multiple access channel* $\mathbf{y} = H_1\mathbf{x}_1 + H_2\mathbf{x}_2 + \mathbf{z}$ *under power constraints* P_1, P_2, *the capacity region* $\mathcal{C}(P_1, P_2)$ *is precisely* $\mathcal{A}(P_1, P_2)$ *without the convex hull and union operations. The capacity region is convex and its extreme points may* *be found by maximizing a weighted sum of data rates $\mu_1 R_1 + \mu_2 R_2$, where $\mu_1 \geq 0$, $\mu_2 \geq 0$ and $\mu_1 + \mu_2 = 1$. Without loss of generality, assume $\mu_1 \leq \mu_2$. Then, the optimization problem is:*

$$\begin{array}{ll} \textit{maximize} & \mu_1 \cdot \frac{1}{2} \log |H_1 S_1 H_1^T + H_2 S_2 H_2^T + S_{zz}| + \\ & (\mu_2 - \mu_1) \cdot \frac{1}{2} \log |H_2 S_2 H_2^T + S_{zz}| - \mu_2 \cdot \frac{1}{2} \log |S_{zz}| \\ \textit{subject to} & \operatorname{tr}(S_1) \leq P_1, \\ & \operatorname{tr}(S_2) \leq P_2, \\ & S_1, S_2 \geq 0. \end{array} \tag{2.15}$$

Proof: The main claim here is that the convex hull operation on the rate regions and the constraints is not necessary, and the capacity region $\mathcal{C}(P_1, P_2)$ is just $\mathcal{A}(P_1, P_2)$. This is a direct consequence of Lemma 2.1. First, let's consider the convex combination of two rate-power pairs $((R_1, R_2), q_1, q_2)$ and $((R_1', R_2'), q_1', q_2')$, where $(R_1, R_2) \in \mathcal{A}(q_1, q_2)$ and $(R_1', R_2') \in \mathcal{A}(q_1', q_2')$. Since $\mathcal{A}(q_1, q_2)$ is a union of pentagons, there exist (S_1, S_2) and (S_1', S_2') such that $(\operatorname{tr}(S_1), \operatorname{tr}(S_2)) \leq (q_1, q_2)$, $(\operatorname{tr}(S_1'), \operatorname{tr}(S_2')) \leq (q_1', q_2')$, $(R_1, R_2) \in \mathcal{B}(S_1, S_2)$ and $(R_1', R_2') \in \mathcal{B}(S_1', S_2')$. ("$\leq$" here means less than or equal to in each component.) Now, consider a convex combination of the rate-power pairs $(0 \leq \alpha \leq 1)$:

$$\alpha((R_1, R_2), q_1, q_2) + (1 - \alpha)((R_1', R_2'), q_1', q_2'). \tag{2.16}$$

For this convex combination to be in $\mathcal{C}(P_1, P_2)$, it must satisfy the power constraint, i.e. $\alpha(q_1, q_2) + (1-\alpha)(q_1', q_2') \leq (P_1, P_2)$, or $\alpha(\operatorname{tr}(S_1), \operatorname{tr}(S_2)) + (1-\alpha)(\operatorname{tr}(S_1'), \operatorname{tr}(S_2')) \leq (P_1, P_2)$.

Now, define $\hat{S}_1 = \alpha S_1 + (1 - \alpha) S_1'$, and $\hat{S}_2 = \alpha S_2 + (1 - \alpha) S_2'$. The claim is that the achievable rates with $\hat{S}_1$ and $\hat{S}_2$ is as large as the convex combination of the original points. First, verify that the power constraints are satisfied.

$$(\operatorname{tr}(\hat{S}_1), \operatorname{tr}(\hat{S}_2)) \leq (P_1, P_2), \tag{2.17}$$

Next, by Lemma 2.1, C_1, C_2 and C_{12} are concave functions. So,

$$\begin{array}{rcccl} \alpha R_1 + (1-\alpha) R_1' & \leq & \alpha C_1(S_1) + (1-\alpha) C_1(S_1') & \leq & C_1(\hat{S}_1) \\ \alpha R_2 + (1-\alpha) R_2' & \leq & \alpha C_2(S_2) + (1-\alpha) C_2(S_2') & \leq & C_2(\hat{S}_2) \\ \alpha(R_1 + R_2) + (1-\alpha)(R_1' + R_2') & \leq & \alpha C_{12}(S_1, S_2) + (1-\alpha) C_{12}(S_1', S_2') & \leq & C_{12}(\hat{S}_1, \hat{S}_2). \end{array}$$

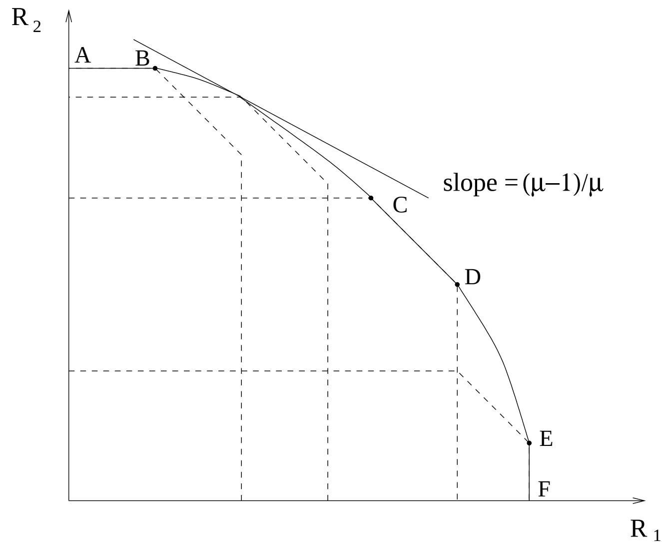


Figure 2.3: Capacity region of a Gaussian vector multiple access channel

Therefore, any rate-power pair that can be achieved via convex combination can also be achieved directly. Thus, the convex hull operation in (2.14) is not necessary. Finally, since $\mathcal{A}(q_1, q_2) \subseteq \mathcal{A}(q_1', q_2')$ whenever $(q_1, q_2) \leq (q_1', q_2')$, the union operation also simplifies, and the capacity region $\mathcal{C}(P_1, P_2)$ is just $\mathcal{A}(P_1, P_2)$.

Note that the above argument also implies that $\mathcal{A}(P_1, P_2)$ is a convex region, and its boundary points can be found by maximizing the weighted sums of the data rates $\mu_1 R_1 + \mu_2 R_2$. Because $\mathcal{A}(P_1, P_2)$ is the union of pentagons, the maximization can be done in two steps: first maximize within each pentagon, then maximize over all pentagons. When $\mu_1 \leq \mu_2$, the maximizing point within each pentagon is the corner point $R_2 = C_2(S_2)$, $R_1 = C_{12}(S_1, S_2) - C_2(S_2)$. Then, the maximization over all pentagons is just a maximization over all such corner points. This gives (2.15). $\square$

Figure 2.3 shows a typical capacity region of a Gaussian vector multiple access channel as a union of pentagons. Each pentagon corresponds to an achievable rate region for a fixed pair of transmit covariance matrices.

Aspects of Theorem 2.1 have been observed in several different contexts. The first part of the theorem is a special case of Theorem 1 in [14], where multiple access channels with memory are treated. The approach here does not use general results from channels with memory [26], and input constraints are dealt with explicitly. Similar results have also appeared in [16], [17] and [18] where the single-antenna fading channel, vector fading channel and CDMA channel are treated respectively. The concavity of the $\log|\cdot|$ function was previously observed in [17] and [27] for sum capacity. The connection between concavity and the ability to remove the convex hull operation is shown explicitly here.

Concavity is a key observation not only in simplifying the capacity expression but also in providing computationally efficient algorithms to compute the capacity numerically. The optimization problem in Theorem 2.1 belongs to a class of convex programming problems for which the global optimum can be found efficiently [24] [25]. In fact, the classical water-filling and the multi-user water-filling algorithm in [14] can be thought of as special purpose convex optimization algorithms.

For the sake of completeness, the analogous result for the general K-user multiple access channel is stated below. The proof is an easy generalization of the two-user case.

Theorem 2.2 *For a K-user multiple access channel $\mathbf{y} = \sum_{i=1}^{K} H_i \mathbf{x}_i + \mathbf{z}$ with power constraints $P_1, \cdots, P_K$, the input distributions that maximize $\sum_{i=1}^{K} \mu_i R_i$, with $0 \leq \mu_1 \leq \cdots \leq \mu_K$ and $\sum_{i=1}^{K} \mu_i = 1$, are Gaussian distributions whose covariance matrices $S_1, \cdots, S_K$ can be found by solving the following optimization problem:*

$$\begin{array}{lll} \textit{maximize} & \mu_1 \cdot \frac{1}{2} \log \left| \sum_{i=1}^{K} H_i S_i H_i^T + S_{zz} \right| - \mu_K \cdot \frac{1}{2} \log |S_{zz}| + & \\ & \sum_{j=2}^{K} (\mu_j - \mu_{j-1}) \cdot \frac{1}{2} \log \left| \sum_{i=j}^{K} H_i S_i H_i^T + S_{zz} \right| & \\ \textit{subject to} & \mathrm{tr}(S_i) \leq P_i, & i = 1, \cdots, K \\ & S_i \geq 0. & i = 1, \cdots, K \end{array} \tag{2.18}$$

2.2 Sum Capacity

The previous section shows that the capacity region for a vector Gaussian multiple access channel may be found by solving a convex programming problem. Although, in theory, there exist efficient numerical algorithms for all convex optimization problems; in practice, the optimization can still be computationally intensive. This is particularly true for the input optimization problem considered here, because the optimization over S_i is performed in the space of positive semi-definite matrices, and the number of scalar variables grows quadratically with the number of input dimensions. In the single-user case, the input optimization problem has a well-known water-filling solution. The water-filling algorithm greatly reduces the computational complexity by taking advantage of the problem structure. The purpose of this section is to show that a similar reduction in computational complexity may be realized for the multiple access channel rate sum maximization problem by extending single-user water-filling to the multi-user case.

2.2.1 Single-User Water-filling

First, let's cast the single-user water-filling problem into a convex programming framework. In the single-user case, the mutual information maximization problem is the following:

$$\begin{array}{ll} \text{maximize} & \frac{1}{2}\log\frac{|HSH^T + S_{zz}|}{|S_{zz}|} \\ \text{subject to} & \mathrm{tr}(S) \leq P, \\ & S \geq 0. \end{array} \tag{2.19}$$

This maximization problem has an analytical solution. First, since S_{zz} is a symmetric positive definite matrix, it has an eigenvalue decomposition $S_{zz} = Q\Delta Q^T$, where Q is an orthogonal matrix $QQ^T = I$, and Δ is a diagonal matrix. Defining $\hat{H} = \Delta^{-\frac{1}{2}}Q^T H$, the objective can then be re-written as

$$\text{maximize} \quad \frac{1}{2}\log|\hat{H}S\hat{H}^T + I|. \tag{2.20}$$

Next, let $\hat{H} = F\Sigma M^T$ be a singular-value decomposition of $\hat{H}$, where F and M are orthogonal matrices, and Σ is a diagonal matrix of singular values $\sigma_1, \sigma_2, \cdots, \sigma_r$, where r is the rank of $\hat{H}$. Consider $\hat{S} = M^T SM$ as the new optimization variable. Since $\mathrm{tr}(S) = \mathrm{tr}(\hat{S})$, the problem can now be transformed into,

$$\begin{array}{ll} \text{maximize} & \frac{1}{2}\log|\Sigma\hat{S}\Sigma^T + I| \\ \text{subject to} & \mathrm{tr}(\hat{S}) \leq P, \\ & \hat{S} \geq 0. \end{array} \tag{2.21}$$

Using Hadamard's inequality [4], it can be shown that the optimal $\hat{S}$ is a diagonal matrix, $\mathrm{diag}(p_1, p_2, \cdots, p_r)$, where the diagonal entries satisfy:

$$p_i + 1/\sigma_i^2 = L, \quad \text{if} \quad 1/\sigma_i^2 < L, \tag{2.22}$$

$$p_i = 0, \quad \text{if} \quad 1/\sigma_i^2 \geq L, \tag{2.23}$$

where L is a constant chosen so that $\sum_i p_i = P$. This solution is called water-filling because $1/\sigma_i^2$ can be thought of as the bottom of a bowl, and p_i may be thought of as the amount of water poured into the bowl.

Thus, to achieve the Gaussian vector channel capacity, the transmitter must align its transmit direction with the right singular-vectors of the effective channel and allocate an appropriate amount of energy in each direction in a water-filling fashion. Solving the single-user input optimization problem via water-filling is more efficient than using general purpose convex programming algorithms because water-filling takes advantage of the problem structure by decomposing the equivalent channel along its eigen-modes.

2.2.2 Simultaneous Water-filling

The first step in generalizing single-user water-filling to the multi-user setting is the following necessary and sufficient condition for the optimal input distribution in a multiple access channel.

Theorem 2.3 *Consider a K-user multiple access channel $\mathbf{y} = \sum_{i=1}^K H_i \mathbf{x}_i + \mathbf{z}$. The set of covariance matrices S_i is a solution to the rate-sum maximization problem*

$$\begin{array}{llr} \textit{maximize} & \frac{1}{2}\log\left|\sum_{i=1}^K H_i S_i H_i^T + S_{zz}\right| - \frac{1}{2}\log|S_{zz}| & \\ \textit{subject to} & \mathrm{tr}(S_i) \leq P_i, & i = 1, \dots, K \\ & S_i \geq 0, & i = 1, \dots, K \end{array} \tag{2.24}$$

if and only if each S_i is the single-user water-filling covariance matrix for the channel H_i with noise $S_{zz} + \sum_{j=1, j\neq i}^K H_j S_j H_j^T$.

Proof: The *only if* part holds for the following reason: Suppose that at the rate-sum optimum, S_i does not satisfy the single-user water-filling condition. Then, a single-user water-filling for S_i would increase the sum capacity, as the single-user rate optimization problem differs from the rate-sum optimization problem only by a constant. Thus, at the optimum, all S_i's must satisfy the single-user water-filling condition.

The *if* part also holds. The proof relies on the Karush-Kuhn-Tucker (KKT) condition for the optimization problem. Basic results on KKT conditions are included in Appendix A. First, (2.24) can be reformulated into an equivalent form:

$$\begin{array}{lll} \text{minimize} & -\log |T| & \\ \text{subject to} & T \leq \sum_{i=1}^{K} H_i S_i H_i^T + S_{zz} & \\ & \operatorname{tr}(S_i) \leq P_i, & i = 1, \ldots, K, \\ & S_i \geq 0, & i = 1, \ldots, K. \end{array} \tag{2.25}$$

The coefficient 1/2 and the constant $\log |S_{zz}|$ are omitted for simplicity. Associate dual variables Γ, $\{\lambda_i\}$, $\{\Psi_i\}$ to each of the constraints. Note that the first and the third constraints are matrix inequalities, so the dual variables Γ and $\{\Psi_i\}$ are positive semi-definite matrices. The dual variables for the power constraints $\{\lambda_i\}$ are real. The Lagrangian for the optimization problem is:

$$\begin{aligned} & L(\{S_i\}, T, \Gamma, \{\lambda_i\}, \{\Psi_i\}) \\ & = -\log |T| + \operatorname{tr}\left[\Gamma\left(T - \sum_{i=1}^{K} H_i S_i H_i^T - S_{zz}\right)\right] + \sum_{i=1}^{K} \lambda_i (\operatorname{tr}(S_i) - P_i) - \sum_{i=1}^{K} \operatorname{tr}(\Psi_i S_i) \\ & = -\log |T| + \operatorname{tr}(\Gamma T) - \operatorname{tr}(\Gamma S_{zz}) - \sum_{i=1}^{K} \lambda_i P_i + \sum_{i=1}^{K} \operatorname{tr}[(\lambda_i I - H_i^T \Gamma H_i - \Psi_i) S_i] \end{aligned} \tag{2.26}$$

where the fact $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ is used. The objective of the dual program is

$$g(\Gamma, \{\lambda_i\}, \{\Psi_i\}) = \inf_{\{S_i\}, T} L(\{S_i\}, T, \Gamma, \{\lambda_i\}, \{\Psi_i\}). \tag{2.27}$$

At the infimum, $\partial L / \partial S_i$ must be zero. This leads to:

$$\lambda_i I = H_i^T \Gamma H_i + \Psi_i, \quad i = 1, 2, \cdots, K. \tag{2.28}$$

Note that the above is equivalent to $\lambda_i I \geq H_i^T \Gamma H_i$, since $\{\Psi_i\}$ only need to be positive semi-definite. Further, $\partial L / \partial T$ must be zero, i.e. $\frac{\partial}{\partial T}(-\log |T| + \operatorname{tr}(\Gamma T)) = 0$. This implies that

$$T^{-1} = \Gamma. \tag{2.29}$$

Therefore, $g(\Gamma, \{\lambda_i\}, \{\Psi_i\}) = \log|\Gamma| + m - \mathrm{tr}(\Gamma S_{zz}) - \sum_{i=1}^{K} \lambda_i P_i$, where m is the dimension of the output vector $\mathbf{y}$. The dual problem of (2.24) is then:

$$\begin{array}{lll} \text{maximize} & \log|\Gamma| + m - \mathrm{tr}(\Gamma S_{zz}) - \displaystyle\sum_{i=1}^{K} \lambda_i P_i & \\ \text{subject to} & \lambda_i I \geq H_i^T \Gamma H_i, & i = 1, \ldots, K \\ & \Gamma \geq 0. & \end{array} \tag{2.30}$$

Because the primal program is convex, the dual problem attains a maximum at the solution of the primal problem.

The primal constraints satisfy Slater's condition, so the KKT condition is sufficient and necessary for optimality. The KKT conditions include the stationarity conditions on the Lagrangian (2.28) and (2.29), as well as the complementary slackness conditions:

$$\mathrm{tr}\left[\Gamma\left(T - \sum_{i=1}^{K} H_i S_i H_i^T - S_{zz}\right)\right] = 0, \tag{2.31}$$

$$\lambda_i(\mathrm{tr}(S_i) - P_i) = 0, \qquad i = 1, \cdots, K \tag{2.32}$$

$$\mathrm{tr}(\Psi_i S_i) = 0, \qquad i = 1, \cdots, K \tag{2.33}$$

Observe that at the optimum, $T = \sum_{i=1}^{K} H_i S_i H_i^T + S_{zz}$, and $\mathrm{tr}(S_i) = P_i, i = 1, \ldots, K$. So, only the last complementary slackness condition (2.33) is useful. Because the stationarity and complementary slackness conditions, together with primal and dual constraints, are necessary and sufficient for optimality, the optimization problem can be transformed into a problem of finding primal variables $\{S_i\}$, T and dual variables Γ, $\{\Psi_i\}$, $\{\lambda_i\}$ that satisfy:

$$\begin{array}{ll} \lambda_i I = H_i^T \left(\displaystyle\sum_{j=1}^{K} H_j S_j H_j^T + S_{zz}\right)^{-1} H_i + \Psi_i, & \\ \mathrm{tr}(S_i) = P_i, & \\ \mathrm{tr}(\Psi_i S_i) = 0, & i = 1, \cdots, K. \\ \Psi_i, S_i, \lambda_i \geq 0. & \end{array} \tag{2.34}$$

Observe that the above set of KKT conditions is also valid for the single-user water-filling problem when K is set to 1. In this case, it is easy to verify that the single-user solution (2.22)-(2.23) satisfies the set of KKT conditions exactly. Now, for the multiple access channel, for each user i, the multi-user KKT condition and the single-user KKT condition differ only by an additional noise term $\sum_{j=1,j\neq i}^{K} H_j S_j H_j^T$. So, if each S_i satisfies the single-user condition while regarding interference from other users as additional noise, then collectively, the set of $\{S_i\}$ must also satisfy the multi-user KKT condition. By the sufficiency of the KKT condition, $\{S_i\}$ must be the optimal covariance matrix for the multi-user problem. This proves the *if* part of the theorem. □

2.2.3 Iterative Water-filling

At the rate-sum optimum, each user's covariance matrix is a water-filling covariance against the combined noise and interference. This suggests that water-filling may be done iteratively to achieve the multiple access sum capacity.

Algorithm 2.1 *Iterative water-filling for a Gaussian vector multiple access channel:*

initialize $S_i = 0$, $i = 1, \ldots K$.

repeat

for i=1 to K

$$Z = \sum_{j=1,j\neq i}^{K} H_j S_j H_j^T + S_{zz};$$

$$S_i = \arg\max_S \frac{1}{2} \log |H_i S H_i^T + Z|, \textit{ subject to } tr(S) \leq P_i;$$

end

until the desired accuracy is reached.

Theorem 2.4 *The iterative water-filling algorithm converges to a limit point from any initial assignment of* S_i. *The limit point maximizes the sum capacity of a Gaussian vector multiple access channel.*

Proof: At each step, the iterative water-filling algorithm finds the single-user water-filling covariance matrix for each user while regarding interference from all other users as additional noise. Since the single-user rate objective differs from the multi-user rate-sum objective by only a constant, the rate-sum objective is non-decreasing after each water-filling step. The rate-sum objective is bounded above, so the rate-sum converges to a limit. At the limit point, all of $S_1, \cdots, S_K$ are simultaneously single-user water-filling covariance matrices while treating the interference from other users as additional noise. Then, by Theorem 2.3, the limit must be rate-sum optimal.

The above proof does not depend on the initial value. So the algorithm converges to the optimum rate sum from any starting point. □

The multiple access channel capacity is achieved with superposition coding and successive decoding. A decoding strategy where each user treats the interference from other users as noise is not capacity achieving. Yet, for the input optimization problem, an iterative procedure where in every step, each user does a single-user water-filling while treating the interference from other users as noise just happens to be the one that converges to an optimal set of transmit covariance matrices.

The set of rate-sum optimal covariance matrices is not necessarily unique. Depending on the initial value, the iterative water-filling algorithm may converge to two different sets of covariance matrices both giving the optimal sum rate. The following is an example when this happens. Let $H_1 = H_2 = S_{zz} = I_{2\times 2}$ and $P_1 = P_2 = 2$. Then, $S_1 = S_2 = I_{2\times 2}$, and $S_1' = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ $S_2' = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ both achieve the same maximum rate-sum.

Figure 2.4 gives a graphical interpretation of the algorithm. The capacity region of a two-user vector multiple access channel is shown in Figure 2.4(a). The sum rate $R_1 + R_2$ reaches the maximum on the line connecting points 'C' and 'D'. Initially, the covariance matrices for the two users, $S_1^{(0)}$ and $S_2^{(0)}$, are zero matrices.

1. The first iteration is shown in Figure 2.4(b). After a single-user water-filling for $S_1^{(1)}$, the rate pair (R_1, R_2) is at point 'F'. Then, treating $S_1^{(1)}$ as noise, a single-user water-filling for $S_2^{(1)}$ moves the rate pair to point 'E'.

2. The second iteration is shown in Figure 2.4(c). First, note that fixing covariance matrices $S_1^{(1)}$ and $S_2^{(1)}$, the capacity region is the pentagon 'abEFO'. So, by changing the decoding order of user 1 and 2, the rate pair can be moved to point 'b' without affecting the rate sum. Once at point 'b', a water-filling for S_1 can be performed treating $S_2^{(1)}$ as noise to get $S_1^{(2)}$. This would increase $I(\mathbf{X}_1; \mathbf{Y})$, while keeping $I(\mathbf{X}_2; \mathbf{Y}|\mathbf{X}_1)$ fixed, thus moving the rate pair to point 'c'.

3. The capacity pentagon with $(S_1^{(2)}, S_2^{(1)})$ is now represented by 'acdeO'. The decoding

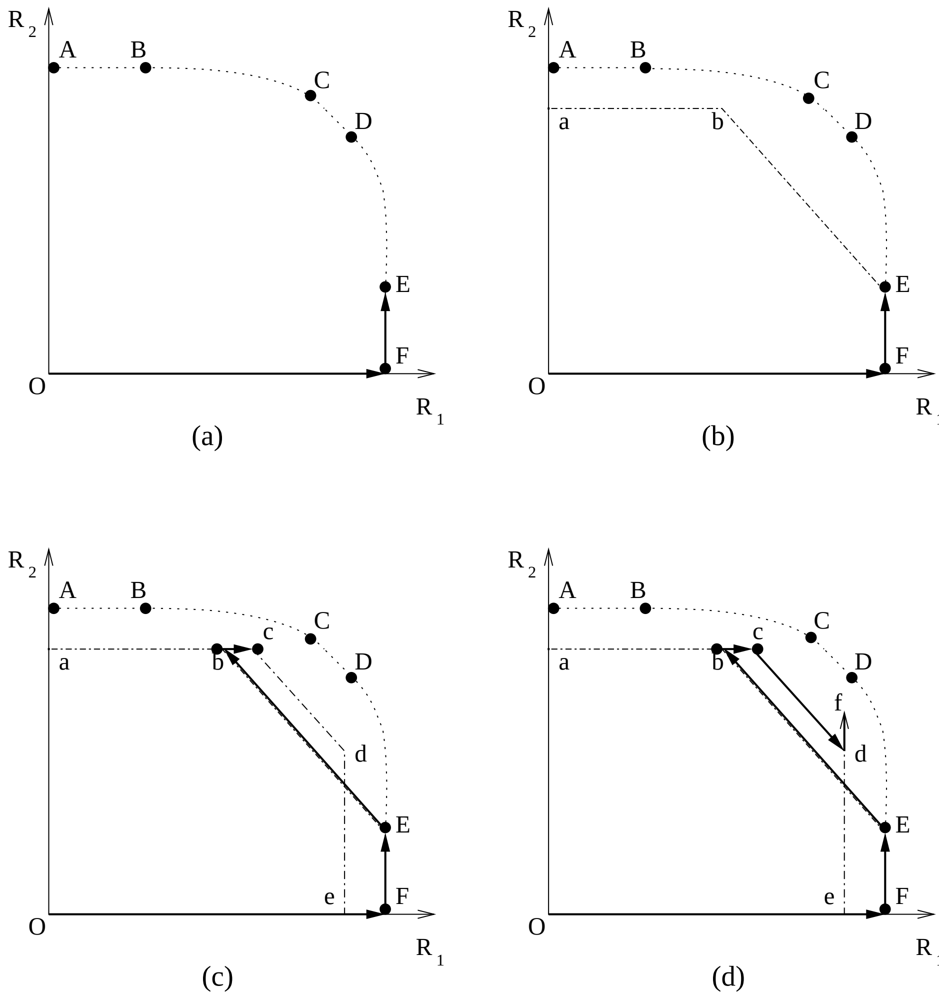


Figure 2.4: Illustration of iterative water-filling

order can be interchanged again to arrive at point 'd'. Another single-user water-filling may then be performed treating $S_2^{(1)}$ as noise. This gives $S_1^{(2)}$ and its corresponding rate-pair point 'f' in Figure 2.4(d). The process continues until it converges to points 'C' and 'D'.

Note that in every step, each user negotiates for itself the best signaling direction as well as the optimal power allocation while regarding the interference generated by all other users as noise. The iterative water-filling algorithm is more efficient than general-purpose convex programming routines, because in each step the algorithm takes advantage of the problem structure by doing an eigen-mode decomposition and water-filling. In fact, the convergence is very fast as the following theorem shows:

Theorem 2.5 *After one iteration of the iterative water-filling algorithm, the set of transmit covariance matrices achieves a sum rate that is at most $(K-1)m/2$ nats away from the sum capacity, where K is the total number of users and m is the dimension of the received vector.*

After the first iteration, the iterative procedure arrives at a corner point of a pentagon. The above theorem states that the corner point is only $1/2$ nats per user per output dimension from the sum capacity. The proof of this theorem uses the duality gap bound in convex analysis and is presented in Appendix B.

2.3 Summary

This chapter considers a Gaussian vector multiple access channel where transmitters do not cooperate but receivers cooperate. The capacity region for the vector multiple access channel is characterized, and the input optimization problem is formulated as a convex programming problem. It is shown that the sum-rate maximization problem can be solved efficiently by an iterative water-filling algorithm, where each step of the algorithm corresponds to a local maximization of one user's individual data rate while treating interference as noise. The iterative process is guaranteed to converge, and it converges to the maximum sum capacity of the multiple access channel. Thus, in a Gaussian vector multiple access channel, a competitive optimum is also a global optimum.

Chapter 3

Broadcast Channel

A communication situation where a single transmitter sends independent information to multiple uncoordinated receivers is referred to as a broadcast channel [6]. Figure 3.1 illustrates a two-user broadcast channel, where independent messages W_1 and W_2 are jointly encoded by the transmitter X, and the receivers Y_1 and Y_2 are each responsible for decoding W_1 and W_2, respectively. A $(n, 2^{nR_1}, 2^{nR_2})$ codebook for a broadcast channel consists of an encoding function $X^n(W_1, W_2)$ where $W_1 \in \{1, \cdots, 2^{nR_1}\}$ and $W_2 \in \{1, \cdots, 2^{nR_2}\}$ and decoding functions $\hat{W}_1(Y_1^n)$ and $\hat{W}_2(Y_2^n)$. An error occurs when $W_1 \neq \hat{W}_1$ or $W_2 \neq \hat{W}_2$. A rate pair (R_1, R_2) is achievable if there exists a sequence of $(n, 2^{nR_1}, 2^{nR_2})$ codebooks for which the average probability of error $P_e^n \to 0$ as $n \to \infty$. The capacity region of a broadcast channel is the union of all achievable rate pairs.

An obvious transmission strategy for the broadcast channel is to send independent information to different receivers at different time-slots. This strategy is called time-division multiplex (TDM). However, as Cover pointed out [6] [28], time-division multiplex is not necessarily optimal, and a strategy that superimposes one user's information on top of another can do strictly better. In fact, this superposition scheme has been shown to be optimal for the class of degraded broadcast channels, where one user's received signal is a noisier version of the other [29] [30]. However, for non-degraded broadcast channels, superposition is in general sub-optimal, and the capacity region is still an unsolved problem. The largest achievable region for the non-degraded broadcast channel is due to Marton [31] [32], but no converse has been established. Over the years, the broadcast channel has become one of the most basic open problems in multi-user information theory [33].

This thesis makes progress on the broadcast channel problem by solving for the sum

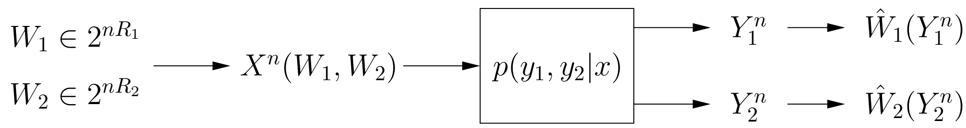


Figure 3.1: Broadcast channel

capacity of a particular class of non-degraded Gaussian vector broadcast channels. The main difficulty in the broadcast channel problem is that a broadcast channel distributes information across several receivers, and without joint processing of the received signals, it is not possible to communicate at a rate equal to the mutual information between the input and the outputs. The contribution of this chapter is to show that for a Gaussian vector broadcast channel, under a certain non-singularity condition, an equivalent of receiver processing can be implemented at the transmitter by precoding, and the optimal precoder takes the form of a generalized decision-feedback equalizer across the user domain. The optimal precoder can also be thought of as a generalization of Tomlinson-Harashima precoder. This generalization is related to shaping codes, and a practical precoding scheme based on trellis shaping is also proposed in this chapter. The solution to the sum-capacity problem for the broadcast channel illustrates the value of cooperation at the receiver. Without receiver cooperation, the capacity of a Gaussian vector channel becomes a saddle-point of a mutual information game, where "nature" effectively puts forth a worst possible noise correlation.

The vector broadcast channel arises in digital subscriber line (DSL) systems where coordination is possible at one end of the telephone cable bundle, but not the other. In this context, Ginis and Cioffi [34] [35] [36] investigated a multi-line precoder that takes advantage of the transmitter coordination to cancel crosstalk interference. An information-theoretical treatment of the Gaussian vector broadcast channel is given by Caire and Shamai [37] [38], who characterized the sum capacity of a broadcast channel with two transmit antennas and two users each equipped with a single antenna. This thesis generalizes these two results to vector channels with an arbitrary number of transmit antennas and an arbitrary number of users each equipped with multiple receive antennas, but under a certain non-singularity condition. Simultaneous and independent work in Gaussian vector broadcast channels has been carried out in [39] [40]. These efforts rely on a duality between the multiple access channel and the broadcast channel and provide alternative proofs for the sum capacity.

The rest of this chapter is organized as follows: In Section 3.1, the Gaussian vector broadcast channel problem is formulated, and a precoding scheme based on channels with transmitter side information is described. In Section 3.2, the optimal precoding structure is shown to be closely related to a generalized decision-feedback equalizer. In Section 3.3, an outer bound for the sum capacity of the Gaussian broadcast channel is computed, and the decision-feedback precoder is shown to achieve the outer bound, thus proving the main capacity result. Section 3.4 summarizes the information theoretical part of chapter by illustrating the value of cooperation for a Gaussian vector channel. Section 3.5 proposes a practical precoding scheme based on trellis shaping codes. Section 3.6 provides a conclusion for the chapter.

3.1 Precoding for Gaussian Broadcast Channels

A Gaussian vector broadcast channel refers to a broadcast channel where the law of the channel transition probability $p(y_1, y_2|x)$ is Gaussian, and where $\mathbf{x}$, $\mathbf{y}_1$ and $\mathbf{y}_2$ are vector valued. Figure 3.2 illustrates a two-user Gaussian vector broadcast channel:

$$\begin{aligned} \mathbf{y_1} &= H_1\mathbf{x} + \mathbf{z_1} \\ \mathbf{y_2} &= H_2\mathbf{x} + \mathbf{z_2}, \end{aligned} \tag{3.1}$$

where $\mathbf{x}$ is the transmit signal, $\mathbf{y_1}$ and $\mathbf{y_2}$ are receive signals, H_1, H_2 are channel matrices, and $\mathbf{z_1}$, $\mathbf{z_2}$ are Gaussian vector noises. Independent information is to be sent to each receiver. This thesis characterizes the maximum sum rate $R_1 + R_2$. The development here is restricted to the two-user case for simplicity. The results can be generalized easily to cases with more than two users.

When a Gaussian broadcast channel has a scalar input and scalar outputs, it can be regarded as a degraded broadcast channel for which the capacity region is well established [4]. A broadcast channel is degraded if $p(\mathbf{y_1}, \mathbf{y_2}|\mathbf{x}) = p(\mathbf{y_1}|\mathbf{x})p(\mathbf{y_2}|\mathbf{y_1})$. Intuitively, this means that one user's signal is a noisier version of the other user's signal. Consider the Gaussian scalar broadcast channel:

$$\begin{aligned} y_1 &= x + z_1 \\ y_2 &= x + z_2, \end{aligned} \tag{3.2}$$

where x is the scalar transmitted signal subject to a power constraint P, y_1 and y_2 are the

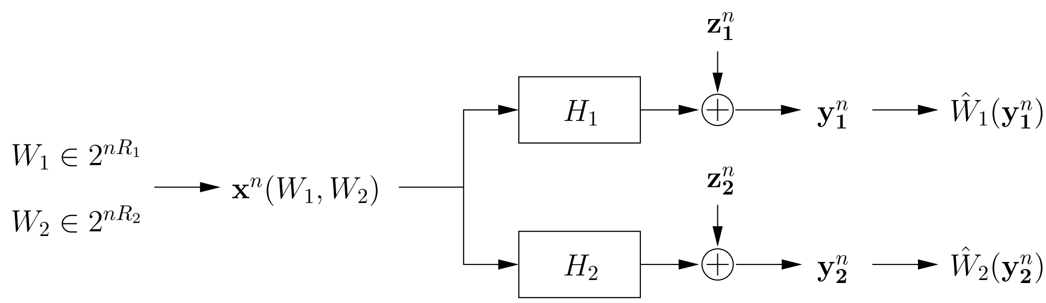


Figure 3.2: Gaussian vector broadcast channel

received signals, and z_1 and z_2 are the additive white Gaussian noises with variances σ_1^2 and σ_2^2, respectively. Without loss of generality, assume $\sigma_1 < \sigma_2$. Then, z_2 can be re-written as $z_2' = z_1 + z'$, where $z' \sim \mathcal{N}(0, \sigma_2^2 - \sigma_1^2)$ is independent of z_1. Since z_2' has the same distribution as z_2, y_2 is now equivalent to $y_1 + z'$. Thus, y_2 can be regarded as a degraded version of y_1. The capacity region for a degraded broadcast channel is achieved using a superposition coding and interference subtraction scheme due to Cover [6]. The idea is to divide the total power into $P_1 = \alpha P$ and $P_2 = (1-\alpha)P$ $(0 \leq \alpha \leq 1)$ and to construct two independent Gaussian codebooks for the two users with powers P_1 and P_2, respectively. To send two independent messages, one codeword is chosen from each codebook, and their sum is transmitted. Because y_2 is a degraded version of y_1, the codeword intended for y_2 can also be decoded by y_1. Thus, y_1 can subtract the effect of the codeword intended for y_2 and can effectively get a cleaner channel with noise power σ_1^2 instead of $\sigma_1^2 + P_2$. It is not difficult to see that the following rate pair is achievable:

$$R_1 = \frac{1}{2}\log\left(1 + \frac{P_1}{\sigma_1^2}\right) \tag{3.3}$$

$$R_2 = \frac{1}{2}\log\left(1 + \frac{P_2}{\sigma_2^2 + P_1}\right). \tag{3.4}$$

In fact, as it was shown by Bergman [30], this superposition and interference subtraction scheme is optimal for the degraded Gaussian broadcast channel.

When a Gaussian broadcast channel has a vector input and vector outputs, it is no longer necessarily degraded, and superposition coding is no longer capacity-achieving. The capacity region for a non-degraded broadcast channel is still an unsolved problem. The

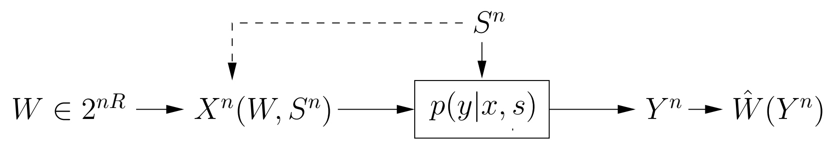


Figure 3.3: Channel with non-causal transmitter side information

largest achievable region in this case is due to Marton [31] [32], and it uses an idea called random binning[1]. For a two-user broadcast channel with independent information for each user, the Marton's region is as follows:

$$R_1 \leq I(\mathbf{U_1}; \mathbf{Y_1}) \tag{3.5}$$

$$R_2 \leq I(\mathbf{U_2}; \mathbf{Y_2}) \tag{3.6}$$

$$R_1 + R_2 \leq I(\mathbf{U_1}; \mathbf{Y_1}) + I(\mathbf{U_2}; \mathbf{Y_2}) - I(\mathbf{U_1}; \mathbf{U_2}) \tag{3.7}$$

where $(\mathbf{U_1}, \mathbf{U_2})$ is a pair of auxiliary random variables, and the mutual information is evaluated under a joint distribution $p(\mathbf{x}|\mathbf{u_1}, \mathbf{u_2})p(\mathbf{u_1}, \mathbf{u_2})$ whose induced marginal distribution $p(\mathbf{x})$ satisfies the input constraint. Although the optimality of Marton's region is not known for the general broadcast channel, it is optimal for the deterministic broadcast channel [33], and by a proper choice of $(\mathbf{U_1}, \mathbf{U_2})$, it gives the capacity region of the scalar Gaussian degraded broadcast channel also. The objective of this chapter of the thesis is to show that a proper choice of $(\mathbf{U_1}, \mathbf{U_2})$ also gives the sum-capacity of a non-degraded Gaussian vector broadcast channel.

As a first step, let's examine the degraded broadcast channel more carefully and give an interpretation of the auxiliary random variables in the degraded case. The connection between the degraded broadcast channel capacity region and Marton's region lies in the study of channels with non-causal transmitter side information. A channel with side information is illustrated in Figure 3.3. The channel output is a function of the input sequence X^n and a channel state sequence S^n. The channel state is not known to the receiver but is known to the transmitter as the side information. Further, the transmitter knows the entire state sequence S^n prior to transmission in a non-causal way. For such a channel, Gel'fand and

[1] Random binning is a coding technique that uses an equivalent class of codewords to transmit information. See Chapter 14 in [4] for a detailed description. The idea of random binning is due to Slepian and Wolf [41].

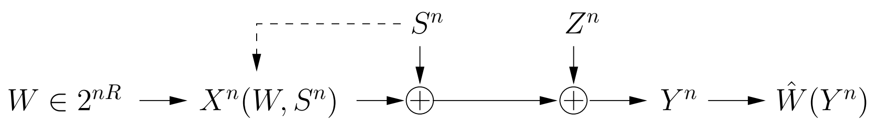


Figure 3.4: Gaussian channel with transmitter side information

Pinsker [42] and Heegard and El Gamal [43] showed that its capacity can be characterized as follows using an auxiliary random variable U:

$$C = \max_{p(u,x|s)} \{I(U;Y) - I(U;S)\}. \tag{3.8}$$

The achievability proof of this result uses a random-binning argument, and it is closely connected to Marton's achievability region for the broadcast channel. Such a connection was noted by Gel'fand and Pinsker in [42], and was further used by Caire and Shamai [38] for the two-by-two Gaussian broadcast channel. The following rough argument illustrates the connection. Fix a pair of auxiliary random variables (U_1, U_2) and a conditional distribution $p(x|u_1, u_2)$. Consider the effective channel $p(y_1, y_2|x)p(x|u_1, u_2)$. Construct a random-coding codebook from U_2 to Y_2 using an i.i.d. distribution according to $p(u_2)$. Evidently, a rate of $R_2 = I(U_2; Y_2)$ is achievable. Now, since U_2 is completely known at the transmitter, the channel from U_1 to Y_1 is a channel with non-causal side information available at the transmitter. Then, Gel'fand and Pinsker's result ensures that a rate of $R_1 = I(U_1; Y_1) - I(U_2; U_1)$ is achievable. This rate pair is precisely a corner point in Marton's region for the broadcast channel. The above rough argument ignores the issue that U_1 now depends on U_2, but for the Gaussian channel, the argument can be made rigorous.

When specialized to the Gaussian channel, the capacity of a channel with side information has an interesting solution. Consider the Gaussian channel shown in Figure 3.4:

$$y = x + s + z, \tag{3.9}$$

where x and y are the transmitted and the received signals respectively, s is a Gaussian interfering signal whose entire non-causal realization is known to the transmitter but not to the receiver, and z is a Gaussian noise independent of s. In a surprising result known as "writing-on-dirty-paper," Costa [44] showed that when s^n and z^n are i.i.d. Gaussian sequences, under a fixed power constraint, the capacity of the channel with interference is the same as if the interference does not exist. In addition, the optimal transmit signal x is statistically independent of s. In effect, interference can be "pre-subtracted" at the transmitter without increase in transmit power.

The "dirty-paper" result gives us another way to derive the degraded Gaussian broadcast channel capacity. Let $x = x_1 + x_2$, where x_1 and x_2 are independent Gaussian signals with average powers P_1 and P_2 respectively, where $P_1 + P_2 = P$. The message intended for y_1 is transmitted through x_1, and the message intended for y_2 is transmitted through x_2. If two independent codebooks are used for x_1 and x_2, each receiver sees the other user's signal as noise. However, the transmitter knows both messages in advance. So, the channel from x_1 to y_1 can be regarded as a Gaussian channel with non-causal side information x_2, for which Costa's result applies. Thus, a transmission rate from x_1 to y_1 that is as high as if x_2 is not present can be achieved, i.e. $R_1 = I(X_1; Y_1 | X_2)$. Further, the optimal x_1 is statistically independent of x_2. Thus, the channel from x_2 to y_2 still sees x_1 as independent noise, and a rate $R_2 = I(X_2; Y_2)$ is achievable. This gives an alternative derivation for the degraded Gaussian broadcast channel capacity in equations (3.3)-(3.4). Curiously, this derivation does not use the fact that y_2 is a degraded version of y_1. In fact, y_1 and y_2 may be interchanged and the following rate pair is also achievable:

$$R_1 = \frac{1}{2}\log\left(1 + \frac{P_1}{\sigma_1^2 + P_2}\right) \tag{3.10}$$

$$R_2 = \frac{1}{2}\log\left(1 + \frac{P_2}{\sigma_2^2}\right). \tag{3.11}$$

It can be shown that, when $\sigma_1 < \sigma_2$, the above rate region is smaller than the true capacity region in equations (3.3)-(3.4).

The idea of subtracting interference at the transmitter instead of at the receiver is attractive because it is also applicable to non-degraded broadcast channels. Consider the following Gaussian vector broadcast channel:

$$\begin{aligned} \mathbf{y_1} &= H_1\mathbf{x} + \mathbf{z_1} \\ \mathbf{y_2} &= H_2\mathbf{x} + \mathbf{z_2}, \end{aligned} \tag{3.12}$$

where $\mathbf{x}$, $\mathbf{y_1}$ and $\mathbf{y_2}$ are vector input and outputs, H_1 and H_2 are channel matrices, and $\mathbf{z_1}$, $\mathbf{z_2}$ are Gaussian vector noises with covariance matrices $S_{z_1 z_1}$ and $S_{z_2 z_2}$, respectively.

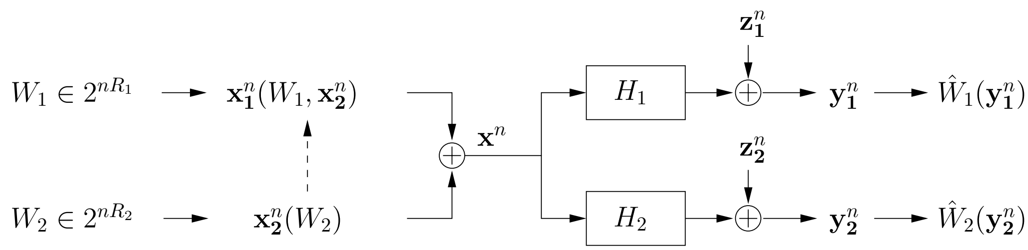


Figure 3.5: Coding for vector broadcast channel

In general, H_1 and H_2 are not degraded versions of each other. Further, they do not necessarily have the same eigenvectors, so it is generally not possible to diagonalize H_1 and H_2 simultaneously[2]. Nevertheless, the "dirty-paper" result can be extended to the vector case to pre-subtract multi-user interference at the transmitter, again with no increase in transmit power.

Lemma 3.1 *Given a fixed power constraint, a Gaussian vector channel with side information* $\mathbf{y} = \mathbf{x}+\mathbf{s}+\mathbf{z}$*, where* $\mathbf{z}$ *and* $\mathbf{s}$ *are independent Gaussian random vectors, and* $\mathbf{s}$ *is known non-causally at the transmitter but not at the receiver, has the same capacity as if* $\mathbf{s}$ *does not exist, i.e.*

$$C = \max_{p(\mathbf{u},\mathbf{x}|\mathbf{s})} \{I(\mathbf{U};\mathbf{Y}) - I(\mathbf{U};\mathbf{S})\} = I(\mathbf{X};\mathbf{Y}|\mathbf{S}). \tag{3.13}$$

Further, the capacity-achieving $\mathbf{x}$ *is statistically independent of* $\mathbf{s}$*.*

This result has been noted by several authors [46] [47] under different conditions. A direct proof is included as Appendix C, where it is shown that the capacity-achieving $p(\mathbf{u},\mathbf{x}|\mathbf{s})$ is such that $\mathbf{x}$ and $\mathbf{s}$ are independent, and $\mathbf{u}$ takes the form of $\mathbf{u} = \mathbf{x} + F\mathbf{s}$, where F is a fixed matrix determined by the covariance matrices of $\mathbf{s}$ and $\mathbf{z}$. Lemma 3.1 suggests a coding scheme for the broadcast channel as shown in Figure 3.5. The following theorem formalizes this idea:

[2] An important exception is when H_1 and H_2 are ISI channels, in which case both are Toeplitz, and can be simultaneously decomposed into scalar channels by discrete Fourier transforms [45].

Theorem 3.1 *Consider the Gaussian vector broadcast channel* $\mathbf{y}_i = H_i\mathbf{x}+\mathbf{z}_i, i = 1, \cdots, K$*, under a power constraint* P*. The following rate region is achievable:*

$$\left\{(R_1, \cdots, R_K) : R_i \leq \frac{1}{2}\log\frac{\left|\sum_{k=i}^{K} H_i S_k H_i^T + S_{z_i z_i}\right|}{\left|\sum_{k=i+1}^{K} H_i S_k H_i^T + S_{z_i z_i}\right|}\right\} \tag{3.14}$$

where $S_{z_i z_i}$ *is the covariance matrix for* $\mathbf{z}_i$*, and* S_i *is a set of positive semi-definite matrices satisfying the constraint:* $\sum_{i=1}^{K} \mathrm{tr}(S_i) \leq P$.

Proof: For simplicity, only the proof for the case $K = 2$ is presented. The extension to the general case is straightforward. Let $\mathbf{x} = \mathbf{x_1} + \mathbf{x_2}$, where $\mathbf{x_1}$ and $\mathbf{x_2}$ are independent Gaussian vectors whose covariance matrices S_1 and S_2 satisfy $\mathrm{tr}(S_1 + S_2) \leq P$. Now, fix $\mathbf{U_2} = \mathbf{x_2}$ and choose the conditional distribution $p(\mathbf{u_1}|\mathbf{u_2}, \mathbf{x_1})$ to be such that it maximizes $I(\mathbf{U_1}; \mathbf{Y_1}) - I(\mathbf{U_1}; \mathbf{U_2})$. By Lemma 3.1, the maximizing distribution is such that $\mathbf{x_1}$ and $\mathbf{U_2}$ are independent. So, assuming that $\mathbf{x_1}$ and $\mathbf{x_2}$ are independent *a priori* is without loss of generality. Further, by (3.13), the maximizing distribution gives $I(\mathbf{U_1}; \mathbf{Y_1}) - I(\mathbf{U_1}; \mathbf{U_2}) = I(\mathbf{X_1}; \mathbf{Y_1}|\mathbf{U_2})$. Using this choice of $(\mathbf{U_1}, \mathbf{U_2})$ in Marton's region (3.5)-(3.7), the following rates are obtained: $R_1 = I(\mathbf{X_1}; \mathbf{Y_1}|\mathbf{X_2})$, $R_2 = I(\mathbf{X_2}; \mathbf{Y_2})$. The mutual information can be evaluated as:

$$R_1 = \frac{1}{2}\log\frac{|H_1 S_1 H_1^T + H_1 S_2 H_1^T + S_{z_1 z_1}|}{|H_1 S_2 H_1^T + S_{z_1 z_1}|} \tag{3.15}$$

$$R_2 = \frac{1}{2}\log\frac{|H_2 S_2 H_2^T + S_{z_2 z_2}|}{|S_{z_2 z_2}|}, \tag{3.16}$$

which is the desired result. □

This theorem is a generalization of an earlier result by Caire and Shamai [38], who essentially considered the set of rank-one S_i in their derivation of the two-by-two broadcast channel sum capacity. Theorem 3.1 restricts $(\mathbf{U_1}, \mathbf{U_2})$ in Marton's region to be of a special form. Although such restriction may be capacity-lossy in general, as the results in the next section show, for achieving the sum capacity of a Gaussian vector broadcast channel, this choice of $(\mathbf{U_1}, \mathbf{U_2})$ is without loss of generality. Note that finding an optimal set of S_i in (3.15)-(3.16) may not be computationally easy. Linear combinations of R_1 and R_2 are non-convex functions of (S_1, S_2). Further, the order of interference pre-subtraction is arbitrary, and it is also possible to split the transmit covariance matrix into more than two users to achieve the rate-splitting points. Caire and Shamai [38] partially circumvented the difficulty for the two-by-two broadcast channel by deriving an outer bound for the sum capacity. They assumed a particular precoding order, and by optimizing over the set of all rank-one S_i, succeeded in proving that Marton's region coincides with the outer bound for the two-user two-antenna broadcast channel. Unfortunately, their procedure does not generalize to the n-user case easily, and it does not reveal the structure of the optimal S_i.

In a separate effort, Ginis and Cioffi [36] demonstrated a precoding technique for an $N \times N$ broadcast channel based on a QR decomposition of the channel matrix. The QR method transforms the matrix channel into a triangular structure, and by doing so, implicitly chooses a set of S_i based on the Q matrix in the QR decomposition. This channel triangularization was also independently considered by Caire and Shamai [38], who further proved that the QR method is rate-sum optimal in both low and high SNR regions. However, this choice of S_i is sub-optimal in general.

A major goal of this chapter is to find an optimal set of S_i in equations (3.15)-(3.16) that maximizes the sum capacity of a Gaussian vector broadcast channel. The key insight is that the optimal precoder has the structure of a decision-feedback equalizer.

3.2 Decision-feedback Precoding

3.2.1 GDFE

Decision-feedback equalization (DFE) is widely used to combat intersymbol interference (ISI) in linear dispersive channels. In a channel with ISI, each input symbol produces a sequence of time-delayed channel outputs, so that each received sample contains contributions from many input symbols. To untangle the interference, a decision-feedback equalizer employs the following strategy. Each input symbol is decoded based on the entire received sequence. After an input symbol is decoded, its effect is subtracted from the received sequence before the decoding for the next symbol begins. Under the assumption of no error propagation and also a channel non-singularity condition (that rarely occurs by accident), a generalization of decision-feedback equalizer (that often consists of several DFE's) can achieve the capacity of a Gaussian linear dispersive channel [48].

The study of the decision-feedback equalizer is related to the study of multiple access channels. If each transmitted symbol in an ISI channel is regarded as a data stream from a separate user, the decision-feedback equalizer can be thought of as a successive interference subtraction scheme for the multiple access channel. This connection can be formalized by considering a decision-feedback structure that operates on a finite block of inputs. This block-based structure, introduced in [49] as the Generalized Decision-Feedback Equalizer (GDFE), was also developed independently in [50] for the multiple access channel. This thesis will eventually use the GDFE structure for the broadcast channel also. As a first step, an information theoretical derivation of the generalized decision-feedback equalizer is given. The derivation is largely based on [49].

Consider a Gaussian vector channel $\mathbf{y} = H\mathbf{x} + \mathbf{z}$, where $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are Gaussian vectors. Let $\mathbf{x} \sim \mathcal{N}(0, S_{xx})$, and without loss of generality, assume $\mathbf{z} \sim \mathcal{N}(0, I)$. Shannon's noisy channel coding theorem suggests that to achieve a rate $R = I(\mathbf{X};\mathbf{Y}) = \frac{1}{2}\log|HS_{xx}H^T + I|$, a random codebook can be constructed, where each codeword in the codebook is a sequence of Gaussian vectors generated from an i.i.d. distribution $\mathcal{N}(0, S_{xx})$. Evidently, sending a message using such a vector codebook requires joint processing of components of $\mathbf{x}$ at the encoder. Now, write $\mathbf{x}$ as $\mathbf{x}^T = [\mathbf{x_1}^T \mathbf{x_2}^T]$, and suppose further that $\mathbf{x_1}$ and $\mathbf{x_2}$ are statistically independent so that the covariance matrix S_{xx} is of the form $\begin{bmatrix} S_{x_1x_1} & 0 \\ 0 & S_{x_2x_2} \end{bmatrix}$. In this case, one might ask, is it possible to achieve a rate $R = I(\mathbf{X};\mathbf{Y})$ using two separate codebooks with the encoding and decoding of $\mathbf{x_1}$ and $\mathbf{x_2}$ done independently? The answer is yes, and the key to do so is to use a receiver based on a generalized decision-feedback equalizer.

The development of GDFE involves three key ideas. The first idea is to recognize that in a Gaussian vector channel $\mathbf{y} = H\mathbf{x} + \mathbf{z}$, the optimal decoding of $\mathbf{x}$ from $\mathbf{y}$ is related to the minimum mean-square error (MMSE) estimation of $\mathbf{x}$ given $\mathbf{y}$. Consider the setting in Figure 3.6, where at the output of the Gaussian vector channel, an MMSE estimator M is applied to $\mathbf{y}$ to generate $\hat{\mathbf{x}}$. First, note that the use of MMSE estimation is capacity lossless. The maximum achievable rate after MMSE estimation is $I(\mathbf{X};\hat{\mathbf{X}})$. The following argument shows that $I(\mathbf{X};\hat{\mathbf{X}}) = I(\mathbf{X};\mathbf{Y})$. The MMSE estimator for a Gaussian process is linear, so M represents a matrix multiplication. Further, let the difference between $\mathbf{x}$ and $\hat{\mathbf{x}}$ be $\mathbf{e}$. From linear estimation theory, $\mathbf{e}$ is Gaussian and is independent of $\hat{\mathbf{x}}$. So, if $I(\mathbf{X};\hat{\mathbf{X}})$ is re-written as $I(\hat{\mathbf{X}};\mathbf{X})$, it can be interpreted as the capacity of a Gaussian channel from $\hat{\mathbf{x}}$ to $\mathbf{x}$ with $\mathbf{e}$ as the additive noise:

$$I(\mathbf{X};\hat{\mathbf{X}}) = I(\hat{\mathbf{X}};\mathbf{X}) = \frac{1}{2}\log\frac{|S_{xx}|}{|S_{ee}|}, \tag{3.17}$$

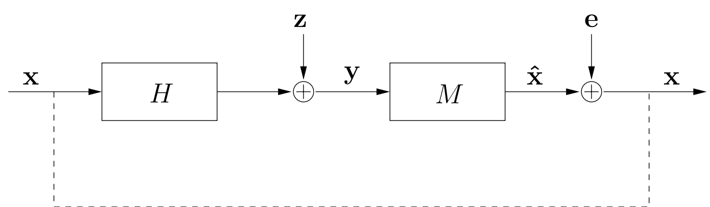


Figure 3.6: MMSE estimation in a Gaussian vector channel

where S_{xx} and S_{ee} are covariance matrices of $\mathbf{x}$ and $\mathbf{e}$ respectively. This mutual information is related to the capacity of the original channel. The key is the following observation [48]:

$$I(\mathbf{X};\mathbf{Y}) = H(\mathbf{Y}) - H(\mathbf{Y}|\mathbf{X}) \quad = \quad \frac{1}{2}\log\frac{|S_{yy}|}{|S_{y|x}|} = \frac{1}{2}\log\frac{|S_{yy}|}{|S_{zz}|}, \tag{3.18}$$

$$I(\mathbf{Y};\mathbf{X}) = H(\mathbf{X}) - H(\mathbf{X}|\mathbf{Y}) \quad = \quad \frac{1}{2}\log\frac{|S_{xx}|}{|S_{x|y}|} = \frac{1}{2}\log\frac{|S_{xx}|}{|S_{ee}|}, \tag{3.19}$$

where $H(\mathbf{Y}|\mathbf{X})$ is the uncertainty in $\mathbf{y}$ given $\mathbf{x}$, so $S_{y|x} = S_{zz}$, and likewise, $H(\mathbf{X}|\mathbf{Y})$ is the uncertainty in $\mathbf{x}$ given $\mathbf{y}$, so $S_{x|y} = S_{ee}$. Since $I(\mathbf{X};\mathbf{Y}) = I(\mathbf{Y};\mathbf{X})$, this implies that

$$I(\mathbf{X};\mathbf{Y}) = I(\mathbf{Y};\mathbf{X}) = I(\mathbf{X};\hat{\mathbf{X}}) = I(\hat{\mathbf{X}};\mathbf{X}). \tag{3.20}$$

Now write $\hat{\mathbf{x}}^T = [\hat{\mathbf{x}}_\mathbf{1}^T \hat{\mathbf{x}}_\mathbf{2}^T]$. Suppose that $\mathbf{x_1}$ and $\mathbf{x_2}$ are independently coded with two separate codebooks. Decoding of $\mathbf{x_1}$ and $\mathbf{x_2}$, however, cannot be done on $\hat{\mathbf{x}}_\mathbf{1}$ and $\hat{\mathbf{x}}_\mathbf{2}$ separately. To see this, write $\mathbf{e_1} = \mathbf{x_1} - \hat{\mathbf{x}}_\mathbf{1}$ and $\mathbf{e_2} = \mathbf{x_2} - \hat{\mathbf{x}}_\mathbf{2}$. Individual detections on $\hat{\mathbf{x}}_\mathbf{1}$ and $\hat{\mathbf{x}}_\mathbf{2}$ achieve $I(\mathbf{X_1};\hat{\mathbf{X}}_\mathbf{1})$ and $I(\mathbf{X_2};\hat{\mathbf{X}}_\mathbf{2})$, respectively. Because $\mathbf{e_1}$ and $\mathbf{e_2}$ are independent of $\mathbf{x_1}$ and $\mathbf{x_2}$ respectively and are both Gaussian, the argument in the previous paragraph may be repeated to conclude that individual detections on $\hat{\mathbf{x}}_\mathbf{1}$ and $\hat{\mathbf{x}}_\mathbf{2}$ achieve $\frac{1}{2}\log\left(|S_{x_1x_1}|/|S_{e_1e_1}|\right)$ and $\frac{1}{2}\log\left(|S_{x_2x_2}|/|S_{e_2e_2}|\right)$, respectively. But, $\mathbf{e_1}$ and $\mathbf{e_2}$ are not necessarily uncorrelated. So, by Hadamard's inequality, $|S_{ee}| \le |S_{e_1e_1}| \cdot |S_{e_2e_2}|$. This implies

$$\frac{1}{2}\log\frac{|S_{x_1x_1}|}{|S_{e_1e_1}|} + \frac{1}{2}\log\frac{|S_{x_2x_2}|}{|S_{e_2e_2}|} \le \frac{1}{2}\log\frac{|S_{xx}|}{|S_{ee}|}. \tag{3.21}$$

Thus, although the decoding of $\mathbf{x}$ based on $\hat{\mathbf{x}}$ is capacity-lossless, the independent decoding of $\mathbf{x_1}$ based on $\hat{\mathbf{x}}_\mathbf{1}$ and decoding of $\mathbf{x_2}$ based $\hat{\mathbf{x}}_\mathbf{2}$ are capacity-lossy.

The goal of GDFE is to use decision-feedback to facilitate the independent decoding of $\mathbf{x_1}$ and $\mathbf{x_2}$. This is accomplished by a diagonalization of the MMSE error $\mathbf{e}$, while preserving the "information" in $\hat{\mathbf{x}}$. First, let's write down the MMSE filter M,

$$\begin{aligned} M &= S_{xy}S_{yy}^{-1} && (3.22) \\ &= S_{xx}H^T(HS_{xx}H^T + I)^{-1} && (3.23) \\ &= (H^T H + S_{xx}^{-1})^{-1}H^T, && (3.24) \end{aligned}$$

where (3.22) follows from standard linear estimation theory and (3.24) follows from the matrix inversion lemma [51], which will be used repeatedly in subsequent developments:

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}. \tag{3.25}$$

Now, it is clear that M may be split into two parts: a matched filter H^T and an estimation filter $(H^T H + S_{xx}^{-1})^{-1}$, as shown in Figure 3.7. This creates a pair of channels. The forward channel goes from $\mathbf{x}$ to $\mathbf{w}$:

$$\mathbf{w} = H^T H\mathbf{x} + H^T\mathbf{z} = R_f\mathbf{x} + \mathbf{z}', \tag{3.26}$$

where $R_f = H^T H$. The backward channel goes from $\mathbf{w}$ to $\mathbf{x}$:

$$\mathbf{x} = (H^T H + S_{xx}^{-1})^{-1}\mathbf{w} + \mathbf{e} = R_b\mathbf{w} + \mathbf{e}, \tag{3.27}$$

where $R_b = (H^T H + S_{xx}^{-1})^{-1}$. The forward channel has the following property: the covariance matrix of the noise $\mathbf{z}'$ is the same as the channel matrix R_f. The second key idea in GDFE is to recognize that the backward channel has the same property as verified below:

$$\begin{aligned} \mathbf{E}[\mathbf{e}\mathbf{e}^T] &= \mathbf{E}[(\mathbf{x} - \hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}})^T] \\ &= \mathbf{E}[(\mathbf{x} - S_{xy}S_{yy}^{-1}\mathbf{y})(\mathbf{x} - S_{xy}S_{yy}^{-1}\mathbf{y})^T] \\ &= S_{xx} - S_{xx}H^T(HS_{xx}H^T + I)^{-1}HS_{xx} \\ &= (H^T H + S_{xx}^{-1})^{-1} \\ &= R_b, && (3.28) \end{aligned}$$

where the matrix inversion lemma (3.25) is again used.

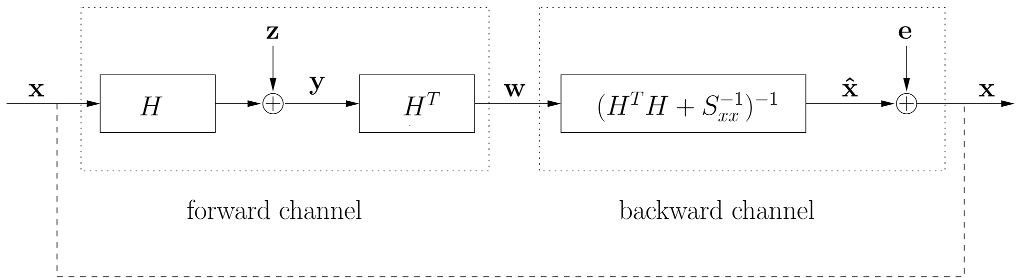


Figure 3.7: Forward and backward channels

The goal is to diagonalize the MMSE error $\mathbf{e}$. The third key idea in GDFE is to recognize that diagonalization may be done using a block Cholesky factorization of R_b, which is simultaneously the backward channel matrix and the covariance matrix of $\mathbf{e}$:

$$R_b = G^{-1}\Delta^{-1}G^{-T}, \tag{3.29}$$

where $G = \begin{bmatrix} I & G_{22} \\ 0 & I \end{bmatrix}$ is a block upper triangular matrix, and $\Delta = \begin{bmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{bmatrix}$ is a block diagonal matrix. The Cholesky factorization diagonalizes $\mathbf{e}$ in the following sense. Define $\mathbf{e}' = G\mathbf{e}$:

$$\begin{bmatrix} \mathbf{e_1'} \\ \mathbf{e_2'} \end{bmatrix} = \begin{bmatrix} I & G_{22} \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{e_1} \\ \mathbf{e_2} \end{bmatrix}. \tag{3.30}$$

Then, the components $\mathbf{e_1'}$ and $\mathbf{e_2'}$ are uncorrelated because

$$S_{e'e'} = \mathbf{E}[\mathbf{e}'\mathbf{e}'^T] = \mathbf{E}[G\mathbf{e}(G\mathbf{e})^T] = GR_bG^T = \Delta^{-1}, \tag{3.31}$$

which is a block-diagonal matrix. Further, the diagonalization preserves the determinant of the covariance matrix:

$$|S_{e'e'}| = |\Delta^{-1}| = |G^{-1}\Delta^{-1}G^{-T}| = |S_{ee}|. \tag{3.32}$$

The next idea is to recognize that the diagonalization can be done directly by modifying the backward channel to form a decision-feedback equalizer. Because the channel matrix

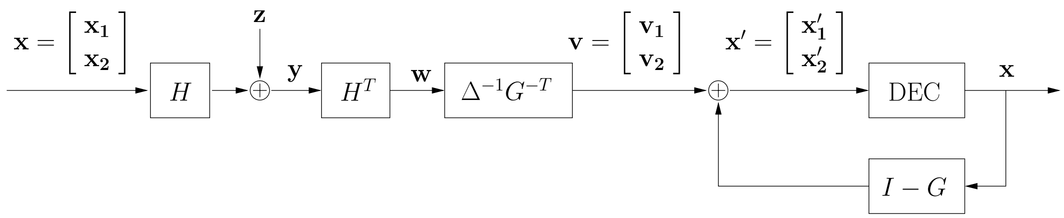


Figure 3.8: Generalized decision feedback equalizer

and the noise covariance matrix are the same, it is possible to split the channel matrix R_b into the following feedback configuration:

$$\begin{aligned} \mathbf{x} &= R_b\mathbf{w} + \mathbf{e} && (3.33)\\ \mathbf{x} &= G^{-1}\Delta^{-1}G^{-T}\mathbf{w} + \mathbf{e} && (3.34)\\ G\mathbf{x} &= \Delta^{-1}G^{-T}\mathbf{w} + G\mathbf{e} && (3.35)\\ \mathbf{x} &= \Delta^{-1}G^{-T}\mathbf{w} + (I-G)\mathbf{x} + \mathbf{e}'. && (3.36) \end{aligned}$$

Writing out the matrix computation explicitly,

$$\begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \end{bmatrix} = \begin{bmatrix} \Delta_{11}^{-1} & 0 \\ 0 & \Delta_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -G_{22}^T & I \end{bmatrix} \begin{bmatrix} \mathbf{w_1} \\ \mathbf{w_2} \end{bmatrix} + \begin{bmatrix} 0 & -G_{22} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \end{bmatrix} + \begin{bmatrix} \mathbf{e_1'} \\ \mathbf{e_2'} \end{bmatrix}.$$

It is now clear that the backward canonical channel is split into two independent sub-channels whose respective noises are uncorrelated. The sub-channel for $\mathbf{x_2}$ is:

$$\mathbf{x_2} = \Delta_{22}^{-1}(-G_{22}^T\mathbf{w_1} + \mathbf{w_2}) + \mathbf{e_2'} \triangleq \mathbf{x_2'} + \mathbf{e_2'}. \quad (3.37)$$

Once $\mathbf{x_2}$ is decoded correctly, $G_{22}\mathbf{x_2}$ can be subtracted from the sub-channel for $\mathbf{x_1}$ to form:

$$\mathbf{x_1} = \Delta_{11}^{-1}\mathbf{w_1} + \mathbf{e_1'} \triangleq \mathbf{x_1'} + \mathbf{e_1'}, \quad (3.38)$$

where $\mathbf{x}'$ is defined as $\mathbf{x}' \triangleq \Delta^{-1}G^{-T}\mathbf{w} + (I-G)\mathbf{x}$, and $\mathbf{x}'^T = [\mathbf{x_1'}^T \mathbf{x_2'}^T]$. This interference subtraction scheme is called a generalized decision-feedback equalizer. The GDFE structure is shown in Figure 3.8. The combination of $\Delta^{-1}G^{-T}$ and H^T is called the feedforward filter; $I-G$ is called the feedback filter.

The key result in the development of the GDFE is that the decision-feedback operation results in equivalent independent channels that have the same capacity as the original vector channel. To see this, note that the maximum achievable rate with a GDFE is $I(\mathbf{X};\mathbf{X}')$. This mutual information can be more easily computed if written as $I(\mathbf{X}';\mathbf{X})$, which can be interpreted as the capacity of the channel $\mathbf{x} = \mathbf{x}' + \mathbf{e}'$. Now, $\mathbf{e}' = G\mathbf{e}$ is independent of $\hat{\mathbf{x}}$, so it is independent of $\mathbf{w}$ and thus independent of $\mathbf{x}'$. Also, $\mathbf{e}'$ is Gaussian, so the capacity of the channel $\mathbf{x} = \mathbf{x}' + \mathbf{e}'$ is just:

$$I(\mathbf{X}';\mathbf{X}) = \frac{1}{2}\log\frac{|S_{xx}|}{|S_{e'e'}|}. \tag{3.39}$$

This is precisely the capacity of the original channel, because by (3.19) and (3.32):

$$I(\mathbf{X};\mathbf{Y}) = \frac{1}{2}\log\frac{|S_{xx}|}{|S_{ee}|} = \frac{1}{2}\log\frac{|S_{xx}|}{|S_{e'e'}|} = I(\mathbf{X};\mathbf{X}'). \tag{3.40}$$

Further, S_{xx} and $S_{e'e'}$ are both diagonal, so, $|S_{xx}| = |S_{x_1x_1}| \cdot |S_{x_2x_2}|$, and $|S_{e'e'}| = |\Delta^{-1}| = |\Delta_{11}^{-1}|\cdot|\Delta_{22}^{-1}| = |S_{e_1'e_1'}|\cdot|S_{e_2'e_2'}|$. Thus, the GDFE structure has decomposed the vector channel into two sub-channels that can be independently encoded and decoded. The capacities of the two sub-channels are:

$$\begin{aligned} R_1 &= I(\mathbf{X_1'};\mathbf{X_1}) = \frac{1}{2}\log\frac{|S_{x_1x_1}|}{|S_{e_1'e_1'}|} && (3.41)\\ R_2 &= I(\mathbf{X_2'};\mathbf{X_2}) = \frac{1}{2}\log\frac{|S_{x_2x_2}|}{|S_{e_2'e_2'}|}. && (3.42)\end{aligned}$$

And the sum capacity is:

$$\begin{aligned} R_1 + R_2 &= I(\mathbf{X_1'};\mathbf{X_1}) + I(\mathbf{X_2'};\mathbf{X_2}) \\ &= \frac{1}{2}\log\frac{|S_{x_1x_1}|}{|S_{e_1'e_1'}|} + \frac{1}{2}\log\frac{|S_{x_2x_2}|}{|S_{e_2'e_2'}|} \\ &= \frac{1}{2}\log\frac{|S_{xx}|}{|S_{ee}|} \\ &= I(\mathbf{X};\mathbf{Y}). && (3.43)\end{aligned}$$

Thus, GDFE is capacity lossless.

3.2.2 Precoding

For a Gaussian vector channel with independent inputs $\mathbf{x_1}$ and $\mathbf{x_2}$, the generalized decision-feedback equalizer decomposes the vector channel into two sub-channels for which encoding and decoding can be performed independently. As long as the decision-feedback operation is error-free, the sum capacity of the two sub-channels is the same as the capacity of the original vector channel. Thus, if $\mathbf{x_1}$ and $\mathbf{x_2}$ are independent, transmitter coordination is not necessary to achieve the mutual information $I(\mathbf{X_1}, \mathbf{X_2}; \mathbf{Y})$. On the other hand, receiver coordination is required in a decision-feedback equalizer. This is so for two reasons. First, the feedforward structure operates on the entire vector $\mathbf{y}$. Second, the feedback operation requires the correct codeword from one sub-channel to be available before the decoding of the other sub-channel. It turns out that the second problem can be averted using ideas from coding for channels with transmitter side information. In this section, a precoding scheme based on "writing-on-dirty-paper" is described. The main result is that the decision-feedback operation can be moved to the transmitter, and it is equivalent to interference "pre-subtraction".

Theorem 3.2 *Consider a Gaussian vector channel* $\mathbf{y} = \sum_{i=1}^{K} H_i \mathbf{x_i} + \mathbf{z}$, *where* $\mathbf{x_i}$*'s are independent Gaussian vectors and* $\mathbf{z} \sim \mathcal{N}(0, I)$. *The sum capacity* $I(\mathbf{X_1}, \cdots, \mathbf{X_K}; \mathbf{Y})$ *with* $R_i = I(\mathbf{X_i}; \mathbf{Y} | \mathbf{X_{i+1}}, \cdots, \mathbf{X_K})$ *is achievable in two ways: either using a decision-feedback structure with the knowledge of* $\mathbf{x_{i+1}}, \cdots, \mathbf{x_K}$ *assumed to be available before the decoding of each* $\mathbf{x_i}$, *or using a precoder structure with the knowledge of* $\mathbf{x_{i+1}}, \cdots, \mathbf{x_K}$ *assumed to be available before the encoding of each* $\mathbf{x_i}$.

Proof: The development in the previous section shows that a generalized decision-feedback equalizer achieves $I(\mathbf{X_1}, \mathbf{X_2}; \mathbf{Y})$. To show the first part of the theorem, it is necessary to compute the individual rates of the two sub-channels. As before, let $\mathbf{x_1}$ and $\mathbf{x_2}$ be independent. Let $H = [H_1 H_2]$[3], $\mathbf{z}^T = [\mathbf{z_1}^T \mathbf{z_2}^T]$, and write the vector channel in the form of a multiple access channel:

$$\mathbf{y} = H\mathbf{x} + \mathbf{z} = [H_1 H_2] \begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \end{bmatrix} + \begin{bmatrix} \mathbf{z_1} \\ \mathbf{z_2} \end{bmatrix}. \tag{3.44}$$

[3]For the rest of this proof only, define $H = [H_1 H_2]$. Elsewhere in the chapter, define $H^T = [H_1^T H_2^T]$.

The block Cholesky factorization (3.29) may be computed explicitly:

$$(S_{xx}^{-1} + H^T H)^{-1} = \begin{bmatrix} S_{x_1x_1}^{-1} + H_1^T H_1 & H_1^T H_2 \\ H_2^T H_1 & S_{x_2x_2}^{-1} + H_2^T H_2 \end{bmatrix}^{-1} = G^{-1}\Delta^{-1}G^{-T}, \tag{3.45}$$

where

$$G = \begin{bmatrix} I & (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} H_1^T H_2 \\ 0 & I \end{bmatrix}, \tag{3.46}$$

and

$$\Delta^{-1} = \begin{bmatrix} (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} & 0 \\ 0 & (S_{x_2x_2}^{-1} + H_2^T H_2 - H_2^T H_1 (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} H_1^T H_2)^{-1} \end{bmatrix}. \tag{3.47}$$

Thus, by (3.32),

$$S_{e_1'e_1'} = \Delta_{11}^{-1} = (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1}. \tag{3.48}$$

So, from (3.41),

$$R_1 = I(\mathbf{X_1'};\mathbf{X_1}) = \frac{1}{2}\log\frac{|S_{x_1x_1}|}{|(S_{x_1x_1}^{-1} + H_1^T H_1)^{-1}|} = \frac{1}{2}\log|H_1 S_{x_1x_1} H_1^T + I|, \tag{3.49}$$

where the matrix identity $|I + AB| = |I + BA|$ is used. Writing it out in another way:

$$R_1 = I(\mathbf{X_1'};\mathbf{X_1}) = I(\mathbf{X_1};\mathbf{Y}|\mathbf{X_2}). \tag{3.50}$$

Also,

$$\begin{aligned} S_{e_2'e_2'} &= (S_{x_2x_2}^{-1} + H_2^T H_2 - H_2^T H_1 (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} H_1^T H_2)^{-1}, & (3.51) \\ &= (S_{x_2x_2}^{-1} + H_2^T (I + H_1 S_{x_1x_1} H_1^T)^{-1} H_2)^{-1}, & (3.52) \end{aligned}$$

where the matrix inversion lemma is used.

Thus, from (3.42),

$$R_2 = I(\mathbf{X_2'};\mathbf{X_2}) = \frac{1}{2}\log\frac{|S_{x_2x_2}|}{|(S_{x_2x_2}^{-1} + H_2^T(I + H_1S_{x_1x_1}H_1^T)^{-1}H_2)^{-1}|} \tag{3.53}$$

$$= \frac{1}{2}\log\frac{|H_1S_{x_1x_1}H_1^T + H_2S_{x_2x_2}H_2^T + I|}{|H_1S_{x_1x_1}H_1^T + I|}, \tag{3.54}$$

which can be verified by directly multiplying out the respective terms and by repeated uses of the identity $|I + AB| = |I + BA|$. Thus,

$$R_2 = I(\mathbf{X_2'};\mathbf{X_2}) = I(\mathbf{X_2};\mathbf{Y}). \tag{3.55}$$

This verifies that the achievable sum rate in the multiple access channel using GDFE is

$$R_1 + R_2 = I(\mathbf{X_1},\mathbf{X_2};\mathbf{Y}) = \frac{1}{2}\log|H_1S_{x_1x_1}H_1^T + H_2S_{x_2x_2}H_2^T + I|. \tag{3.56}$$

Therefore, the generalized decision feedback equalizer not only achieves the sum capacity of a multiple access channel, it also achieves the individual rates of a corner point in the multiple access capacity region. Interchanging the order of $\mathbf{x_1}$ and $\mathbf{x_2}$ achieves the other corner point. This, together with time-sharing or rate-splitting, allows GDFE to achieve the entire capacity region of the multiple access channel.

An induction argument generalizes the above result to more than two users. Assume that a GDFE achieves $R_i = I(\mathbf{X_i};\mathbf{Y}|\mathbf{X_{i+1}},\cdots,\mathbf{X_K})$ for a K-user multiple access channel. In a $(K+1)$-user channel, users 1 and 2 can first be considered as a super-user, and the GDFE result can be applied to the resulting K-user channel with $R_i = I(\mathbf{X_i};\mathbf{Y}|\mathbf{X_{i+1}},\cdots,\mathbf{X_{K+1}})$ for $i = 3,\cdots,K$, and $R_1 + R_2 = I(\mathbf{X_1},\mathbf{X_2};\mathbf{Y}|\mathbf{X_3},\cdots,\mathbf{X_{K+1}})$. Then, a separate two-user GDFE can be applied to users 1 and 2 to obtain $R_i = I(\mathbf{X_i};\mathbf{Y}|\mathbf{X_{i+1}},\cdots,\mathbf{X_{K+1}})$, for $i = 1, 2$.

Next, it is shown that the same rate-tuple can be achieved using a precoding structure for channels with side information at the transmitter. Consider the output of the feedforward filter, the vector $\mathbf{v}$ in Figure 3.8. Write $\mathbf{v}^T = [\mathbf{v_1}^T\mathbf{v_2}^T]$, and consider the capacity of the two sub-channels: one from $\mathbf{x_1}$ to $\mathbf{v_1}$ and the other from $\mathbf{x_2}$ to $\mathbf{v_2}$. Note that $\mathbf{v_2} = \mathbf{x_2'}$. So, the sub-channel from $\mathbf{x_2}$ to $\mathbf{v_2}$ is the same as in a GDFE:

$$R_2 = I(\mathbf{X_2};\mathbf{V_2}) = I(\mathbf{X_2};\mathbf{X_2'}) = I(\mathbf{X_2};\mathbf{Y}). \tag{3.57}$$

Now, consider the sub-channel from $\mathbf{x_1}$ to $\mathbf{v_1}$ with $\mathbf{x_2}$ available at the transmitter. Because $\mathbf{x_2}$ is Gaussian and is independent of $\mathbf{x_1}$, Lemma 3.1 applies. The capacity of this sub-channel is then $R_1 = I(\mathbf{X_1}; \mathbf{V_1}|\mathbf{X_2})$. The rest of the proof shows that this conditional mutual information is equal to the corresponding data rate in GDFE: $I(\mathbf{X_1}; \mathbf{X'_1})$. Toward this end, it is necessary to explicitly compute $\mathbf{v_1}$. Since

$$\mathbf{v} = \Delta^{-1} G^{-T} H^T (H\mathbf{x} + \mathbf{z}), \tag{3.58}$$

using (3.47) and (3.46), $\mathbf{v_1}$ can be expressed as:

$$\mathbf{v_1} = (S_{x_1 x_1}^{-1} + H_1^T H_1)^{-1} H_1^T (H_1 \mathbf{x_1} + H_2 \mathbf{x_2}) + \mathbf{z'_1}, \tag{3.59}$$

where $\mathbf{z'} = \Delta^{-1} G^{-T} H^T \mathbf{z}$, $\mathbf{z'}^T = [\mathbf{z'_1}^T \mathbf{z'_2}^T]$. It can be shown that $\mathbf{z'_1}$ has a covariance matrix:

$$\mathbf{E}[\mathbf{z'_1} \mathbf{z'_1}^T] = (S_{x_1 x_1}^{-1} + H_1^T H_1)^{-1} H_1^T H_1 (S_{x_1 x_1}^{-1} + H_1^T H_1)^{-1}. \tag{3.60}$$

So, $\mathbf{v_1}$ is equivalent to

$$\mathbf{v_1} = (S_{x_1 x_1}^{-1} + H_1^T H_1)^{-1} H_1^T (H_1 \mathbf{x_1} + H_2 \mathbf{x_2} + \mathbf{z_1}), \tag{3.61}$$

On the other hand, $\mathbf{x'_1}$ can be computed explicitly from $\mathbf{x'} = \mathbf{v} + (I - G)\mathbf{x}$.

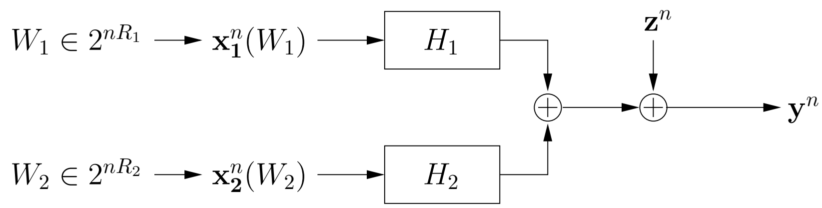
$$\mathbf{x'_1} = (S_{x_1 x_1}^{-1} + H_1^T H_1)^{-1} H_1^T (H_1 \mathbf{x_1} + \mathbf{z_1}). \tag{3.62}$$

Since $\mathbf{x_1}$, $\mathbf{x_2}$ and $\mathbf{z_1}$ are jointly independent, it follows from (3.61) and (3.62) that

$$R_1 = I(\mathbf{X_1}; \mathbf{V_1}|\mathbf{X_2}) = I(\mathbf{X_1}; \mathbf{X'_1}) = I(\mathbf{X_1}; \mathbf{Y}|\mathbf{X_2}). \tag{3.63}$$

Therefore, a precoder achieves the same capacity as a decision-feedback equalizer. This proof generalizes to the K-user case by a similar induction argument as before. □

Figure 3.9 and Figure 3.10 illustrate the two coding strategies for the Gaussian vector channel. Figure 3.9 illustrates the decision-feedback configuration. $\mathbf{x_1}$ and $\mathbf{x_2}$ are coded independently. After $\mathbf{x_2}$ is decoded, its effect, namely $(S_{x_1 x_1}^{-1} + H_1^T H_1)^{-1} H_1^T H_2 \mathbf{x_2}$, is subtracted before $\mathbf{x_1}$ is decoded. This decision-feedback configuration achieves the



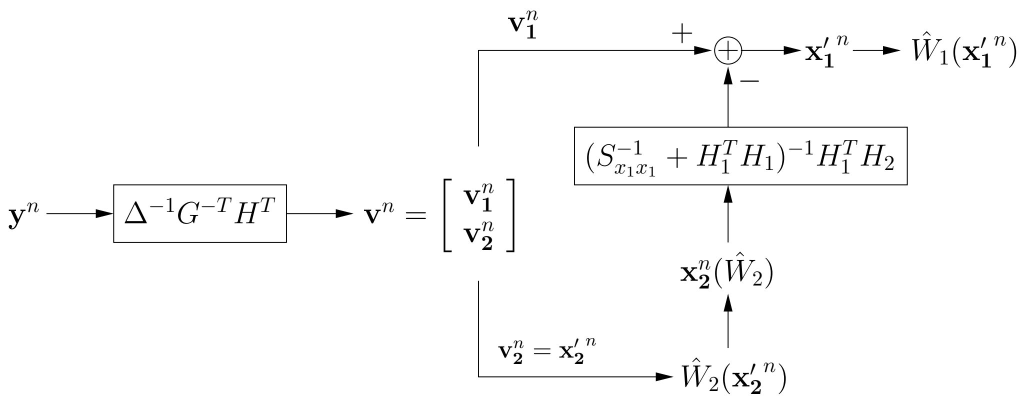


Figure 3.9: Decision feedback decoding

vector channel capacity in the sense that $I(\mathbf{X_1}, \mathbf{X_2}; \mathbf{Y}) = I(\mathbf{X_1}; \mathbf{Y}|\mathbf{X_2}) + I(\mathbf{X_2}, \mathbf{Y}) = I(\mathbf{X_1}; \mathbf{X_1'}) + I(\mathbf{X_2}; \mathbf{X_2'})$. Figure 3.10 illustrates the precoder configuration. In this case, $\mathbf{x_2}$ is coded as before. The channel for $\mathbf{x_1}$ is a Gaussian channel with transmitter side information $\mathbf{x_2}$, whose effect can be completely pre-subtracted. This precoder configuration achieves the vector channel capacity in the sense that $I(\mathbf{X_1}, \mathbf{X_2}; \mathbf{Y}) = I(\mathbf{X_1}; \mathbf{Y}|\mathbf{X_2}) + I(\mathbf{X_2}; \mathbf{Y}) = I(\mathbf{X_1}; \mathbf{V_1}|\mathbf{X_2}) + I(\mathbf{X_2}; \mathbf{V_2})$. In the decision-feedback configuration, $\mathbf{x_2}$ is assumed to be decoded correctly before its interference is subtracted. This implies a decoding delay between the two users. Further, if an erroneous decision on $\mathbf{x_2}$ is made, error would propagate. In the precoding configuration, error propagation never occurs. However, because non-causal side information is needed, $\mathbf{x_1}$ cannot be encoded until $\mathbf{x_2}$ is available. This implies an encoding delay. The two situations are symmetric, and they are both capacity-achieving.

The decision-feedback configuration does not require transmitter coordination. So, it is naturally suited for a multiple access channel. In the precoder configuration, the feedback operation is moved to the transmitter. So, one might hope that it corresponds to a broadcast

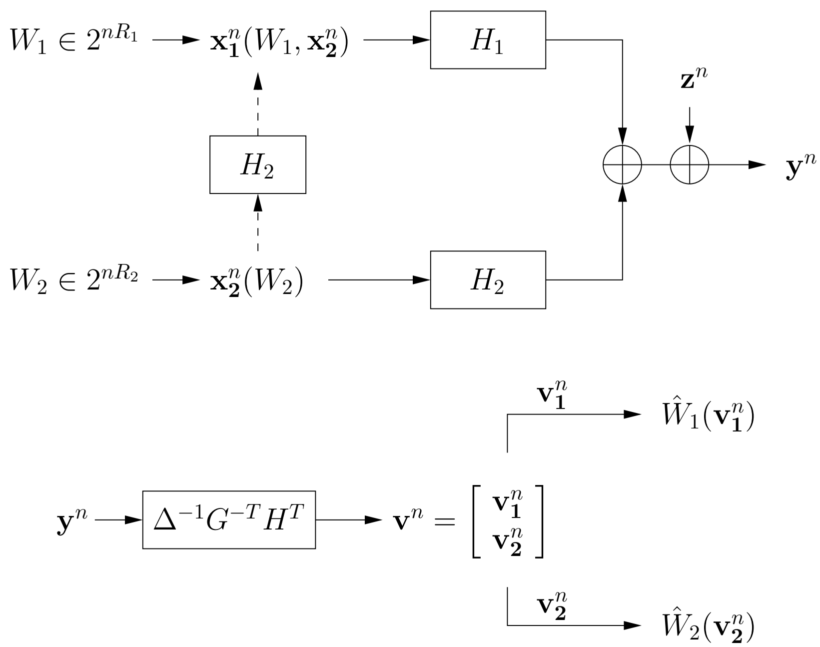


Figure 3.10: Decision feedback precoding

channel where receiver coordination is not possible. This is, however, not yet true in the present setting. The capacity-achieving precoder requires a feedforward filter that acts on the entire received vector, so receiver coordination is still needed. However, under certain conditions, the feedforward filter degenerates into a diagonal matrix, which eliminates the need for receiver coordination entirely. The condition under which this happens is the focus of the next section.

3.3 Broadcast Channel Sum Capacity

3.3.1 Least Favorable Noise

The key in deriving of the broadcast channel sum capacity is to find a tight capacity outer bound. Consider the broadcast channel

$$\begin{bmatrix} \mathbf{y_1} \\ \mathbf{y_2} \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{z_1} \\ \mathbf{z_2} \end{bmatrix}, \tag{3.64}$$

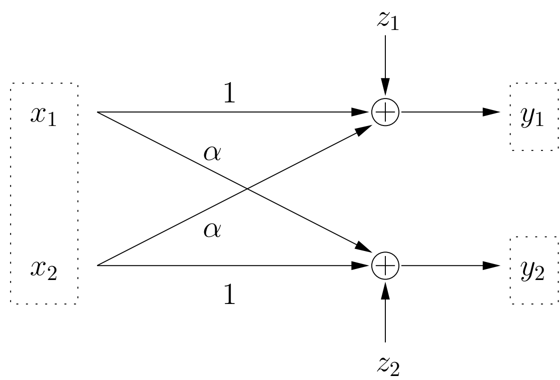


Figure 3.11: A simple two-user broadcast channel

where $\mathbf{y_1}$ and $\mathbf{y_2}$ do not cooperate. Fix an input distribution $p(\mathbf{x})$. The sum capacity of the broadcast channel is clearly bounded by the capacity of the vector channel $I(\mathbf{X};\mathbf{Y_1},\mathbf{Y_2})$ where $\mathbf{y_1}$ and $\mathbf{y_2}$ cooperate. As recognized by Sato [52], this bound can be further tightened. Because $\mathbf{y_1}$ and $\mathbf{y_2}$ cannot coordinate in a broadcast channel, the broadcast channel capacity does not depend on the joint distribution $p(\mathbf{z_1},\mathbf{z_2})$, but only on the marginals $p(\mathbf{z_1})$ and $p(\mathbf{z_2})$. This is so because two broadcast channels with the same marginals but with different joint distribution can use the same encoder and decoders and maintain the same probability of error. Therefore, the sum capacity of a broadcast channel must be bounded by the minimum mutual information:

$$R_1 + R_2 \leq \min I(\mathbf{X};\mathbf{Y_1},\mathbf{Y_2}), \tag{3.65}$$

where the minimization is over all $p(\mathbf{z_1},\mathbf{z_2})$ that has the same marginal distributions as the actual noise. The minimizing noise distribution is called the "least-favorable" noise. Sato's bound is the basis for the computation of two-by-two broadcast channel capacity by Caire and Shamai [38].

The following example illustrates Sato's bound. Consider the two-user two-terminal broadcast channel shown in Figure 3.11, where the channel from x_1 to y_1 and the channel from x_2 to y_2 have unit gain, and the cross-over channels have a gain α. Assume that x_1 and x_2 are independent Gaussian signals, and z_1 and z_2 are Gaussian noises all with unit variance. The broadcast channel capacity is clearly bounded by $I(X_1,X_2;Y_1,Y_2)$.

This mutual information is a function of the cross-over channel gain α and the correlation coefficient ρ between z_1 and z_2. Consider the case $\alpha = 0$. In this case, the least favorable noise correlation is $\rho = 0$. This is because if z_1 and z_2 were correlated, decoding of y_1 would reveal z_1 from which z_2 can be partially inferred. Such inference is possible, of course, only if y_1 and y_2 can cooperate. In a broadcast channel where y_1 and y_2 cannot take advantage of such correlation, the capacity with correlated z_1 and z_2 is the same as with uncorrelated z_1 and z_2. Thus, regardless of the actual correlation between z_1 and z_2, the broadcast channel capacity is bounded by the mutual information $I(X_1, X_2; Y_1, Y_2)$ evaluated assuming uncorrelated z_1 and z_2. Consider another case $\alpha = 1$. The least favorable noise here is the perfectly correlated noise with $\rho = 1$. This is because $\rho = 1$ implies $z_1 = z_2$ and $y_1 = y_2$. So, one of y_1 and y_2 is superfluous. If z_1 and z_2 were not perfectly correlated, (y_1, y_2) collectively would reveal more information than y_1 or y_2 alone would. Since $\rho = 1$ is the least favorable noise correlation, the broadcast channel sum capacity is bounded by the mutual information $I(X_1, X_2; Y_1, Y_2)$ assuming $\rho = 1$. This example illustrates that the least favorable noise correlation depends on the structure of the channel. The rest of this section is devoted to a characterization of the least favorable noise.

Consider the Gaussian vector channel $\mathbf{y_i} = H_i\mathbf{x} + \mathbf{z_i}, i = 1, \cdots, K$. Assume for now that $\mathbf{x}$ is a Gaussian vector signal with a fixed covariance matrix S_{xx}, and $\mathbf{z_1}, \cdots, \mathbf{z_K}$ are jointly Gaussian noises each with a marginal distribution $\mathbf{z_i} \sim \mathcal{N}(0, I)$. Then, the task of finding the least favorable noise correlation can be formulated as the following optimization problem. Let $H^T = [H_1^T \cdots H_K^T]$. The minimization problem is:

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|} \\ \text{subject to} \quad & S_{zz}^{(i)} = I, \quad i = 1, \cdots, K, \\ & S_{zz} \geq 0, \end{aligned} \tag{3.66}$$

where S_{zz} is the covariance matrix for $\mathbf{z}$ with $\mathbf{z}^T = [\mathbf{z_1}^T \cdots \mathbf{z_K}^T]$, and $S_{zz}^{(i)}$ refers to the ith block-diagonal term of S_{zz}. The optimization is over all off-diagonal terms of S_{zz} subject to the constraint that S_{zz} is positive semi-definite.

In writing down the optimization problem (3.66), it has been tacitly assumed that the minimizing S_{zz} is strictly positive definite, so that $|S_{zz}| > 0$. This is an additional assumption that will be made throughout this chapter. In fact, it is possible for the minimizing

S_{zz} to be singular. For example, for the two-user broadcast channel considered earlier with $\alpha = 1$, the least favorable noise has a covariance matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, which is singular. Note that a sufficient condition for the minimizing S_{zz} to be non-singular is that $|HS_{xx}H^T| > 0$. This is because whenever $|S_{zz}| = 0$, it must also be that $|HS_{xx}H^T + S_{zz}| = 0$, (as otherwise the mutual information goes to infinity.) But $|HS_{xx}H^T + S_{zz}|$ cannot be zero unless $|HS_{xx}H^T|$ is zero. Thus, $|HS_{xx}H^T| > 0$ is sufficient to ensure that $|S_{zz}| > 0$. This sufficient condition holds, for example, when both H and S_{xx} are full rank. This condition always applies to downstream digital subscriber line (DSL) multi-line channels, but it does not necessarily apply to wireless multi-antenna downlink channels unless the number of users is smaller than the number of base-station antennas.

The following lemma characterizes an optimality condition for the least favorable noise assuming that such a noise is non-singular. For now, the transmit signal for the broadcast channel $\mathbf{x}$ is assumed to be Gaussian with a fixed covariance matrix. It will be shown later that this restriction is without loss of generality.

Lemma 3.2 *Consider a Gaussian vector broadcast channel* $\mathbf{y_i} = H_i\mathbf{x} + \mathbf{z_i}, i = 1, \cdots, K$, *where* $\mathbf{x} \sim \mathcal{N}(0, S_{xx})$ *and* $\mathbf{z_i} \sim \mathcal{N}(0, I)$. *Let* $H^T = [H_1^T \cdots H_K^T]$. *Then, the least favorable noise distribution that minimizes* $I(\mathbf{X}; \mathbf{Y_1}, \cdots, \mathbf{Y_K})$ *is jointly Gaussian. Further, if the minimizing* S_{zz} *is non-singular, then, the least favorable noise has a covariance matrix* S_{zz} *such that* $S_{zz}^{-1} - (HS_{xx}H^T + S_{zz})^{-1}$ *is a block-diagonal matrix. Conversely, any Gaussian noise with a covariance matrix* S_{zz} *that satisfies the diagonalization condition and has* $S_{zz}^{(i)} = I$ *is a least favorable noise.*

Proof: Fix a Gaussian input distribution $\mathbf{x} \sim \mathcal{N}(0, S_{xx})$, and fix a noise covariance matrix S_{zz}. Let $\mathbf{z} \sim \mathcal{N}(0, S_{zz})$ be a Gaussian random vector, and let $\mathbf{z'}$ be any other random vector with the same covariance matrix, but with possibly a different distribution. Then, $I(\mathbf{X}; H\mathbf{X} + \mathbf{Z}) \leq I(\mathbf{X}; H\mathbf{X} + \mathbf{Z'})$. This fact is proved in [53] and [54]. Thus, to minimize $I(\mathbf{X}; \mathbf{Y_1}, \cdots, \mathbf{Y_K})$, it is without loss of generality to restrict attention to $\mathbf{z_1}, \cdots, \mathbf{z_K}$ that are jointly Gaussian. In this case, the cooperative capacity is just $\frac{1}{2} \log |HS_{xx}H^T + S_{zz}|/|S_{zz}|$. So, the least favorable noise is the solution to the optimization problem (3.66).

The objective function in the optimization problem is convex in the set of semi-definite matrices S_{zz}. The constraints are convex in S_{zz}, and they satisfy the constrained quantification condition. Thus, the Karush-Kuhn-Tucker (KKT) condition is a necessary and sufficient condition for optimality. To derive the KKT condition, form the Lagrangian:

$$L(S_{zz}, \Psi_1, \cdots, \Psi_K, \Phi) = \log|HS_{xx}H^T + S_{zz}| - \log|S_{zz}| + \sum_{i=1}^{K} \mathrm{tr}(\Psi_i(S_{zz}^{(i)} - I)) - \mathrm{tr}(\Phi S_{zz}), \tag{3.67}$$

where $(\Psi_1, \cdots, \Psi_K)$ are dual variables associated with the block-diagonal constraints, and Φ is a dual variable associated with the semi-definite constraint. ($\Psi_1, \cdots, \Psi_K, \Phi$ are positive semi-definite matrices.) The coefficient $\frac{1}{2}$ is omitted for simplicity. Setting $\partial L/\partial S_{zz}$ to zero:

$$0 = \frac{\partial L}{\partial S_{zz}} = (HS_{xx}H^T + S_{zz})^{-1} - S_{zz}^{-1} + \begin{bmatrix} \Psi_1 & & 0 \\ & \ddots & \\ 0 & & \Psi_K \end{bmatrix} - \Phi. \tag{3.68}$$

The minimizing S_{zz} is assumed to be positive definite. So, by the complementary slackness condition (see Appendix A), $\Phi = 0$. Thus, at the optimum, the following block-diagonal condition must be satisfied:

$$S_{zz}^{-1} - (HS_{xx}H^T + S_{zz})^{-1} = \begin{bmatrix} \Psi_1 & & 0 \\ & \ddots & \\ 0 & & \Psi_K \end{bmatrix}. \tag{3.69}$$

Conversely, this block-diagonal condition combined with the constraints in the original problem form the KKT condition, which is sufficient for optimality. Thus, if a noise covariance matrix satisfies (3.69), it must be a least favorable noise. □

Note that the diagonalization condition may be written in a different form. If assuming, in addition, that $HS_{xx}H^T$ is non-singular and $\Psi_1, \cdots, \Psi_K$ are invertible, (3.69) may be re-written using the matrix inversion lemma as follows:

$$S_{zz} + S_{zz}(HS_{xx}H^T)^{-1}S_{zz} = \begin{bmatrix} \Psi_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \Psi_K^{-1} \end{bmatrix}. \tag{3.70}$$

Curiously, this equation resembles a Ricatti equation. Although neither (3.69) nor (3.70) appears to have a closed-form solution, the structure of the least favorable noise is useful in deriving the broadcast channel capacity result.

3.3.2 GDFE with Least Favorable Noise

The main result of this chapter is that the cooperative capacity of the Gaussian vector channel with a least favorable noise is achievable for the Gaussian broadcast channel. Toward this end, it is shown that a generalized decision feedback precoder designed for the least favorable noise does not require receiver coordination in the sense that not only can the feedback operation be moved to the transmitter by precoding, but the feedforward matrix can also be made to have a block-diagonal structure that totally eliminates the need for receiver coordination.

Consider a generalized decision-feedback equalizer designed for the Gaussian vector channel $\mathbf{y} = H\mathbf{x} + \mathbf{z}$. For now, assume that $\mathbf{x}$ is Gaussian, and in addition, assume that H is a square matrix. If the noise covariance matrix S_{zz} is not block-diagonal, an implementation of the GDFE requires noise whitening as a first step. Suppose that the noise covariance matrix has an eigenvalue decomposition:

$$S_{zz} = Q^T \Lambda Q, \tag{3.71}$$

where Q is an orthogonal matrix and Λ is a diagonal matrix, then $\frac{1}{\sqrt{\Lambda}}Q$ is the appropriate noise whitening filter. If in addition, the transmitter covariance matrix S_{xx} is also not block-diagonal, then a Gaussian source $\mathbf{u}$ and a transmit filter B can be created such that $S_{uu} = I$ and $\mathbf{x} = B\mathbf{u}$. Let

$$S_{xx} = V\Sigma V^T \tag{3.72}$$

be an eigenvalue decomposition of the transmit covariance matrix S_{xx}. The appropriate transmit filter must be of the form:

$$B = V\sqrt{\Sigma}M \tag{3.73}$$

where M is an arbitrary orthogonal matrix, so that $S_{xx} = BS_{uu}B^T = V\Sigma V^T$. A different generalized decision-feedback equalizer can be designed for each choice of M.

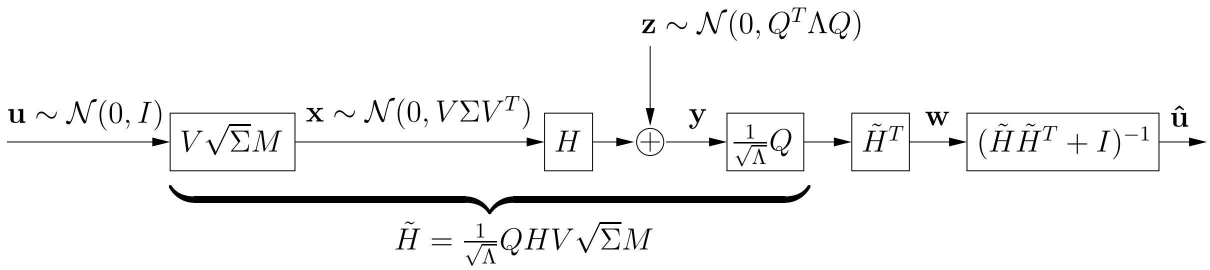


Figure 3.12: GDFE with transmit filter

Lemma 3.3 *Consider the Gaussian vector channel* $\mathbf{y} = H\mathbf{x} + \mathbf{z}$*, where* H *is a square matrix and* $\mathbf{x} \sim \mathcal{N}(0, S_{xx})$*. Fix a Gaussian source* $\mathbf{u} \sim \mathcal{N}(0, I)$*. There exists a transmit filter* B *such that* $\mathbf{x} = B\mathbf{u}$ *has a covariance matrix* S_{xx} *and the induced generalized decision-feedback equalizer has a block-diagonal feedforward filter if and only if the noise covariance matrix* S_{zz} *is such that* $S_{zz}^{-1} - (S_{zz} + HS_{xx}H^T)^{-1}$ *is block-diagonal.*

Proof: The GDFE configuration is as shown in Figure 3.12. Let $S_{xx} = V\Sigma V^T$ and $S_{zz} = Q^T \Lambda Q$. As stated before, the transmit filter must be of the form $B = V\sqrt{\Sigma}M$, where M is an orthogonal matrix. The noise whitening filter is $\frac{1}{\sqrt{\Lambda}}Q$. The combined transmit filter and the noise whitening filter give the following effective channel:

$$\widetilde{H} = \frac{1}{\sqrt{\Lambda}}QHV\sqrt{\Sigma}M. \tag{3.74}$$

The GDFE depends on the following Cholesky factorization:

$$\begin{aligned}
G^{-1}\Delta^{-1}G^{-T} &= (\widetilde{H}^T\widetilde{H} + I)^{-1} &(3.75)\\
&= \left(M^T\sqrt{\Sigma}V^T H^T Q^T \Lambda^{-1} QHV\sqrt{\Sigma}M + I\right)^{-1} &(3.76)\\
&= M^T\left(\sqrt{\Sigma}V^T H^T Q^T \Lambda^{-1} QHV\sqrt{\Sigma} + I\right)^{-1} M. &(3.77)
\end{aligned}$$

Now, choose a square matrix R such that

$$R^T R = \left(\sqrt{\Sigma}V^T H^T Q^T \Lambda^{-1} QHV\sqrt{\Sigma} + I\right)^{-1}. \tag{3.78}$$

(For example, R can be chosen to be a triangular matrix using a Cholesky factorization.) Now, because the right-hand side of the above is positive definite, all square matrices C that satisfy $C^TC = \left(\sqrt{\Sigma}V^TH^TQ^T\Lambda^{-1}QHV\sqrt{\Sigma}+I\right)^{-1}$ must be of the form $C = UR$ where U is an orthogonal matrix [23]. Therefore, the Cholesky factorization (3.77) can be written as:

$$G^{-1}\Delta^{-1}G^{-T} = M^TR^TU^TURM, \tag{3.79}$$

where URM is a block-lower-triangular matrix. For a fixed M, it is possible to choose a U to make URM block-triangular. Such a U can be found via a block QR-factorization of RM. Similarly, for each fixed U, it is possible to choose a M that makes URM block-triangular. Such a M can be found by a block QR-factorization of $(UR)^T$.

The feedforward filter of a GDFE, denoted as F, can now be computed as follows:

$$\begin{aligned} F &= \Delta^{-1}G^{-T}\widetilde{H}^T\frac{1}{\sqrt{\Lambda}}Q & (3.80)\\ &= \Delta^{-\frac{1}{2}}URMM^T\sqrt{\Sigma}V^TH^TQ^T\Lambda^{-1}Q & (3.81)\\ &= \Delta^{-\frac{1}{2}}UR\sqrt{\Sigma}V^TH^TQ^T\Lambda^{-1}Q. & (3.82)\end{aligned}$$

Next, it is shown that the condition under which there exists a suitable U to make the feedforward filter F a block-diagonal matrix is the same as the diagonalization condition on the noise covariance matrix. First, assume that $S_{zz}^{-1} - (S_{zz} + HS_{xx}H^T)^{-1}$ is block-diagonal. Then,

$$\begin{aligned} \begin{bmatrix} \Psi_1 & & 0 \\ & \ddots & \\ 0 & & \Psi_K \end{bmatrix} &= S_{zz}^{-1} - (S_{zz} + HS_{xx}H^T)^{-1} & (3.83)\\ &= Q^T\Lambda^{-1}Q - (Q^T\Lambda Q + HV\Sigma V^TH^T)^{-1} \\ &= Q^T\Lambda^{-\frac{1}{2}}\left(I - \left(I + \Lambda^{-\frac{1}{2}}QHV\Sigma V^TH^TQ^T\Lambda^{-\frac{1}{2}}\right)^{-1}\right)\Lambda^{-\frac{1}{2}}Q \\ &= Q^T\Lambda^{-1}QHV\sqrt{\Sigma}\left(I + \sqrt{\Sigma}V^TH^TQ^T\Lambda^{-1}QHV\sqrt{\Sigma}\right)^{-1} \\ &\quad \sqrt{\Sigma}V^TH^TQ^T\Lambda^{-1}Q \end{aligned}$$

where the matrix inversion lemma is used in the last step. Now, substituting (3.78) into the above:

$$Q^T\Lambda^{-1}QHV\sqrt{\Sigma}R^T R\sqrt{\Sigma}V^T H^T Q^T\Lambda^{-1}Q = \begin{bmatrix} \Psi_1 & & 0 \\ & \ddots & \\ 0 & & \Psi_K \end{bmatrix}. \tag{3.84}$$

Because H is assumed to be a square matrix, $R\sqrt{\Sigma}V^T H^T Q^T$ is also square. So, it must be of the form $U'D$, where U' is orthogonal and $D = \mathrm{diag}\{\sqrt{\Psi}_1, \cdots, \sqrt{\Psi}_K\}$:

$$R\sqrt{\Sigma}V^T H^T Q^T\Lambda^{-1}Q = U' \begin{bmatrix} \sqrt{\Psi}_1 & & 0 \\ & \ddots & \\ 0 & & \sqrt{\Psi}_K \end{bmatrix}. \tag{3.85}$$

But, this is exactly the diagonalization condition for F. By choosing $U = U'^T$ in (3.82), F becomes:

$$\begin{aligned} F &= \Delta^{-\frac{1}{2}}U'^T R\sqrt{\Sigma}V^T H^T Q^T\Lambda^{-1}Q && (3.86) \\ &= \Delta^{-\frac{1}{2}} \begin{bmatrix} \sqrt{\Psi}_1 & & 0 \\ & \ddots & \\ 0 & & \sqrt{\Psi}_K \end{bmatrix}. && (3.87) \end{aligned}$$

which is block-diagonal. Finally, an appropriate transmit filter B can be found by finding an M that makes URM block lower-triangular. This is possible by performing the following QR-factorization: $R^T U^T = MT$, where T is upper-triangular and M is orthogonal. Then, $URM = T^T$ is lower-triangular.

Conversely, if there exists a transmit filter that makes F block-diagonal, then a suitable U can be found in (3.82). Further, by setting $U' = U^T$ in (3.85), the appropriate $\sqrt{\Psi}_1, \cdots, \sqrt{\Psi}_K$ can be found to satisfy the noise covariance diagonalization. □

Combining Lemma 3.2 and Lemma 3.3, it is now clear that the Gaussian vector channel with the least favorable noise admits a GDFE structure whose feedforward filter is block-diagonal if the least favorable noise is non-singular. This means that at the feedforward stage, only individual processing of $\mathbf{y_i}$ is needed. This, together with the fact that decision-feedback can be moved to the transmitter as a precoder, completely eliminates the need for receiver cooperation. Thus, in a broadcast channel, a rate equal to the cooperative capacity with the least favorable noise correlation, $\min_{S_{zz}} \frac{1}{2} \log |HS_{xx}H^T + S_{zz}|/|S_{zz}|$ is achievable. This rate is achieved under a fixed input covariance S_{xx}. So, one might expect the capacity of the broadcast channel to be the above rate maximized over all S_{xx} subject to a power constraint. This is proved next.

3.3.3 Sum Capacity

The development so far contains the simplifying assumption that the input distribution is Gaussian. To see that the restriction is without loss of generality, a result concerning the saddle-point is useful. Consider the mutual information expression $I(\mathbf{X}; H\mathbf{X}+\mathbf{Z})$, where $\mathbf{X}$ and $\mathbf{Z}$ are independent. Let $\mathcal{K}_x$ and $\mathcal{K}_z$ be constraint sets for $\mathbf{X}$ and $\mathbf{Z}$. If some $(p(\mathbf{x}), p(\mathbf{z}))$ is such that for all $p(\mathbf{x}') \in \mathcal{K}_x$ and $p(\mathbf{z}') \in \mathcal{K}_z$,

$$I(\mathbf{X}'; H\mathbf{X}' + \mathbf{Z}) \leq I(\mathbf{X}; H\mathbf{X} + \mathbf{Z}) \leq I(\mathbf{X}; H\mathbf{X} + \mathbf{Z}') \tag{3.88}$$

then $(p(\mathbf{x}), p(\mathbf{z}))$ is called a saddle-point. The main result concerning the saddle-point is the following:

Lemma 3.4 ([54]) *The mutual information expression $I(\mathbf{X}; H\mathbf{X} + \mathbf{Z})$, where $p(\mathbf{x}) \in \mathcal{K}_x$ and $p(\mathbf{z}) \in \mathcal{K}_z$ are convex constraints, has at least one saddle-point. Further, there exists a saddle-point whose distributions are Gaussian.*

The proof of this result can be found in [54]. It goes as follows: First, it is shown that the search for the saddle-point can be restricted to Gaussian distributions without loss of generality. With Gaussian distributions, the mutual information can be written as $\frac{1}{2} \log |HS_{xx}H^T + S_{zz}|/|S_{zz}|$. Because $\log|\cdot|$ is a concave function over the set of positive definite matrices, $\frac{1}{2} \log |HS_{xx}H^T + S_{zz}|/|S_{zz}|$ is convex in S_{zz} and concave in S_{xx}. The constraints are convex. So, from a minimax theorem in game theory [55], there exists a saddle-point (S_{xx}, S_{zz}) such that

$$\frac{1}{2} \log \frac{|HS'_{xx}H^T + S_{zz}|}{|S_{zz}|} \leq \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|} \leq \frac{1}{2} \log \frac{|HS_{xx}H^T + S'_{zz}|}{|S'_{zz}|}, \tag{3.89}$$

for all (S'_{xx}, S'_{zz}) in the constraint sets.

A saddle-point (when exists) is the solution to the following max-min problem:

$$\max_{p(\mathbf{x})} \min_{p(\mathbf{z})} I(\mathbf{X}; H\mathbf{X} + \mathbf{Z}). \tag{3.90}$$

This can be easily seen as follows. Suppose $(\mathbf{X}, \mathbf{Z})$ is a saddle-point. Then, $\min_{p(\mathbf{z}'')} I(\mathbf{X}'; H\mathbf{X}' + \mathbf{Z}'') \leq I(\mathbf{X}'; H\mathbf{X}' + \mathbf{Z}) \leq I(\mathbf{X}; H\mathbf{X} + \mathbf{Z})$. So $\max_{p(\mathbf{x}')} \min_{p(\mathbf{z}')} I(\mathbf{X}'; H\mathbf{X}' + \mathbf{Z}') \leq I(\mathbf{X}; H\mathbf{X} + \mathbf{Z})$. On the other hand, fixing $p(\mathbf{x})$ gives $\min_{p(\mathbf{z})} I(\mathbf{X}; H\mathbf{X} + \mathbf{Z}') = I(\mathbf{X}; H\mathbf{X} + \mathbf{Z})$. So, $\max_{p(\mathbf{x}')} \min_{p(\mathbf{z}')} I(\mathbf{X}'; H\mathbf{X}' + \mathbf{Z}') = I(\mathbf{X}; H\mathbf{X} + \mathbf{Z})$. By the same argument, the saddle-point is also the solution to the min-max problem:

$$\min_{p(\mathbf{z})} \max_{p(\mathbf{x})} I(\mathbf{X}; H\mathbf{X} + \mathbf{Z}). \tag{3.91}$$

For any arbitrary function $f(x, y)$, it is always true that $\min_x \max_y f(x, y) \geq \max_y \min_x f(x, y)$. However, if a saddle-point exists, then max-min equals min-max:

$$\max_{S_{xx}} \min_{S_{zz}} \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|} = \min_{S_{zz}} \max_{S_{xx}} \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|}. \tag{3.92}$$

The main result of this chapter is that max-min corresponds to achievability, min-max corresponds to the converse, and the saddle-point corresponds to the sum capacity of a Gaussian vector broadcast channel.

Theorem 3.3 *Consider a Gaussian vector broadcast channel* $\mathbf{y_i} = H_i\mathbf{x} + \mathbf{z_i}$, $i = 1, \cdots, K$. *Let* $H^T = [H_1^T \cdots H_K^T]$. *The sum capacity under a power constraint is a saddle-point of the mutual information* $\frac{1}{2} \log |HS_{xx}H^T + S_{zz}|/|S_{zz}|$, *if the saddle-point is such that* $S_{zz} > 0$. *Here, the saddle point is computed with the following constraints:* S_{zz} *has block-diagonal entries that are the covariance matrices of* $\mathbf{z_1}, \cdots, \mathbf{z_K}$, *and* S_{xx} *satisfies* $\mathrm{tr}(S_{xx}) \leq P$.

Proof: First, the converse: Sato's outer bound states that the broadcast channel sum capacity is bounded by the capacity of any discrete memoryless channel whose noise marginal distributions are equal to $p(\mathbf{z_i})$. The tightest outer bound is then the capacity of the channel with the least favorable noise correlation. The capacity of a discrete memoryless channel is $\max_{p(\mathbf{x})} I(\mathbf{X}; \mathbf{Y_1}, \cdots, \mathbf{Y_K})$, so:

$$C \leq \min_{p(\mathbf{z})} \max_{p(\mathbf{x})} I(\mathbf{X}; H\mathbf{X} + \mathbf{Z}), \tag{3.93}$$

where the maximization is over the power constraint $\mathbf{E}[\mathbf{X}^T\mathbf{X}] \le P$, and the minimization is over all noise distributions whose marginals are the same as the actual noise. The solution to this minimax problem is the saddle-point for $I(\mathbf{X}; H\mathbf{X} + \mathbf{Z})$. Since the constraint sets are convex, by Lemma 3.4, a saddle-point exists. Further, the saddle-point can be chosen to be Gaussian, so the outer bound can be written as

$$C \le \min_{S_{zz}} \max_{S_{xx}} \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|}, \tag{3.94}$$

where S_{xx} belongs to the set of positive semi-definite matrices satisfying the power constraint $\mathrm{tr}(S_{xx}) \le P$, and S_{zz} belongs to the set of noise covariance matrices with $S_{zz}^{(i)} = \mathbf{E}[\mathbf{z_i}\mathbf{z_i}^T]$, $i = 1, \cdots, K$, as block-diagonal terms.

Next, the achievability: the existence of a saddle-point implies that min-max equals max-min. So, it is only necessary to show that

$$C \ge \max_{S_{xx}} \min_{S_{zz}} \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|}. \tag{3.95}$$

Since the saddle-point can be chosen to be Gaussian, the development leading to the theorem, which restricts consideration to Gaussian inputs, is without loss of generality. Further, Lemma 3.3 requires the channel matrix to be square. If there are more receive antennas than transmit antennas, zeros can be padded to H to make H a square matrix without affecting capacity. If there are more transmit antennas than receive antennas, because S_{xx} is a water-filling covariance matrix with respect to H, the rank of S_{xx} is bounded by the number of receive antennas. Then, the null space of S_{xx} may be deleted, and H can be made equivalent to a square matrix. In either case, the condition in Lemma 3.3 that H is square can be satisfied.

Now, at the saddle-point, S_{zz} is a least favorable noise for S_{xx}. So, by Lemma 3.2 and the assumption $S_{zz} > 0$, it must satisfy the condition that $S_{zz}^{-1} - (S_{zz} + HS_{xx}H^T)^{-1}$ is block-diagonal. By Lemma 3.3, this implies that there is an appropriate transmit filter B such that a GDFE designed for this B and S_{zz} has a block-diagonal feedforward filter. Consider now the precoding configuration of the GDFE. The feedforward section is block-diagonal. The feedback section is moved to the transmitter. So, the decoding operations of $\mathbf{y_1}, \cdots, \mathbf{y_K}$ are completely independent of each other. Further, because the feedback filter is block-diagonal, the GDFE receiver is oblivious of the correlation between $\mathbf{z_i}$'s. Thus, although the actual noise distribution may not have the same joint distribution as the least favorable noise, because the marginal distributions are the same, a GDFE precoder designed for the least favorable noise performs as well as with the actual noise. Since by Theorem 3.2, this GDFE precoder achieves $I(\mathbf{X}; H\mathbf{X} + \mathbf{Z})$, so $\min_{S_{zz}} I(\mathbf{X}; H\mathbf{X} + \mathbf{Z})$ is achievable. Further, it is possible to maximize the above over S_{xx}. Therefore, the outer bound (3.95) is achievable. $\square$

Note that the GDFE transmit filter B designed for the least favorable noise also identifies the set of sum capacity-achieving S_i in Theorem 3.1. Let $B = [B_1 \cdots B_K]$. Set $S_1 = B_1 B_1^T$, $\cdots$, $S_K = B_K B_K^T$. Then, it is easy to verify that the sum capacity is achieved with $R_i = \frac{1}{2} \log |\sum_{k=i}^{K} H_i S_k H_i^T + I| / |\sum_{k=i+1}^{K} H_i S_k H_i^T + I|$.

Theorem 3.3 suggests the following game-theoretical interpretation of the Gaussian vector broadcast channel. There are two players in the game. A signal player chooses a S_{xx} to maximize $I(\mathbf{X}; H\mathbf{X} + \mathbf{Z})$ subject to the constraint $\mathrm{tr}(S_{xx}) \leq P$. A noise player chooses a fictitious noise correlation in S_{zz} to minimize $I(\mathbf{X}; H\mathbf{X} + \mathbf{Z})$ subject to the constraint $S_{zz}^{(i)} = I$. A Nash equilibrium in the game is a set of strategies such that each player's strategy is the best response to the other player's strategy. The Nash equilibrium in this mutual information game exists, and the Nash equilibrium is the sum capacity of the Gaussian vector broadcast channel.

The saddle-point property of the Gaussian broadcast channel sum capacity implies that the capacity achieving (S_{xx}, S_{zz}) is such that S_{xx} is the water-filling covariance matrix for S_{zz}, and S_{zz} is the least favorable noise covariance matrix for S_{xx}. In fact, the converse is also true. If a set of (S_{xx}, S_{zz}) can be found such that S_{xx} is the water-filling covariance for S_{zz}, and S_{zz} is the least favorable noise for S_{xx}, then (S_{xx}, S_{zz}) constitutes a saddle-point. This is because the mutual information is a concave-convex function, and the two KKT conditions, corresponding to the two optimization problems are, collectively, sufficient and necessary at the saddle-point [56] [57]. Thus, the computation of the saddle-point is equivalent to simultaneously solving the water-filling problem and the least favorable noise problem.

One might suspect that the following algorithm can be used to find a saddle-point numerically. The idea is to iteratively compute the best input covariance matrix S_{xx} for a given noise covariance, then compute the least favorable noise covariance matrix S_{zz} for the given input covariance. If the iterative process converges, both KKT conditions are satisfied, and the limit must be a saddle-point of $\frac{1}{2}\log|HS_{xx}H^T + S_{zz}|/|S_{zz}|$. However, such an iterative min-max procedure is not guaranteed to converge for a general game even when the pay-off function is concave-convex. But, the iterative procedure appears to work well in practice for this particular problem. The convex-concave nature of the problem also suggests that general-purpose numerical convex programming algorithms can be used to solve for the saddle-point with polynomial complexity [56] [58] [59].

3.4 Value of Cooperation

A principal aim of this thesis to illustrate the value of cooperation in a Gaussian vector channel $\mathbf{y} = H\mathbf{x} + \mathbf{z}$, where the transmit signal $\mathbf{x}$ and the receive signal $\mathbf{y}$ are both vector valued. Let S_{zz} be the noise covariance matrix. When cooperation is possible both among the transmit terminals and among the receive terminals, the channel capacity under a power constraint is the solution to the following optimization problem:

$$\begin{array}{ll} \text{maximize} & \dfrac{1}{2}\log\dfrac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|} \\ \text{subject to} & \mathrm{tr}(S_{xx}) \leq P, \\ & S_{xx} \geq 0. \end{array} \tag{3.96}$$

When cooperation is possible at the receiver, but not at the transmitter, the sum capacity is still a maximization of $I(\mathbf{X};\mathbf{Y})$, but with an additional constraint:

$$\begin{array}{ll} \text{maximize} & \dfrac{1}{2}\log\dfrac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|} \\ \text{subject to} & \mathrm{tr}(S_{xx}) \leq P, \\ & S_{xx}(i,j) = 0, \quad \forall (i,j) \text{ uncoordinated} \\ & S_{xx} \geq 0. \end{array} \tag{3.97}$$

Here $S_{xx}(i,j)$ denotes the (i,j)-entry of S_{xx}. Thus, in terms of capacity, the value of cooperation at the transmitter lies in the ability for the transmitters to send correlated signals.

In a broadcast channel where cooperation is possible at the transmitter but not at the receiver, the capacity is now the solution to a minimax problem (assuming that the solution is such that $S_{zz} > 0$):

$$\begin{aligned} \max_{S_{xx}} \min_{S'_{zz}} \quad & \frac{1}{2} \log \frac{|HS_{xx}H^T + S'_{zz}|}{|S'_{zz}|} \\ \text{subject to} \quad & \mathrm{tr}(S_{xx}) \le P, \\ & S'_{zz}(i,j) = S_{zz}(i,j), \quad \forall (i,j) \text{ coordinated} \\ & S_{xx}, S_{zz} \ge 0. \end{aligned} \tag{3.98}$$

Because of the lack of coordination, the receivers cannot distinguish between different noise correlations. So, the capacity is as if "nature" has chosen a least favorable noise correlation. Thus, the value of cooperation at the receiver lies in its ability to recognize and to take advantage of the true correlation among the noise signals.

When full cooperation is possible at both the transmitter and at the receiver, a Gaussian vector channel can be decomposed into non-interfering scalar sub-channels that can be independently encoded and decoded. With coordination at one side only, the vector channel can only be decomposed into a series of scalar sub-channels each interfering into subsequent sub-channels. Thus, from a coding point of view, the value of cooperation lies in the ability to eliminate the need to either pre-subtract or post-subtract interference. When full coordination is not possible, the generalized decision-feedback equalizer emerges as a unifying structure that is able to achieve both the multiple access channel capacity and the broadcast channel sum-capacity.

3.5 Practical Precoding

An information theoretical treatment of the broadcast channel has been given so far. The broadcast channel is shown to be closely related to the channel with transmitter side information, and the optimal precoding structure is shown to be that of a generalized decision feedback equalizer. The implementation of the precoder depends critically on finding powerful codes that can approach the "dirty-paper" capacity region. The rest of this chapter is devoted to practical precoding schemes. Finding good "dirty-paper" codes is still an open research area (see e.g. [60]). This section focuses on codes that work well in the high SNR region.

The development of the Gaussian vector broadcast channel sum capacity involves a vector version of "writing-on-dirty-paper". However, from a coding perspective, finding

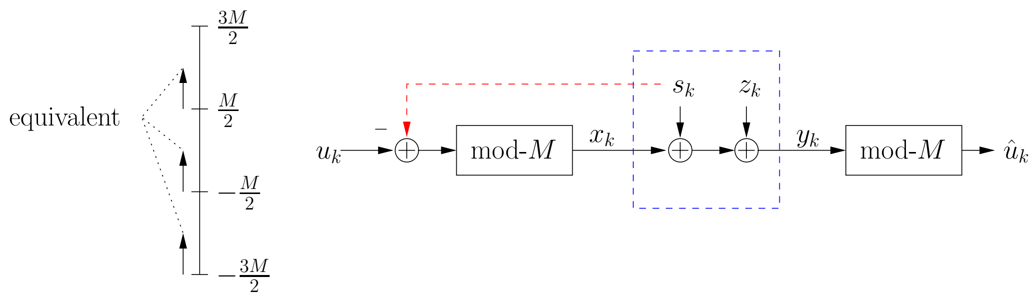


Figure 3.13: Tomlinson-Harashima precoder

good scalar "dirty-paper" codes is sufficient as the following argument shows. This argument is due to Lapidoth [61]. Consider a Gaussian vector channel with side information $\mathbf{Y} = \mathbf{X} + \mathbf{S} + \mathbf{Z}$. The vector channel may be decomposed into parallel scalar channels along the eigenvectors of the noise covariance matrix S_{zz}. The resulting set of Gaussian scalar channels have independent noises, but may have correlated interference. However, since interference is completely known at the transmitter, scalar "writing-on-dirty-paper" codes can be applied individually to each of the scalar channels regardless of interference correlation. Collectively, the set of scalar channels achieves a vector channel capacity as if interference does not exist. For this reason, this section considers only scalar "dirty-paper" codes. The key to finding good scalar codes is to recognize that "dirty-paper" coding is related to Tomlinson-Harashima precoding.

3.5.1 Tomlinson-Harashima Precoding

The purpose of a precoder is to pre-subtract known interference at the transmitter. A purely arithmetic subtraction requires additional transmit power. The Tomlinson-Harashima precoder [62] [63], originally developed for the intersymbol interference channel, is a clever scheme to pre-subtract interference with minimal extra power. Figure 3.13 illustrates a Tomlinson-Harashima precoder, where s_k is the known interference signal at time instant k. In order to convey an intended symbol u_k, a precoder sends $x_k = u_k - s_k$ to compensate for the interference. However, s_k may be large, so x_k may exceed the power constraint. The idea is to constrain the intended symbol u_k to lie within $[-M/2, M/2)$. Instead of sending x_k, the encoder sends x_k modulo-M. Likewise, the decoder also performs a modulo-M

operation. In effect, all transmit symbols that differ by an integer multiple of M are regarded as equivalent by the decoder. The modulo-M operation reduces the transmit power to approximately $M^2/12$ regardless of the interference strength, while maintaining distinguishability of x_k. The process of encoding a symbol can now be interpreted as a process of modifying s_k to one of the equivalent representatives of the intended symbol. The modulo-M geometry ensures that there is one representative within distance M from each possible s_k. Tomlinson-Harashima precoder is in fact a one-dimensional implementation of "writing-on-dirty-paper".

The connection between "writing-on-dirty-paper" and Tomlinson-Harashima precoding was first observed by Erez, Shamai and Zamir [64]. There are two main differences between a Tomlinson-Harashima precoder and an optimal "dirty-paper" precoder. First, Tomlinson precoder performs a pre-subtraction $x_k = u_k - s_k$, while the optimal "dirty-paper" code does $x_k = u_k - \alpha s_k$, where $\alpha = P/(P + N)$ (P is the transmit power constraint and N is the noise variance.) Thus, Tomlinson-Harashima precoder cannot achieve the "dirty-paper" capacity unless $\frac{P}{N} \to \infty$. Finding practical codes in the low SNR region is still very much an open research area. This section concentrates on codes for the high SNR region. The second difference between a Tomlinson-Harashima precoder and an optimal "dirty-paper" precoder is that pre-subtraction in Tomlinson precoding is done on a symbol-by-symbol basis and it takes only the current s_k into account, while "dirty-paper" precoding requires the entire s_k sequence. As pointed out in [64], in the limit of large interference, the capacity loss when using only causal side information is exactly the shaping loss (up to 1.53dB). This is related to the fact that the input to a Tomlinson precoder must be constrained between $-M/2$ and $M/2$. This results an input distribution that is uniform inside a cubic shape, and it incurs a shaping loss when compared to the spherical shape of an optimal Gaussian code. To recover the shaping loss, non-causal side information must be used to perform the modulo operation on high-dimensional spheres. This is possible using a precoder based on an optimal vector quantizer. In the next section, a practical precoder based on trellis codes is presented. It is a generalization of Tomlinson-Harashima precoder and is capable of achieving "dirty-paper" channel capacity at high SNR.

3.5.2 Trellis Precoding

A conceptual model for a vector quantization-based precoder is shown in Figure 3.14. It works as follows. First, a codeword sequence U^n is generated by an error-correcting code.

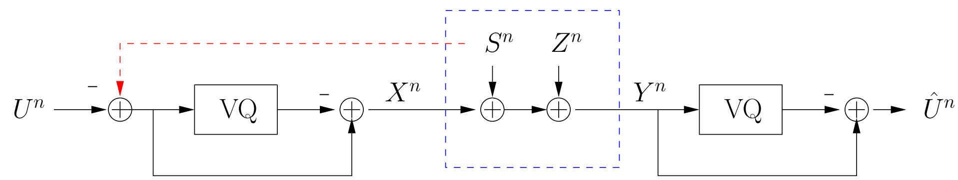


Figure 3.14: Precoding via vector quantization

The additive interference sequence S^n is pre-subtracted from the codeword, and the difference sequence is then quantized by a vector quantizer. The quantization noise is sent as the input to the channel. The channel adds interference and noise. At the decoder, the received sequence is first quantized by the same quantizer. The quantization noise is sent to the decoder of the error-correcting code to recover the message. Figure 3.15 illustrates the relation between U^n, X^n and S^n. The spheres denote the Voronoi regions associated with the quantizer outputs. The codeword sequence U^n is designed to be confined within the Voronoi region. Each codeword is given multiple equivalent representatives corresponding to multiple quantizer outputs. The equivalent representatives are illustrated by the arrows in Figure 3.15. Since the interference sequence S^n is known non-causally, the precoder can construct an input sequence X^n to steer S^n to the closest representative of U^n. It is easy to see that as long as the codeword is confined to the Voronoi region, perfect reconstruction is possible in the absence of noise. This vector quantization approach can be viewed as a generalization of Tomlinson-Harashima precoding. The one-dimensional modulo-M operation is replaced by a vector quantizer which performs a modulo operation with respect to a Voronoi region. After the modulo operation, the precoder outputs are uniformly distributed in the Voronoi region. The Voronoi region of an ideal vector quantizer is a high dimensional sphere, thus achieving a shaping gain.

The idea of using Voronoi region for shaping was proposed by Forney in [65], where a trellis code is used as a vector quantizer to achieve a shaping gain on an additive white Gaussian noise (AWGN) channel. For an AWGN channel, the capacity achieving distribution over a large block length is a uniform distribution over a high dimensional sphere. However, the traditional rectangular constellation-based trellis codes cover a high dimensional cube uniformly. A cubic shape suffers from a shaping loss up to 1.53dB when compared to a spherical shape. To recover the shaping loss, Forney proposed to expand the constellation

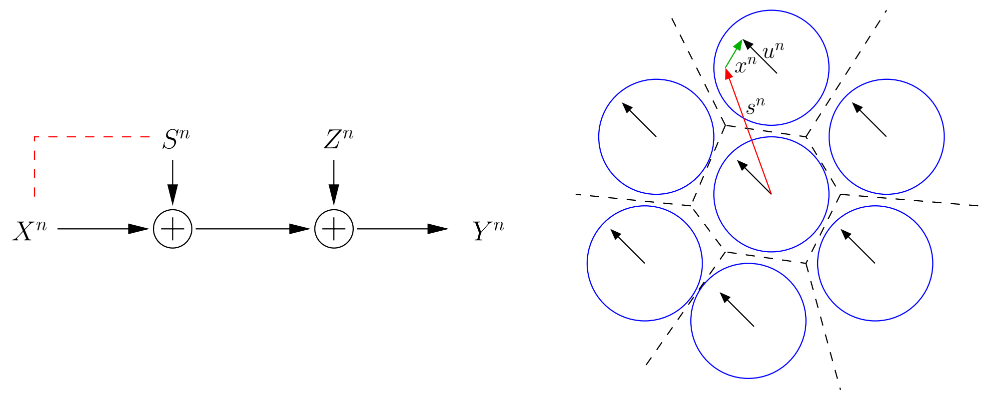


Figure 3.15: Precoding using spherical regions

size slightly first, then modulo the expanded sequence with respect to the Voronoi region of a trellis shaping code. It can now be seen that shaping for AWGN channel is very similar to shaping for channel with side-information. In fact, Eyuboglu and Forney [66] further extended trellis shaping by combining shaping with one-dimensional Tomlinson precoding and trellis coding for the ISI channel. This combined structure can be easily modified for channels with non-causal side information.

A trellis shaping code for channels with side information is now described. A trellis code can be viewed as a collection of paths through the constellation. These paths have a good minimum distance property, and they can be represented by a finite state machine, thus allowing efficient decoding. Such a collection of trellis paths can also be used as reconstruction values in a vector quantizer. The operation of the precoder is shown in Figure 3.16. Two codes are working independently. C_c is a trellis channel code with G_c as its k_c/n_c convolutional coset encoder. C_s is a k_s/n_s convolutional shaping code, whose Voronoi region is used as the basis of modulo operation. The input bits are divided into two groups. The first q bits are inputs to the trellis channel code, where k_c bits are encoded by the convolutional code G_c, and the rest are uncoded bits selecting constellation points within each coset. The output of the trellis code is a constellation of size $2^{q-k_c+n_c}$. This constellation is repeated 2^{n_s} times in two dimensions resulting in 2^{n_s} non-overlapping regions. At any given time instance, one of these regions is selected by the second group of r_s inputs and the shaping convolutional code C_s. Therefore, the entire codeword can be

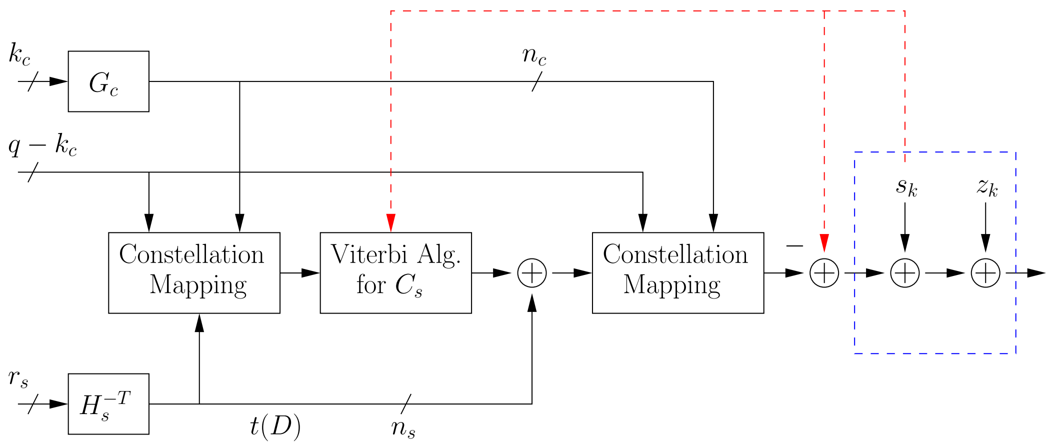


Figure 3.16: Trellis implementation of Voronoi precoder

thought of as a combination of 2^q trellis paths within each region, and a sequence of regions. Two sequences of regions are deemed equivalent if their difference is a valid codeword in C_s. The initial sequence is determined by r_s input bits and an inverse syndrome decoder H_s^{-T} for C_s. In a convolutional code, sequences that differ by a valid codeword all have the same syndrome. Thus, using r_s input bits as the syndrome uniquely determines an equivalent class of n_s-bit sequences $t(D)$. Every sequence in this equivalent class can be represented by $t(D)$ plus some valid codeword in G_s. So, the shaping encoder only needs to select the appropriate codeword to be added. The appropriate codeword is selected by the Viterbi algorithm for C_s. The criterion for such selection can in principle be anything as far as decoding is concerned. To minimize transmit power for channels with side information, the criterion here is the minimization of the square difference between the output sequence and the side information sequence. Thus, the Viterbi algorithm for C_s compares the side information sequence with the constellation sequence determined jointly by the trellis encoder C_c and the syndrome sequence for C_s, and outputs a codeword for C_s which modifies the sequence of region selects so that the resulting codeword is as close to the side information as possible. The encoder sends out the difference between the two to the channel, and the side information is then added back. On the decoder side, bits are recovered with a usual trellis decoder for C_c and a syndrome mapper for C_s. The decoder is identical to the one in [66].

There are two principal differences between a trellis precoder for a channel with side-information and a trellis shaping precoder for an ISI channel as described in [66]. First, a shaping code selects a region sequence so that the output codeword has the minimum energy, while a precoder selects a region sequence so that it takes the minimum energy to steer the side information sequence to a correct codeword. The second difference is more subtle. In a shaping code for an ISI channel, a small amount of constellation expansion suffices, thus a rate 1/2 shaping code is sufficient. For precoding for a channel with side information, it is desirable to have a code rate such as 3/4 or 5/6 to force constellation expansion. The reason for expansion is to ensure that the side information sequence lies entirely within the expanded constellation. In practice where the magnitude of the side information sequence is not known in advance, it is necessary to add a modulo-M operation outside of the expanded constellation. Note that the actual transmitted constellation is not the expanded constellation, but its difference with the side information sequence. Therefore, the expanded constellation does not pose a practical concern.

The shaping gain for a Voronoi precoder depends on the shape of the Voronoi region of the shaping code. The trellis shaping codes for AWGN channels reported in [65] can be used directly for the broadcast channel with exactly the same shaping gain. In particular, a simple 4-state trellis shaping code already achieves almost 1dB shaping gain.

3.6 Summary

To summarize, this chapter deals with a class of non-degraded Gaussian vector broadcast channels. The sum capacity is characterized as a saddle-point of a Gaussian mutual information game where a signal player chooses a signal covariance matrix to maximize the mutual information, and a noise player chooses a fictitious noise correlation to minimize the mutual information. This result holds under the condition that the noise covariance matrix at the saddle-point is non-singular. The broadcast channel sum capacity is achieved using a precoder with the structure of a generalized decision-feedback equalizer. A trellis precoder for the broadcast channel is proposed. The precoder is a generalization of the Tomlinson-Harashima precoder, and it is capable of achieving the broadcast channel sum capacity at high SNR.

Chapter 4

Interference Channel

In a Gaussian channel with multiple transmitters and multiple receivers, when neither transmitters nor receivers cooperate, the vector channel becomes an interference channel. The capacity region of the interference channel is a long-standing open problem in information theory. Even for the simplest two-user additive white Gaussian interference channel, only partial results are available. The largest achievable region for the interference channel is due to Han and Kobayashi [67], and it is based on superposition coding and interference subtraction. In fact, when the interference level is very high, interference subtraction is optimal and it achieves the same data rate as if interference is completely removed [68] [69]. For this to happen, however, interference coupling must be stronger than the direct channel, which typically does not happen in realistic applications. When the interference level is low, interference subtraction is difficult to do, and the capacity region is unknown. In this light, this thesis restricts attention to transmission techniques where no interference subtraction takes place. This restriction is realistic in many practical systems. With this assumption, the transmission strategy for each user is simply its power allocation, and multi-user interference is treated as noise. The principal aim of this chapter is to develop power allocation algorithms that are able to optimize the joint performance of multiple users in the presence of mutual interference.

This chapter uses the digital subscriber line (DSL) system as the motivating example. Digital subscriber line is a local access technology that brings high-speed data connection to home via ordinary telephone twisted-pairs. The DSL transmission environment is traditionally thought of as a single-user environment because each user is connected to the central office via a pair of dedicated wires. However, a central office typically serves hundreds of thousands of homes, and twisted-pairs from different homes are bundled together on the way to the central office. In the bundled environment, because of the physical proximity, the twisted-pairs emit electromagnetic interference into each other. Such interference is called crosstalk, and it can be the dominant noise source in a line. For this reason, the DSL environment is more accurately modeled as a multi-user environment.

In most current DSL systems, the transmit power spectral densities (PSD) for all modems are fixed regardless of the loop environment. This is a single-user design approach. The goal of this chapter is to show that in many cases, a multi-user system design that dynamically optimizes each modem's power spectral density based on the loop environment can significantly improve the system performance. Further, a simple distributed power allocation scheme that is implementable in existing modems can be used to realize much of the gain. This dynamic spectrum management approach is becoming increasingly important especially as high-speed DSL systems evolve toward higher frequency bands, where the crosstalk problem is more pronounced, and as optical network units (ONU) are increasingly deployed closer to customer premises, where they can potentially emit strong crosstalk into neighboring lines.

The power control problem in DSL systems differs from the more widely-studied power control problem in wireless systems (e.g. [70] [71] [72] [73]) in two key aspects. First, although the DSL transmission environment varies from line to line, it does not vary over time. Fading and mobility are not issues. Consequently, the assumption of perfect channel knowledge is realistic and is made here. On the other hand, unlike narrowband wireless applications where flat-fading can often be assumed, the DSL lines are severely frequency selective. Thus, the optimal power allocation scheme needs to consider not only the total amount of power allocated to each user, but also the allocation of power over frequencies. Nevertheless, power control schemes designed for wireless systems [70] [72] [74] can still provide considerable insight. In particular, DSL systems suffer from a near-far problem similar to that in CDMA systems. The near-far problem arises when two transmitters located at different distances attempt to communicate with the same central office. When one transmitter is much closer to the central office than the other, the interference generated by the closer transmitter can overwhelm the signal from the other transmitter. The power control algorithm proposed in this chapter is capable of overcoming this problem.

The proposed power control algorithm is based on the formulation of the multi-user environment as a non-cooperative game. This game-theory point of view has appeared in several recent works on the power control problem for wireless networks [75] [76] [77] [78]. However, these existing works focus on the CDMA system, and the power control algorithms studied there consider only flat-fading channels. In a DSL environment, the frequency-selective nature of the channel is crucial, and it must be dealt with explicitly. The main result of this chapter is that under a wide range of conditions, the frequency-selective interference channel game has a unique Nash equilibrium. This result leads to a power control algorithm based on the concept of competitive optimality, and it further suggests that power control can be implemented distributively and asynchronously with minimal centralized control. Distributed power control schemes have important advantages over centralized schemes, especially as the local access market moves toward "unbundling" where competing service providers can potentially share the same binder.

The rest of this chapter is organized as follows. Section 4.1 models the DSL environment as an interference channel. Section 4.2 defines and characterizes the competitive equilibrium in the interference network and devises an iterative method to achieve the equilibrium. Section 4.3 proposes a distributed power allocation method based on the idea of competitive equilibrium. System performance is characterized in Section 4.4, and concluding remarks are made in Section 4.5.

4.1 DSL Environment

Telephone twisted-pairs are severely frequency selective channels. To combat intersymbol interference (ISI), the DSL technology uses Discrete Multitone (DMT) modulation to divide the frequency band into a large number of ISI-free sub-channels, each of which is used to carry a separate data stream. DMT modulation is standardized for asymmetric digital subscriber lines (ADSL) in [79] and for very high-speed digital subscriber lines (VDSL) in [80]. The use of DMT modulation allows arbitrary power assignment in each frequency, thus making spectral shaping easy to realize.

Figure 4.1 illustrates a typical DSL bundle. There are two types of crosstalk interference. Near-end crosstalk (NEXT) refers to crosstalk created by transmitters located on the same side as the receiver. Far-end crosstalk (FEXT) refers to crosstalk created by transmitters located on the opposite end of the line. NEXT is usually much stronger than FEXT. To avoid NEXT, DSL transmission uses either frequency-division duplexing (FDD), where all lines transmit in the same direction in every frequency, or time-division duplexing (TDD),

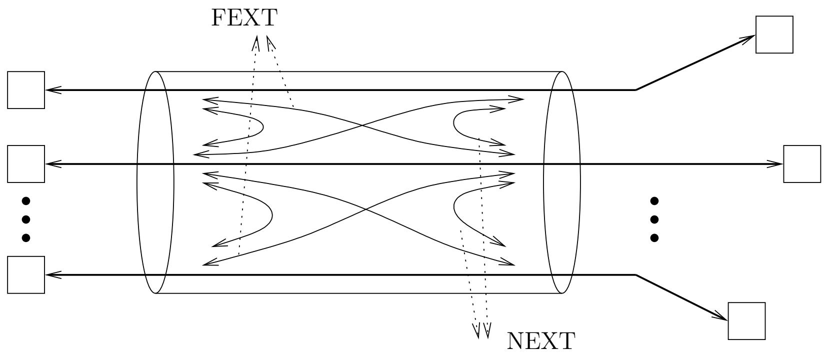


Figure 4.1: DSL crosstalk environment

where all lines transmit in the same direction in every time slot. North American and European standards use FDD, while TDD is used in Japan. This thesis mainly considers frequency-division duplexing systems such as the one standardized in [80], where the entire frequency spectrum is divided into four to six upstream and downstream bands.

The DSL environment can be modeled as an interference channel. The interference channel model is appropriate when neither transmitters nor receivers cooperate. In future DSL systems where remote fiber-fed-terminal modems can coordinate in encoding or decoding messages, a multiple access or broadcast channel model will become applicable. In this part of the thesis, such coordination is not assumed.

Figure 4.2 shows an interference channel model. There are K transmitters and K receivers. The transfer function of the channel from transmitter i to receiver j is denoted as $H_{ij}(f)$, where $0 \le f \le F_s$, $F_s = \frac{1}{2T_s}$, and T_s is the sampling rate. The noise power spectral density for the receiver i is denoted as $\sigma_i(f)$. Denote the transmit power spectral density for the transmitter i as $P_i(f)$. $P_i(f)$ must satisfy a power constraint:

$$\int_0^{F_s} P_i(f)df \le \mathbf{P_i}. \tag{4.1}$$

Fixing $P_i(f)$ and treating interference as noise, the following data rate is achievable:

$$R_i = \int_0^{F_s} \log_2 \left(1 + \frac{P_i(f)|H_{ii}(f)|^2}{\Gamma \left(\sigma_i(f) + \sum_{j \ne i} P_j(f)|H_{ji}(f)|^2 \right)} \right) df, \tag{4.2}$$

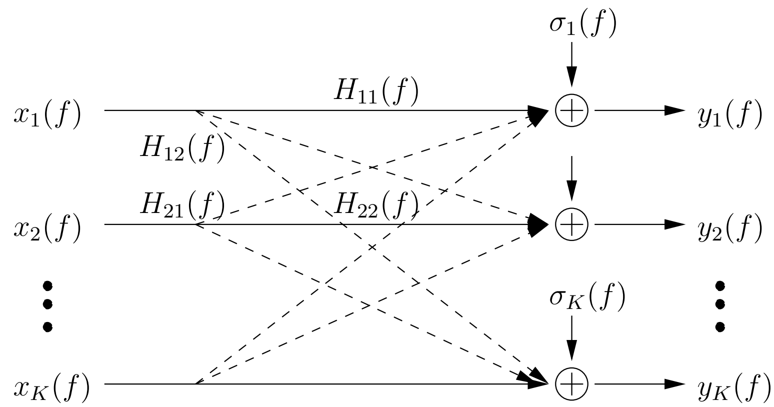


Figure 4.2: Gaussian interference channel model

where Γ is the SNR-gap[1]. The objective of the system design is to "jointly" maximize the set of rates $R_1, \cdots R_K$ subject to power constraints $\mathbf{P_1}, \cdots \mathbf{P_K}$. Notice that for each transmitter, increasing its power spectral density increases its data rate, but this also increases its interference into other users. Thus, a system design must consider the tradeoff among the data rates of all users, and a single figure of merit is inadequate to represent the system performance. For example, it is not enough to consider just the maximization of the sum rate. The sum-rate optimal power allocation is often the one that gives high data rates to users closer to the central office, and low data rates to users farther away, which may not be fair. As realistic DSL deployments could require an arbitrary level of service for each user, it is necessary to characterize fully the performance tradeoff among all users. A convenient way to characterize the tradeoff is to use the concept of a rate region:

$$\mathcal{R} = \{(R_1, \cdots, R_K) : \exists (P_1(f), \cdots P_K(f)) \text{ satisfying (4.1) and (4.2)}\}. \tag{4.3}$$

The rate region characterizes all possible data rate combinations among all users subject to the power constraints.

Despite its attractiveness, the above rate region is not so easy to compute. This is because the capacity expression is a non-convex function of power allocations. So, although

[1]The SNR-gap denotes the gap between a practical coding and modulation scheme and the channel capacity. The SNR-gap depends on the target probability of error and the coding and modulation scheme used. At theoretical capacity, $\Gamma = 0$dB.

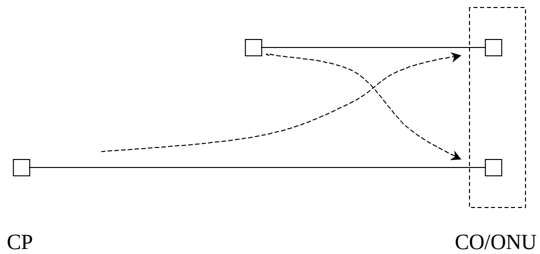


Figure 4.3: Near-far problem

in theory, the rate region can be found by an exhaustive search through all possible power allocations or by a series of optimization steps involving weighted sums of data rates, the computational complexity of doing so is prohibitively high. This thesis circumvents the difficulty by taking a different approach. Instead of solving the optimization problem globally, the interference channel is viewed instead as a non-cooperative game among the competing users, and the focus is shifted to the notion of competitive optimality. The main point of this chapter is to show that although a competitively optimal point may not be the global optimum, it nevertheless gives substantial improvement in performance over current DSL systems.

As mentioned earlier, current DSL systems are designed with each modem transmitting at a fixed power spectral density. The fixed power-spectral-density mask limits the worst-case interference level, and the modems are designed to withstand the worst-case noise. Such a design is conservative in the sense that the actual interference is often much smaller than the worst-case noise in realistic deployment scenarios. Moreover, the same power-spectral-density mask is applied to all modems regardless of their geographic locations. This is problematic because of the near-far problem mentioned before. Figure 4.3 illustrates a configuration in which two DSL lines in the same binder emanate from the central office (CO) to the customer premises (CP). When both transmitters at the CP-side transmit with the same power spectral density, due to the difference in line attenuation, the FEXT caused by the short line can severely affect the transmission on the long line. To remedy this spectral incompatibility problem, the short line must reduce its upstream transmit power spectral density. This reduction in upstream power spectral density is known as upstream power back-off (UPBO), and it has been under intensive study in VDSL standardization bodies [81] [82] [83]. Note that in this specific configuration, the downstream direction does not suffer the same problem. Although all transmitters at the CO-side also transmit at the same power spectral density, the FEXT they create into each other is identical at any fixed distance from CO [82]. This downstream FEXT level is always much weaker than the data signals, so it does not pose a serious problem to downstream transmission.

Several upstream power back-off algorithms have been proposed for VDSL. An extensive review of these methods can be found in [83]. All current power back-off methods attempt to reduce the interference emission of shorter lines by forcing them to emulate the behavior of a longer line. For example, in the *constant power back-off* method, the PSD is reduced by a constant factor across the upstream transmission bands, so that at a particular reference frequency the received PSD level from shorter lines is the same as the received PSD level from a longer reference line. A generalization of this method is the *reference length* method, where variable amount of back-off is implemented across frequency so that the received PSD for a shorter line is the same as some longer reference line at all frequencies. Imposing the same-PSD criterion for shorter lines across the entire frequency band may sometimes be too restrictive. In the *multiple reference length* method, a different reference length is set for different frequency bands. The three methods mentioned so far equalize the PSD level of a shorter line to the PSD level of some longer reference line. While this may be easy to implement, better performance can be obtained if the interference levels themselves are equalized instead. Examples of such approaches are the *equalized-FEXT* method, which forces the FEXT emission by shorter line to be equal to the FEXT from a longer reference line, and the *reference noise* method, which forces the FEXT emission to be equal to a more general reference noise.

All previously discussed power back-off methods require the power or noise spectrum of the short lines to comply with a reference line or a reference noise. Although the optimal choice of reference may not be easy to make, once the references are standardized, these approaches are simple to implement, because they only require each line to adjust its power spectrum according to a pre-determined reference and thus do not require knowledge of the network configuration. If, however, the line characteristics of the network are known, more sophisticated adaptive power control methods can be implemented, and much better performance is possible. For example, the optimal power allocation can be pre-computed by some centralized agent, and an appropriate power spectral density can be assigned to each user. However, as mentioned before, finding the optimal power spectrum is computationally complex. This is because the optimization problem involves a large number of variables, and due to the non-convex nature of the problem, many local minima exist. Early attempts in solving this problem have resorted to imposing additional power-spectral-density constraints [84] [85], and more recent work has focused on advance techniques such as simulated annealing [86]. Moreover, these existing approaches require the use of a centralized agent, and they can only be implemented if a single service provider controls the entire bundle. In an unbundled environment where loops within the same bundle may be operated by competing service providers, loop information is typically not shared among the service providers, and it is impractical for a centralized control agent to enforce spectral compatibility.

For the reasons stated above, the approach outlined in this thesis focuses on distributed power control algorithms that do not require centralized control. Each line is assumed to have the knowledge of its own channel transfer function and noise profile, and each DSL modem is allowed to optimize its performance locally. This locally optimized power control scheme leads to a game-theoretic characterization of the interference channel. The locally optimized power allocation is a Nash equilibrium in the game, and it has the intuitive appeal of being the operating point where all users have an incentive to move toward. The Nash equilibrium is computationally easy to characterize, and the power control algorithm offers the following advantages when compared to previous methods:

- The power control algorithm can be implemented distributively without the need of a centralized agent.

- Unlike previous methods that set a PSD level for each transmitter based solely on its interference emission level, the new power allocation method strikes a balance between maximizing each user's own data rate and minimizing its interference emission. In particular, it deals with the frequency-selective nature of the channel explicitly.

- The line transfer functions and cross-couplings are implicitly taken into account, and the new method offers the lines an opportunity to negotiate the best use of power and frequency with each other.

- The usual PSD constraint, which is in place for the purpose of controlling interference, is no longer needed. Only total power constraints apply.

- Unlike previous methods, which fix a data rate for each line regardless of actual service requirement, the new method naturally supports multiple service requirements in different lines.

- The proposed method does not involve arbitrary decisions on reference noise or reference length.

- Although not globally optimum, the proposed method performs much better than existing methods.

4.2 Competitive Optimality

The traditional information-theoretic view of an interference channel allows the transmitters, while sending independent data streams, to be cooperative in their respective coding strategies, so that multi-user detection may take place in the receivers. If such cooperation cannot be assumed, the interference channel can alternatively be modeled as a non-cooperative game. Under this viewpoint, each user competes for data rates with the sole objective of maximizing its *own* performance regardless of all other users. Since each modem has a fixed power budget, the data rate maximization is done by adjusting the power allocation over frequencies. If such power adjustment is done continuously by all users at the same time, it is natural to ask: Will the users eventually reach an equilibrium? Such an equilibrium is called a Nash equilibrium, and it is defined as a set of strategies in which each player's strategy is an optimal response to the other player's strategy [87]. The goal of this section is to characterize the Nash equilibrium in the Gaussian interference channel game and to determine conditions for its existence and uniqueness in realistic channels.

Consider a two-user interference channel:

$$\mathbf{y}_1 = \mathbf{x}_1 + \mathcal{A}_2 \mathbf{x}_2 + \mathbf{n}_1 \tag{4.4}$$

$$\mathbf{y}_2 = \mathbf{x}_2 + \mathcal{A}_1 \mathbf{x}_1 + \mathbf{n}_2, \tag{4.5}$$

where $\mathbf{x}_i$ and $\mathbf{y}_i$ are user i's input and output signals, $\mathcal{A}_1$ and $\mathcal{A}_2$ are the interference channels with their transfer functions denoted as $\alpha_1(f)$ and $\alpha_2(f)$ respectively, and $\mathbf{n}_1$ and $\mathbf{n}_2$ are additive noise with power spectral densities denoted as $N_1(f)$ and $N_2(f)$ respectively. The two senders are considered as two players in a game. The structure of the game, i.e., the interference coupling functions and the noise power spectra, are assumed to be common knowledge to both players. The strategies for the two players[2] are the transmit power spectra $P_1(f)$ and $P_2(f)$, subject to power constraints $\int_0^{F_s} P_1(f)df \leq \mathbf{P}_1$, and $\int_0^{F_s} P_2(f)df \leq \mathbf{P}_2$. The payoffs for the players are the respective data rates. Under the assumption that no interference subtraction is performed regardless of interference strength, the following rates are achievable:

$$R_1 = \int_0^{F_s} \log_2 \left(1 + \frac{P_1(f)}{N_1(f) + \alpha_2(f)P_2(f)}\right) df, \tag{4.6}$$

$$R_2 = \int_0^{F_s} \log_2 \left(1 + \frac{P_2(f)}{N_2(f) + \alpha_1(f)P_1(f)}\right) df. \tag{4.7}$$

Comparing the above expression with equation (4.2), it is easy to identify:

$$N_1(f) = \frac{\Gamma\sigma_1(f)}{|H_{11}(f)|^2} \tag{4.8}$$

$$\alpha_2(f) = \frac{\Gamma|H_{21}(f)|^2}{|H_{11}(f)|^2}, \tag{4.9}$$

and similarly for $N_2(f)$ and $\alpha_1(f)$. Thus, this simplified model incurs no loss of generality. The interference channel game considered here is not a zero-sum game, i.e. one player's loss is not equal to the other player's gain.

The main objective here is to characterize all pure-strategy Nash equilibria in an interference channel game. At a Nash equilibrium, each user's strategy is the optimal response to the other player's strategy. So, fixing $P_2(f)$, the optimal $P_1(f)$ must be the solution to the following optimization problem:

$$\begin{aligned} \text{maximize} \quad & \int_0^{F_s} \log_2 \left(1 + \frac{P_1(f)}{N_1(f) + \alpha_2(f)P_2(f)}\right) df, \\ \text{subject to} \quad & \int_0^{F_s} P_1(f)df \leq \mathbf{P_1}, \\ & P_1(f) \geq 0, \ \forall f. \end{aligned} \tag{4.10}$$

The solution to this problem is the well-known water-filling power allocation. More precisely,

[2]Only deterministic or pure strategies are considered here.

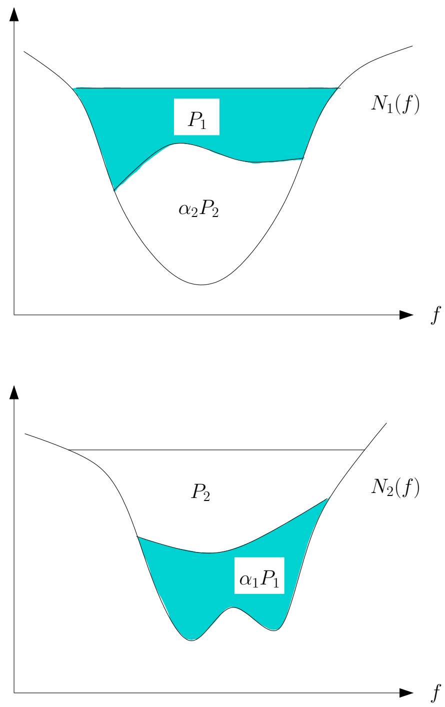


Figure 4.4: Simultaneous water-filling

let $\tilde{N}(f) = N_1(f) + \alpha_2(f)P_2(f)$. Then, the water-filling power allocation is:

$$P_1(f) = \begin{cases} 0, & \text{if } \tilde{N}(f) \geq L_1 \\ L_1 - \tilde{N}(f), & \text{if } \tilde{N}(f) \leq L_1 \end{cases} \tag{4.11}$$

where L_1 is a constant chosen so that the power constraint is met. Likewise, fixing $P_1(f)$, the optimal $P_2(f)$ should also be a water-filling power allocation against the combined interference from $P_1(f)$ and the noise. Thus, a Nash equilibrium is reached if and only if the water-filling condition is simultaneously satisfied for both users. The characterization of Nash equilibria is therefore equivalent to a characterization of "simultaneous water-filling" points. The idea of simultaneous water-filling is illustrated in Figure 4.4. The following theorem offers several sufficient conditions for the existence and uniqueness of the Nash equilibrium in the two-user case.

Theorem 4.1 *Suppose that $\alpha_1(f)\alpha_2(f) < 1$, $\forall f$, then the two-user Gaussian interference* *game has at least one pure strategy Nash equilibrium. Further, let* $\epsilon_0 = \sup\{\alpha_1(f)\}\sup\{\alpha_2(f)\}$, $\epsilon_1 = \sup\{\alpha_1(f)\alpha_2(f)\}$, $\epsilon_2 = \sup\{\alpha_1(f)\}\frac{1}{F_s}\int_0^{F_s}\alpha_2(f)df$, *and* $\epsilon_3 = \sup\{\alpha_2(f)\}\frac{1}{F_s}\int_0^{F_s}\alpha_1(f)df$. *If any of the following conditions,* $\epsilon_0 < 1$, $\epsilon_1 + \epsilon_2 < \frac{1}{2}$, *or* $\epsilon_1 + \epsilon_3 < \frac{1}{2}$, *is satisfied, then the Nash equilibrium is unique and is stable.*

The proof of Theorem 4.1 is lengthy and it is included in Appendix E. The basic idea is to approach a Nash equilibrium by successively letting each user optimize his power spectrum while regarding the interference from other users as noise. The main purpose of Theorem 4.1 is to characterize conditions under which such an iterative water-filling procedure converges. The following corollary is a direct consequence of the theorem.

Corollary 4.1 *If the condition for the existence and uniqueness of a Nash Equilibrium in Theorem 4.1 is satisfied, then the iterative water-filling algorithm for the two-user Gaussian interference game, where in every step, each user updates its power spectral density regarding all interference as noise, converges from any starting point, and it converges to the unique Nash equilibrium.*

The condition of Theorem 4.1 is not a mere technicality. The following simple example illustrates a case where Nash equilibrium is not unique. Consider a two-user case where there are only two frequencies of concern. Let $\alpha_1(f_1) = \alpha_1(f_2) = \alpha_2(f_1) = \alpha_2(f_2) = 2$. Let power constraints and background noise all be 1. The set of power allocations $P_1(f_1) = P_2(f_2) = 1$ and $P_1(f_2) = P_2(f_1) = 0$ is a Nash equilibrium. The set of power allocations $P_1(f_1) = P_2(f_2) = 0$ and $P_1(f_2) = P_2(f_1) = 1$ is a different Nash equilibrium.

4.3 Distributed Power Control

Because of the frequency-selective nature of the DSL channel, power control algorithms for DSL applications need to allocate power optimally not only among different users, but also in the frequency domain. This brings in many extra variables and makes the design of optimal power control for DSL challenging. However, if only competitively optimal power allocations are considered, the total power alone is sufficient for power control purposes. Assuming that the condition for Theorem 4.1 is satisfied, then a unique Nash equilibrium exists for each set of power constraints. Thus, it is possible to reach all possible competitive equilibria by adjusting only the total power constraints despite of the frequency selective

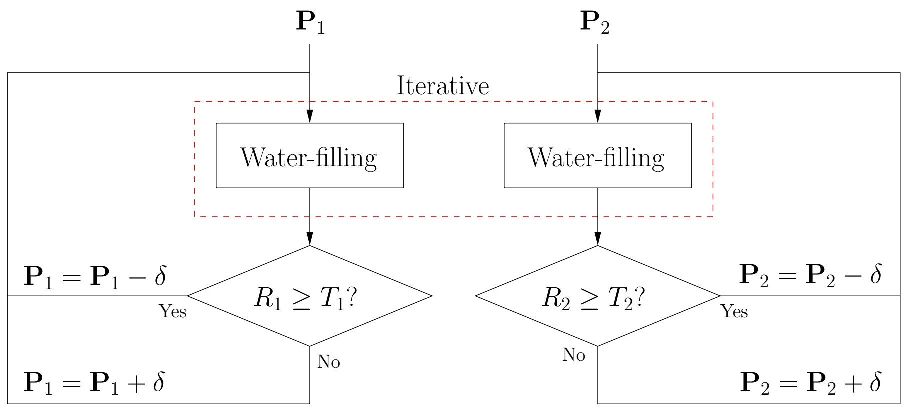


Figure 4.5: Distributed power control by iterative water-filling

nature of the channel. Although competitively optimal solutions are in general not globally optimal, it will be shown that they give significant improvements over current methods.

The following is a description of the proposed power control algorithm. A set of target rates is set for the users. The adaptive algorithm runs in two stages. The inner loop takes a set of power constraints for each user and derives the competitively optimal power allocation and its associated data rates using the iterative water-filling procedure. The outer loop finds the optimal total power constraint for each user. Each user's total power is adjusted based on the outcome of the iterative water-filling. If a user's data rate is below its target rate, its power is increased, (unless this exceeds the power constraint.) If a user's data rate is much above its target rate, its power is decreased. If the data rate is only slightly above the target rate, its power remains unchanged. The outer loop converges when the set of target rates is achieved. The algorithm is summarized in the following, and a simplified illustration is shown in Figure 4.5:

Algorithm 4.1 *Iterative water-filling for a Gaussian interference channel: Consider a K-user interference channel where each user has a power constraint $\mathbf{P}$. Let T_i be the target rate for user i.*

Initialize $\mathbf{P}_i = \mathbf{P}$, $P_i(f) = 0$, $i = 1, \dots K$.

repeat

repeat

for $i = 1$ *to* K,

$N(f) = \sum_{j=1, j\neq i}^{K} |H_{ji}(f)|^2 P_j(f) + \sigma_i(f)$;

Set $P_i(f)$ to be the water-filling spectrum with noise $N(f)$ and total power $\mathbf{P}_i$.

Set R_i to be the resulting data rate.

end

until the desired accuracy is reached.

for $i = 1$ to K,

If $R_i > T_i + \epsilon$, set $\mathbf{P}_i = \mathbf{P}_i - \delta$.

If $R_i < T_i$, set $\mathbf{P}_i = \mathbf{P}_i + \delta$.

If $\mathbf{P}_i > \mathbf{P}$, set $\mathbf{P}_i = \mathbf{P}$.

end

until $R_i > T_i$ for all i.

Although Theorem 4.1 only gives a sufficient condition for the convergence of iterative water-filling in the two-user case, it is observed in practice that iterative water-filling converges for DSL channels with more than two users also. The outer iteration is an ad-hoc method to find the appropriate power constraint for each user. Since data rates are monotonic functions of total power, the linear adjustment used in the above algorithm converges as long as the set of target rates is reasonable. The algorithm is found to work well with parameters $\delta = 3$dB and ϵ equal to roughly 10% of the target rate.

The outer loop of the power control algorithm essentially attempts to find the minimum amount of power that is needed to support the target data rate. In fact, the inner and outer loops can be combined. The usual water-filling maximizes the achievable data rate under a fixed power constraint. This is referred to as a "rate-adaptive" water-filling. On the other hand, a "power-adaptive" water-filling minimizes the total transmission power subject to a fixed rate constraint. The proposed algorithm can be alternatively thought of as each user doing "power-adaptive" water-filling against each other. Most ADSL modems deployed today already have the capability to do various types of water-filling. In fact, because the Nash equilibrium points are stable, the iterative procedure can be done asynchronously. Thus, the proposed power control algorithm is easy to implement in practice.

To implement the proposed power control algorithm distributively, each user must know its target data rate *a priori*. It is important for the target rates to be within the achievable rate region, as otherwise, some or all of the users would operate with negative margin.

Unfortunately, the set of achievable target rates cannot be determined distributively. Some centralized agent with the full knowledge of channel and interference transfer functions has to decide, by running through all possible total power constraints, which sets of target rates can be deployed in a DSL bundle. However, this happens in the loop planning stage and needs to be done only once. For situations where crosstalk coupling is not well known, conservative estimates of the rate region can instead be used.

Compared to conventional methods, the key advantage of this new power control algorithm is the following: the iterative water-filling algorithm offers an opportunity for different lines in a binder to negotiate the best use of frequency with each other. Thus, each line has an incentive to move away from frequency bands where interference is strong and to concentrate on frequency bands that it can most efficiently utilize. This method of controlling the interference removes the arbitrary power-spectral-density constraint, and it is able to bring a large overall improvement in system performance.

4.4 Performance

The performance of the distributed power control algorithm is examined in this section. The upstream power back-off problem for VDSL is used as a first example. Figure 4.6 shows the channel and crosstalk transfer functions for two modems located at 3000 feet and 1000 feet away from the central office. H_{ij} refers to the transfer function from user i to user j. The crosstalk transfer function is computed using the FEXT crosstalk models [80] where cross-coupling increases with frequency as f^2. The 26 gauge or 0.4mm lines are modeled here. Observe that at high frequencies, the crosstalk transfer function is actually larger than the direct channel. However, it is always true that:

$$\alpha_1(f)\alpha_2(f) = \frac{\Gamma|H_{12}(f)|^2}{|H_{22}(f)|^2} \cdot \frac{\Gamma|H_{21}(f)|^2}{|H_{11}(f)|^2} < 1, \tag{4.12}$$

where Γ is about 16dB for an uncoded QAM transmission with 6dB margin. Further, in the frequency range of interest (up to 12 MHz), all three sufficient conditions in Theorem 4.1 are satisfied. So, if the binder consists of two users only, the existence and uniqueness of Nash equilibrium would have been guaranteed.

A maximum transmit power of 11.5dBm is applied to each modem [80]. The usual power-spectral-density constraint is not applied, except below 1.1MHz where ADSL and

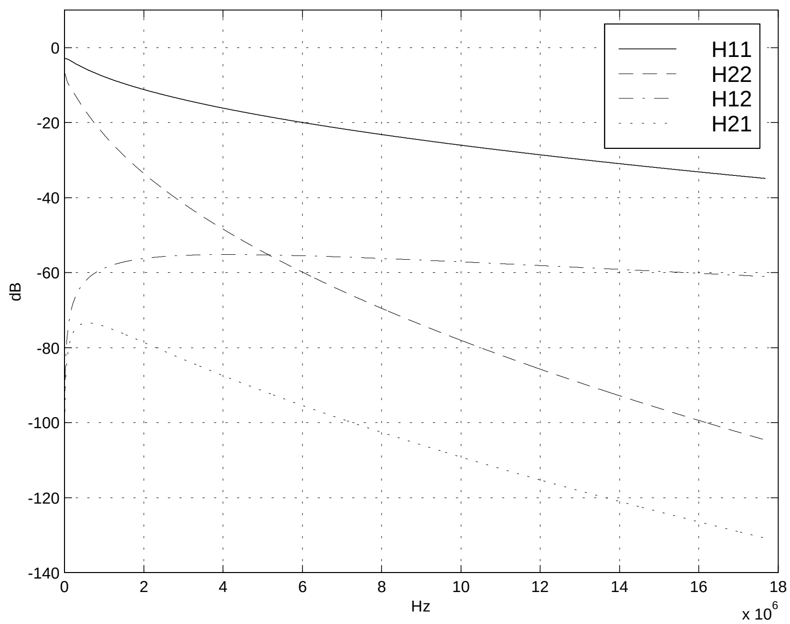


Figure 4.6: Channel and crosstalk transfer functions of 3000ft and 1000ft lines

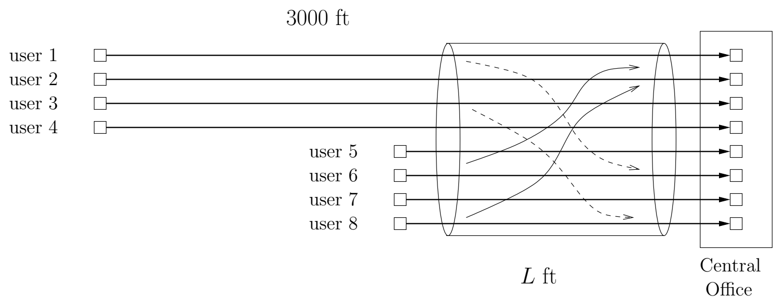


Figure 4.7: Loop topology for VDSL upstream power back-off

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other services need to be protected. A number of non-VDSL disturbers are also included. This includes 10 ADSL, 4 HDSL and 10 ISDN disturbers, comprising the so-called noise A model [88]. The loop topology is shown in Figure 4.7. It consists of 8 VDSL lines, 4 of which are at a distance of 3000 feet from the central office, and the other 4 are at a distance of L from the central office, with L varying from 500 feet to 2500 feet. The North American frequency plan (so-called Plan 998) [89] is used to separate upstream and downstream signals. The 998 plan uses the $3.75-5.2$MHz band, $8.5-12.0$MHz band and an optional $30-138$kHz band for upstream transmission. Frequency bands corresponding to the amateur radio frequencies [80] are notched off.

The iterative water-filling algorithm is applied to the 8-user scenario. Although Theorem 4.1 applies only to a 2-user case, the author has observed that iterative water-filling always converges in DSL channels regardless of the number of lines. Figure 4.8 shows the convergence behavior for a bundle with four 1000ft lines each with a power budget of -15.5dBm and four 3000ft lines each with a power budget of 11.5dBm. The top four lines in Figure 4.8 correspond to the four 1000ft lines. The lower four lines correspond to the four 3000ft lines. The iterative algorithm successively performs water-filling for each line while holding the power allocations of the other 7 lines fixed. As the figure shows, after the first water-filling, a 1000ft line achieves a data rate of 32Mbps as there is no interference at this point yet. The subsequent lines achieve smaller data rates due to the interference coming from lines that were previously water-filled. At the 9th water-fill, the first line is re-visited. It also drops its data rate in response to the interference from other lines. The algorithm converges after just two water-fillings for each user.

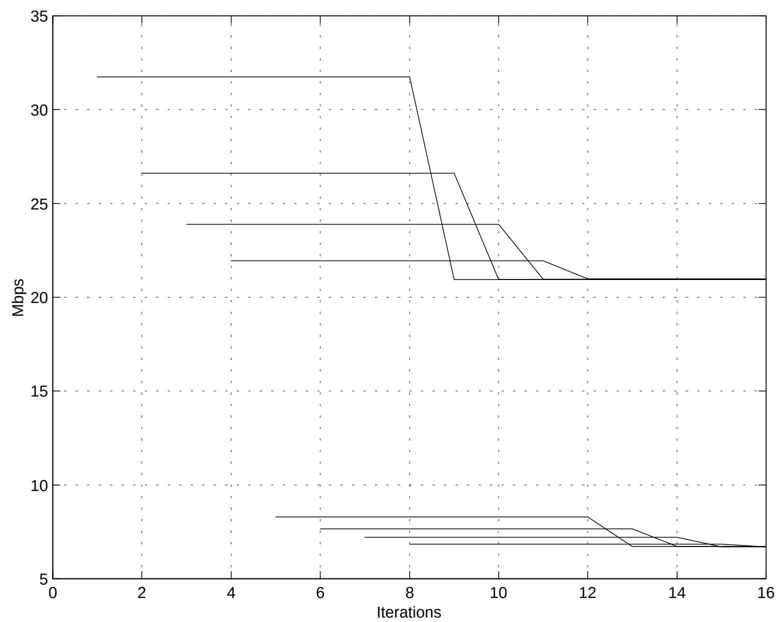

Figure 4.8: Convergence of iterative water-filling algorithm

Different rate-tuples are achievable with different power constraints. The set of all possible rate-tuples is the rate region. Figure 4.9 shows the upstream rate regions for the 8 VDSL lines. The data rates within each set of 4 users of the same length are the same, so the rate region can be plotted in a two-dimensional graph. Each curve in the figure corresponds to a different loop topology. The outer-most curve corresponds to the topology with 4 lines at 500ft and 4 lines at 3000ft. The next curve corresponds to the topology with 4 lines at 1000ft and 4 lines at 3000ft, and so on. The rate region illustrates the data rate tradeoff among the users. For example, with 4 lines at 500ft and the other 4 lines at 3000ft, it is possible to achieve 18Mbps in 500ft lines and 7.8Mbps in 3000ft lines, or 26Mbps on 500ft lines and 7Mbps on 3000ft lines, etc. This ability to provide many classes of different service levels is inherent in the proposed power control method.

The proposed power control algorithm compares favorably with existing power back-off methods. As an example, consider the reference noise method, where the reference noise is chosen to be equal to the FEXT caused by a 3000ft line. This choice of the reference noise forces all lines to emit the same amount of interference as a 3000ft line, regardless

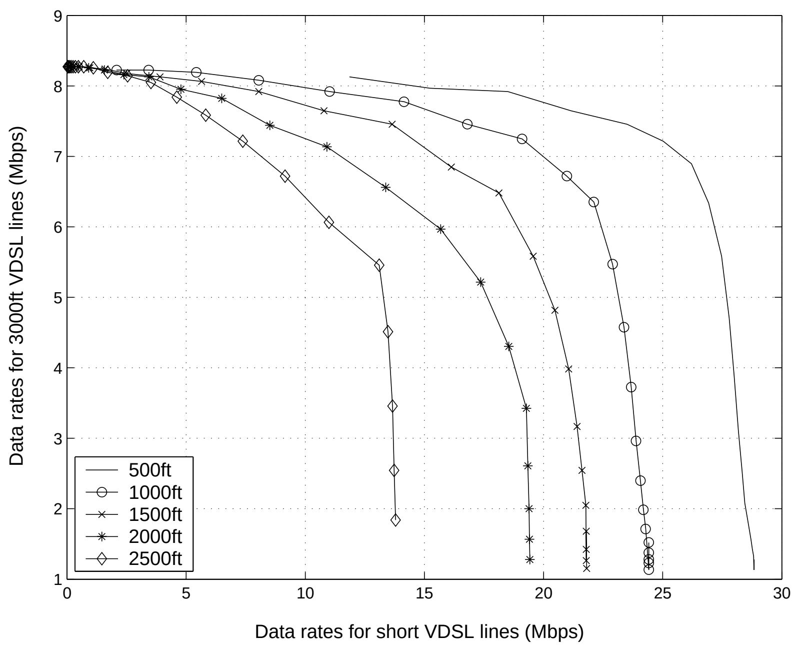

Figure 4.9: Competitively optimal upstream rate regions in VDSL

line length (ft)	reference noise (Mbps)	competitive optimum (Mbps)
500	12.5	26.5
1000	10.1	21.0
1500	8.9	16.5
2000	8.0	12.5
2500	7.3	9.0

Table 4.1: Reference noise power back-off vs. competitively optimal power control. The data rates for four 3000ft lines are fixed at 6.7Mbps. The data rates for the other four lines are shown.

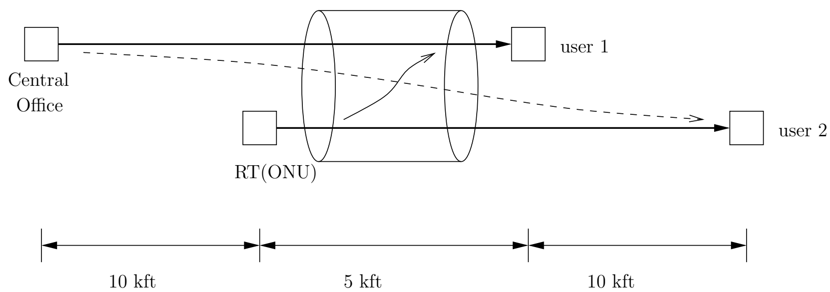

Figure 4.10: CO-based ADSL vs. RT-based ADSL

of their actual lengths. (This is also called the equalized-FEXT method.) Using this reference noise, the four 3000ft lines always achieve a data rate of 6.7Mbps each. Table 4.1 tabulates the performance of the other four lines. As the results show, the competitively optimal power allocation, although not globally optimal, nevertheless offers a significant improvement in performance over current static spectrum management methods. This improvement is possible because the iterative water-filling algorithm implicitly takes into account the interaction among the users.

As a second example, Figure 4.10 shows an ADSL scenario where downstream power control is necessary. This deployment configuration consists of a central office (CO) based ADSL line residing in the same binder as a remote terminal (RT) based ADSL line. This type of scenario is becoming increasingly common as service providers install optical network units (ONU) to be located close to customer homes in order to enlarge the service area. However, as downstream transmitters are now located in geographically different places, downstream power control needs to be implemented. This is true in the example illustrated, as the RT-based ADSL emits strong downstream interference into the CO-based ADSL. In fact, without power control, the CO-based ADSL does not function at all. Figure 4.11 illustrates the rate region for the two ADSL lines when the power control algorithm based on iterative water-filling is used. Again, a graceful tradeoff between the two lines is possible. In fact, a data rate of 1.4Mbps can be supported in both lines.

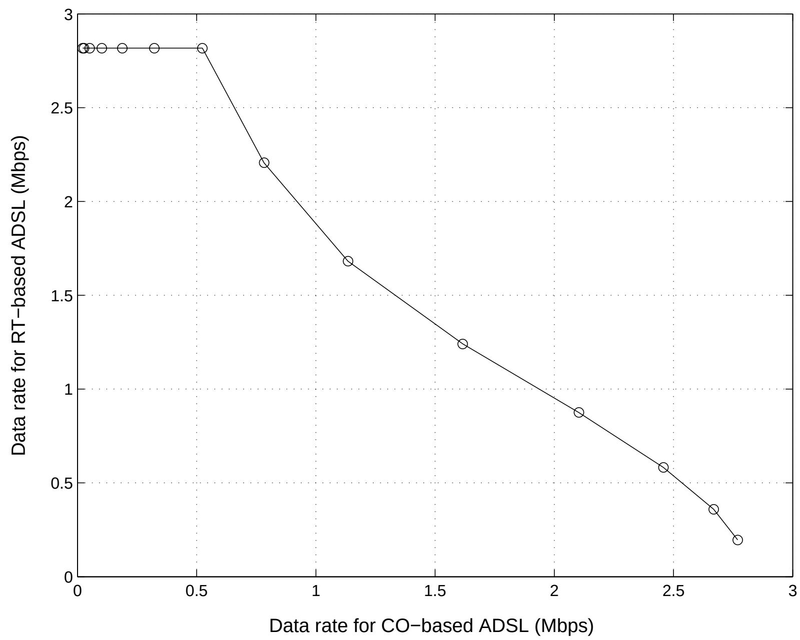

Figure 4.11: Competitively optimal downstream rate region of two ADSL lines

4.5 Summary

This chapter considers the power control problem in a frequency-selective multi-user interference channel. The interference channel is viewed as a non-cooperative game, and the Nash equilibrium of the game is characterized under a set of sufficient conditions. The Nash equilibrium corresponds to a set of competitively optimal power allocations, and it leads to a distributed power control algorithm based on iterative water-filling. The iterative water-filling algorithm implicitly takes into account the loop transfer functions and cross-couplings, and it allows the lines to negotiate the best use of power and frequency with each other. The new power control algorithm does not require centralized control, and it is implementable in today's existing modems. When applied to the VDSL upstream power back-off problem and the ADSL downstream spectral compatibility problem, the new method is shown to provide a significant performance improvement comparing to existing methods.

Chapter 5

Conclusion

This thesis illustrates the role of competition and the value of cooperation in a multi-user communication environment by treating three multi-user channels. First, it is shown that in a Gaussian vector multiple access channel, where transmitters do not cooperate but receivers cooperate, the spectrum optimization problem can be solved by an iterative water-filling procedure. The rate-sum optimal transmission strategy is one where each transmitter maximizes its own data rate while treating interference from other users as noise. Thus, a competitive optimum in a multiple access channel is also a global optimum. Second, in a non-degraded Gaussian vector broadcast channel, where transmitters cooperate but receivers do not cooperate, under a certain non-singularity condition, the sum capacity is shown to be the solution to a minimax problem. The sum capacity is achieved using a precoding strategy for channels with side information, and the optimum structure of the precoder is shown to correspond to a decision-feedback equalizer. Further, the sum capacity can be interpreted as a saddle-point of a mutual information game, where a signal player chooses a transmit covariance matrix to maximize the mutual information, and a fictitious noise player chooses a noise covariance matrix to minimize the mutual information. Thus, the sum capacity of a Gaussian vector broadcast channel is a competitive equilibrium. Third, it is shown that in a Gaussian interference channel, where neither transmitters nor receivers cooperate, under a certain set of conditions, a competitive equilibrium exists and is unique. Further, although a competitive equilibrium is not a global optimum, it nevertheless corresponds to a desirable operating point. This leads to a distributed dynamic power control algorithm for digital subscriber lines. In all three cases, a game-theory perspective has provided useful insights into the optimal transmission problems in multi-user channels.

Appendix A

Convex Optimization

A convex optimization problem is of the form:

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0 \quad i = 1, \dots, K,
\end{array}
\tag{A.1}
$$

where $x \in \mathbb{R}^n$ is the optimization variable, and $f_0, \cdots, f_K$ are convex functions. Associate a dual variable λ_i with each constraint $f_i(x) \leq 0$. The dual variable belongs to the dual space of the constraint space, and each dual variable defines a linear functional (or an inner product) from the constraint space to the real line. For example, when the constraint space is real, the dual variable is also real, and the inner product is just the usual product. When the constraint space is the set of positive semi-definite matrices, the dual space (more precisely, the dual cone) is the set of positive semi-definite matrices, and the inner product in this case is the trace of the matrix product. The dual variables always take on non-negative values.

The Lagrangian of an optimization problem is a linear combination of the objective and the inner product defined by the dual variables:

$$
L(x, \lambda) = f_0(x) + \sum_{i=1}^{K} \langle \lambda_i, f_i(x) \rangle, \tag{A.2}
$$

where $\langle \cdot, \cdot \rangle$ denotes an inner product. The dual objective is defined to be

$$
g(\lambda) = \inf_x L(x, \lambda). \tag{A.3}
$$

It is easy to see that $g(\lambda)$ is a lower bound on the optimal $f_0(x)$:

$$\begin{aligned}
f_0(x) &\geq f_0(x) + \sum_{i=1}^{K} \langle \lambda_i, f_i(x) \rangle && \text{(A.4)} \\
&\geq \inf_z \left(f_0(z) + \sum_{i=1}^{K} \langle \lambda_i, f_i(z) \rangle \right) && \text{(A.5)} \\
&\geq g(\lambda). && \text{(A.6)}
\end{aligned}$$

So,

$$\max_\lambda g(\lambda) \leq \min_x f_0(x), \tag{A.7}$$

where the minimization is over the original constraint set and is called the "primal" problem, and the maximization is over all non-negative λ_i's and is called the "dual" problem. The difference between the minimum of the primal objective $f_0(x)$ and the maximum of the dual objective $g(\lambda)$ is called the duality-gap. A central result in convex analysis [24] is that when $f_0, \cdots f_K$ are convex, under some technical conditions (called constraint qualifications) [56], the duality gap reduces to zero at the optimum, i.e. (A.7) is achieved with equality for some (x^*, λ^*). One version of constraint qualification is Slater's condition, which is satisfied when there exists x such that $f_i(x) < 0$, $i = 1, \cdots, K$ [24] [56].

Consider a convex optimization problem that satisfies constraint qualifications. Let x^* and λ^* be the primal and dual variables at the optimum. Since $f_0(x^*) = g(\lambda^*)$, the chain of inequalities (A.4)-(A.6) is satisfied with equality. Now, $\langle \lambda_i, f_i(x) \rangle \leq 0$, because $\lambda_i \geq 0$ and $f_i(x) \leq 0$. So, to have equality in (A.4), it must be that $\langle \lambda_i^*, f_i(x^*) \rangle = 0$. This is the so-called complementary slackness condition. Moreover, the inequality in (A.5) is also satisfied with equality. So, when the functions $f_0, \cdots f_K$'s are differentiable, at the infimum, $\nabla L(x^*, \lambda^*) = 0$. These two facts, together with the primal and dual constraints, form the Karush-Kuhn-Tucker (KKT) conditions, which are necessary and sufficient for optimality:

$$\begin{aligned}
f_i(x^*) &\leq 0 && \text{(A.8)} \\
\lambda_i^* &\geq 0 && \text{(A.9)} \\
\nabla f_0(x^*) + \sum_{i=1} \nabla \langle \lambda_i^*, f_i(x^*) \rangle &= 0 && \text{(A.10)} \\
\langle \lambda_i^*, f_i(x^*) \rangle &= 0. && \text{(A.11)}
\end{aligned}$$

Appendix B

Duality Gap in Water-filling

The proof of Theorem 2.5 is presented here.

Lemma B.1 *Let X and Y be positive semi-definite matrices. The followings are true:*

1. *if $X \geq Y$, then $\mathrm{tr}(X) \geq \mathrm{tr}(Y)$;*
2. $\mathrm{tr}(XY) \geq 0$*;*
3. *if $X \geq Y$, then $\max \mathrm{eig}(X) \geq \max \mathrm{eig}(Y)$.*

Proof: The trace of a matrix is the sum of eigenvalues. Eigenvalues of a positive semi-definite matrix are non-negative, so its trace is non-negative. If $X \geq Y$, then $X - Y \geq 0$. So, $\mathrm{tr}(X - Y) \geq 0$, and $\mathrm{tr}(X) \geq \mathrm{tr}(Y)$. Further, positive semi-definite matrices may be represented by their square roots: $X = AA^T$ and $Y = BB^T$. So, $\mathrm{tr}(XY) = \mathrm{tr}(AA^TBB^T) = \mathrm{tr}((B^TA)(A^TB)) \geq 0$. Lastly, if $X \geq Y$, then $v^TXv \geq v^TYv$ for all unit vectors v. So, $\max \mathrm{eig}(X) = \max v^TXv \geq \max v^TYv = \max \mathrm{eig}(Y)$, where the middle two maximizations are over all unit vectors v. □

Proof of Theorem 2.5: The idea is to use the fact that the dual objective is always a bound on the primal objective (c.f. equation (A.7)). Thus, the difference between the primal and dual objectives, the so-called "duality gap" is a upper bound on how far away the true optimum is from the present primal objective.

Start with $S_i = 0$. The first iteration of the algorithm consists of K water-fillings: S_1 is the single-user water-filling covariance with respect to noise Z alone, S_2 is the water-filling of noise plus interference from S_1, and so on. S_K is the water-filling of noise plus interference from all other users. For this set of primal feasible S_i, the difference between the primal problem (2.25) and the dual problem (2.30), which is denoted as γ, is:

$$\gamma = \operatorname{tr}\left[\left(\sum_{i=1}^{K} H_i S_i H_i^T + Z\right)^{-1} Z\right] + \sum_{i=1}^{K} \lambda_i P_i - m. \tag{B.1}$$

The bound holds for all dual feasible λ_i's. The bound is tightest when λ_i is chosen to be:

$$\lambda_i = \max \operatorname{eig}\left[H_i^T \left(\sum_{j=1}^{K} H_j S_j H_j^T + Z\right)^{-1} H_i\right], \quad i = 1, \cdots, K. \tag{B.2}$$

which is the smallest non-negative value that satisfies the dual constraints in (2.30).

Now, since S_1 is a single-user water-filling, its duality gap must be zero. Thus,

$$\operatorname{tr}[(H_1 S_1 H_1^T + Z)^{-1} Z] + \lambda_1' P_1 - m = 0, \tag{B.3}$$

where

$$\lambda_1' = \max \operatorname{eig}[H_1^T (H_1 S_1 H_1^T + Z)^{-1} H_1]. \tag{B.4}$$

More generally, S_i is a single-user water-filling regarding $\sum_{j=1}^{i-1} H_j S_j H_j^T + Z$ as noise. So,

$$\operatorname{tr}\left[\left(\sum_{j=1}^{i} H_j S_j H_j^T + Z\right)^{-1} \left(\sum_{j=1}^{i-1} H_j S_j H_j^T + Z\right)\right] + \lambda_i' P_i - m = 0, \tag{B.5}$$

$$\lambda_i' = \max \operatorname{eig}\left[H_i^T \left(\sum_{j=1}^{i} H_j S_j H_j^T + Z\right)^{-1} H_i\right]. \tag{B.6}$$

Lemma B.1 is now used to prove the following three facts. First

$$\operatorname{tr}\left(\sum_{j=1}^{K} H_j S_j H_j^T + I\right)^{-1} \leq \operatorname{tr}(H_1 S_1 H_1^T + I)^{-1} \tag{B.7}$$

This is a straightforward consequence of Lemma B.1(1). Secondly,

$$\lambda_i \leq \lambda_i'. \tag{B.8}$$

This follows from their definitions (B.2) and (B.6). Since

$$H_i^T \left(\sum_{j=1}^{K} H_j S_j H_j^T + Z \right)^{-1} H_i \leq H_i^T \left(\sum_{j=1}^{i} H_j S_j H_j^T + Z \right)^{-1} H_i, \tag{B.9}$$

$\lambda_i \leq \lambda_i'$ by Lemma B.1(3). Thirdly,

$$\lambda_i' P_i \leq m. \tag{B.10}$$

This follows from (B.5). The two matrices in the trace expression in (B.5) are both positive semi-definite, so Lemma B.1(2) implies that $\lambda_i' P_i \leq m$.

Now, putting everything together,

$$\begin{aligned}
\gamma &= \operatorname{tr}\left[\left(\sum_{i=1}^{K} H_i S_i H_i^T + Z \right)^{-1} Z \right] + \sum_{i=1}^{K} \lambda_i P_i - m && \text{(B.11)} \\
&\leq \operatorname{tr}\left[\left(\sum_{i=1}^{K} H_i S_i H_i^T + Z \right)^{-1} Z \right] + \sum_{i=1}^{K} \lambda_i' P_i - m && \text{(B.12)} \\
&= \operatorname{tr}\left[\left(\sum_{i=1}^{K} H_i S_i H_i^T + Z \right)^{-1} Z \right] + \lambda_1' P_1 - m + \sum_{i=2}^{K} \lambda_i' P_i && \text{(B.13)} \\
&\leq \sum_{i=2}^{K} \lambda_i' P_i && \text{(B.14)} \\
&\leq (K-1)m, && \text{(B.15)}
\end{aligned}$$

where the first inequality follows from (B.8), the second inequality follows from (B.7) and (B.3), and the last inequality follows from (B.10). Recall that a factor of $\frac{1}{2}$ was omitted in the original statement of the primal and dual problems: (2.25) and (2.30). Therefore the true duality gap is at most $(K-1)m/2$ nats. $\square$

Appendix C

Writing on Colored Paper

When an additive white Gaussian channel is corrupted by a Gaussian interfering signal whose entire sample sequence is known non-causally to the transmitter, but not to the receiver, Costa [44] showed that, surprisingly, the capacity of the channel is the same as if the interference were not present. Costa's result is derived under the assumption that both noise and interference are i.i.d. Gaussian processes. This result is known as "writing on dirty paper" because the transmitter can be thought of as encoding on top of known "dirt" on a piece of paper. This section extends Costa's result to dirt and noise sequences that are colored Gaussian vector random variables, hence the name "writing on colored paper".

Consider a Gaussian vector channel with side information as shown in Figure 3.4:

$$\mathbf{Y} = \mathbf{X} + \mathbf{S} + \mathbf{Z}, \tag{C.1}$$

where $\mathbf{S}$ and $\mathbf{Z}$ are independent Gaussian vectors with covariance matrices S_{ss} and S_{zz} respectively. The sequence $\mathbf{S}^n$ is known non-causally to the transmitter but not to the receiver, the sequence $\mathbf{Z}^n$ is known neither to the transmitter nor to the receiver. A power constraint P is imposed on the input vector $\mathbf{X}$, i.e. $\mathbf{E}[\mathbf{X}^T\mathbf{X}] \leq P$. A codeword $\mathbf{X}^n(W, \mathbf{S}^n)$ is an encoding function that maps a codeword index $W \in \{1, \cdots, 2^{nR}\}$ and a side information sequence $\mathbf{S}^n$ to a block of n transmissions. A decoding function $\hat{W}(\mathbf{Y}^n)$ maps the channel output to a codeword index. The probability of error is defined to be the average probability that $W \neq \hat{W}(\mathbf{Y}^n)$.

Gel'fand and Pinsker [42] showed that the capacity of a channel with non-causal side information can be characterized using an auxiliary random variable $\mathbf{U}$:

$$C = \max_{p(\mathbf{u},\mathbf{x}|\mathbf{s})} \{I(\mathbf{U};\mathbf{Y}) - I(\mathbf{U};\mathbf{S})\}. \tag{C.2}$$

The main result of this section is to identify the optimal choice of the auxiliary random variable for a Gaussian channel to be $\mathbf{U} = \mathbf{X} + F\mathbf{S}$, where $\mathbf{X}$ and $\mathbf{S}$ are independent Gaussian random vectors. This is analogous to the i.i.d. case where $U = X + \alpha S$, and $\alpha = P/(P+N)$ [44]. Curiously, the optimal F takes the form of an optimal non-causal Wiener filter. In the i.i.d. case, $\alpha = P/(P+N)$ can be interpreted as the optimal filter to estimate X from $X + Z$. In the vector case, the optimal F takes the form $F = S_{xx}(S_{xx} + S_{zz})^{-1}$, and it can be interpreted as the optimal filter to estimate $\mathbf{X}$ from $\mathbf{X} + \mathbf{Z}$. Further, neither optimal F nor capacity depends on the distribution of $\mathbf{S}$. In fact, the capacity of the channel is the same as if $\mathbf{S}$ does not exist. This result is stated as Lemma 3.1. A proof is presented as follows:

Proof of Lemma 3.1: Let $\mathbf{U} = \mathbf{X} + F\mathbf{S}$, where $\mathbf{X}$ and $\mathbf{S}$ are independent Gaussian vectors with covariance matrices S_{xx} and S_{ss}, respectively. Compute:

$$\begin{aligned} I(\mathbf{U};\mathbf{Y}) &= H(\mathbf{U}) + H(\mathbf{Y}) - H(\mathbf{U};\mathbf{Y}) \\ &= \frac{1}{2}\log \frac{|S_{xx} + FS_{ss}F^T| \cdot |S_{xx} + S_{ss} + S_{zz}|}{\begin{vmatrix} S_{xx} + FS_{ss}F^T & S_{xx} + FS_{ss} \\ S_{xx} + S_{ss}F^T & S_{xx} + S_{ss} + S_{zz} \end{vmatrix}}, \end{aligned} \tag{C.3}$$

where $H(\mathbf{U})$, $H(\mathbf{Y})$ and $H(\mathbf{U};\mathbf{Y})$ are computed using the relation $\mathbf{U} = \mathbf{X} + F\mathbf{S}$, where $\mathbf{X}$ and $\mathbf{S}$ are independent. Also, to compute $I(\mathbf{X};\mathbf{S})$, $\mathbf{S}$ can be viewed as the input and $\mathbf{U}$ as the output of a Gaussian channel with $\mathbf{X}$ as noise. So,

$$I(\mathbf{U};\mathbf{S}) = \frac{1}{2}\log \frac{|S_{xx} + FS_{ss}F^T|}{|S_{xx}|} \tag{C.4}$$

Define the function:

$$\begin{aligned} R(F) &= I(\mathbf{U};\mathbf{Y}) - I(\mathbf{U};\mathbf{S}) \\ &= \frac{1}{2}\log \frac{|S_{xx}| \cdot |S_{xx} + S_{ss} + S_{zz}|}{\begin{vmatrix} S_{xx} + FS_{ss}F^T & S_{xx} + FS_{ss} \\ S_{xx} + S_{ss}F^T & S_{xx} + S_{ss} + S_{zz} \end{vmatrix}}. \end{aligned} \tag{C.5}$$

The task is to maximize $R(F)$ over F. By expanding the denominator using Schur's complement formula for matrix determinant $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D| \cdot |A - BD^{-1}C|$:

$$R(F) = \frac{1}{2} \log \frac{|S_{xx}|}{|S_{xx} + FS_{ss}F^T - (S_{xx} + FS_{ss})(S_{xx} + S_{ss} + S_{zz})^{-1}(S_{xx} + S_{ss}F^T)|}. \tag{C.6}$$

So, to maximize $R(F)$ over F, it is only necessary to minimize the denominator in the above equation. The denominator is a quadratic function of F, so it can be minimized with the standard "complete-the-square" technique. Setting the denominator as $(Fa-b)(Fa-b)^T+c$, where a, b and c are $n \times n$ matrices, and comparing the coefficients:

$$\begin{aligned} aa^T &= S_{ss} - S_{ss}(S_{xx} + S_{ss} + S_{zz})^{-1}S_{ss}, && \text{(C.7)} \\ ba^T &= S_{xx}(S_{xx} + S_{ss} + S_{zz})^{-1}S_{ss}, && \text{(C.8)} \\ bb^T + c &= S_{xx} - S_{xx}(S_{xx} + S_{ss} + S_{zz})^{-1}S_{xx}. && \text{(C.9)} \end{aligned}$$

Then, the minimizing F is,

$$\begin{aligned} F &= ba^{-1} \\ &= ba^T(aa^T)^{-1} \\ &= S_{xx}(S_{xx} + S_{ss} + S_{zz})^{-1}S_{ss}(S_{ss} - S_{ss}(S_{xx} + S_{ss} + S_{zz})^{-1}S_{ss})^{-1} \\ &= S_{xx}(S_{xx} + S_{zz})^{-1}. && \text{(C.10)} \end{aligned}$$

The minimum value of the denominator is c. To find c, solve for bb^T:

$$\begin{aligned} bb^T &= Faa^TF^T \\ &= S_{xx}(S_{xx} + S_{zz})^{-1}(S_{ss} - S_{ss}(S_{xx} + S_{ss} + S_{zz})^{-1}S_{ss})(S_{xx} + S_{zz})^{-1}S_{xx} \\ &= S_{xx}(S_{xx} + S_{zz})^{-1}(S_{ss}^{-1} + (S_{xx} + S_{zz})^{-1})^{-1}(S_{xx} + S_{zz})^{-1}S_{xx} \\ &= S_{xx}(S_{ss}^{-1}(S_{xx} + S_{zz}) + I)^{-1}(S_{xx} + S_{zz})^{-1}S_{xx} \\ &= S_{xx}(S_{xx} + S_{zz} + S_{ss})^{-1}S_{ss}(S_{xx} + S_{zz})^{-1}S_{xx}, && \text{(C.11)} \end{aligned}$$

where the identity $(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$ is used. The minimum value of the denominator in (C.6) can then be found:

$$\begin{aligned} c &= S_{xx} - S_{xx}(S_{xx} + S_{ss} + S_{zz})^{-1} S_{xx} - bb^T \\ &= S_{xx} - S_{xx}(S_{xx} + S_{zz} + S_{ss})^{-1}(I + S_{ss}(S_{xx} + S_{zz})^{-1}) S_{xx} \\ &= S_{xx} - S_{xx}(S_{xx} + S_{zz})^{-1} S_{xx}. \end{aligned} \tag{C.12}$$

Thus,

$$\max_F R(F) = \frac{1}{2} \log \frac{|S_{xx}|}{|S_{xx} - S_{xx}(S_{xx} + S_{zz})^{-1} S_{xx}|}. \tag{C.13}$$

It remains to evaluate the above. Using the determinant formula for Schur's complement again, $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D| \cdot |A - BD^{-1}C| = |A| \cdot |D - CA^{-1}B|$:

$$|S_{xx} + S_{zz}| \cdot |S_{xx} - S_{xx}(S_{xx} + S_{zz})^{-1} S_{xx}| = |S_{xx}| \cdot |S_{zz}|. \tag{C.14}$$

Thus,

$$\max_F R(F) = \frac{1}{2} \log \frac{|S_{xx} + S_{zz}|}{|S_{zz}|}. \tag{C.15}$$

This is the mutual information formula for the vector Gaussian channel without the interfering $\mathbf{S}$. Thus $\max_F I(\mathbf{U};\mathbf{Y}) - I(\mathbf{U};\mathbf{S}) = I(\mathbf{X};\mathbf{Y}|\mathbf{S})$. The original assumption that $(\mathbf{U},\mathbf{S})$ takes the form $\mathbf{U} = \mathbf{X} + F\mathbf{S}$, where $\mathbf{X}$ and $\mathbf{S}$ are independent Gaussian is without loss of generality. The optimal $F = S_{xx}(S_{xx} + S_{zz})^{-1}$. The capacity achieving S_{uu} is equal to $S_{xx} + S_{xx}(S_{xx} + S_{zz})^{-1} S_{ss} (S_{xx} + S_{zz})^{-1} S_{xx}$. $\square$

Appendix D

Broadcast Channel Example

A numerical example for the Gaussian vector broadcast channel is presented. Consider the following broadcast channel:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1.0 & -0.3 & 0.2 \\ -0.4 & 2.0 & 0.5 \\ -0.1 & 0.2 & 3.0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ z_2 \end{bmatrix}, \tag{D.1}$$

where y_1, y_2, and y_3 are uncoordinated receivers, and z_1, z_2, and z_3 are i.i.d. Gaussian noises with variance 1. The total power constraint is set to 5. The iterative algorithm described at the end of the section 3.3 is used to solve for the saddle point (S_{xx}, S_{zz}). The water-filling step is standard. The least favorable noise problem is solved using an interior-point method. The algorithm converged in 3-4 steps. The numerical solution is:

$$S_{xx} = \begin{bmatrix} 1.0762 & -0.2327 & -0.0074 \\ -0.2327 & 1.8635 & 0.0387 \\ -0.0074 & 0.0387 & 2.0603 \end{bmatrix}, \quad S_{zz} = \begin{bmatrix} 1.0000 & -0.1286 & 0.0493 \\ -0.1286 & 1.0000 & 0.0311 \\ 0.0493 & 0.0311 & 1.0000 \end{bmatrix}. \tag{D.2}$$

To verify that the above solution satisfies the KKT conditions:

$$S_{zz}^{-1} - \left(S_{zz} + H S_{xx} H^T\right)^{-1} = \Psi = \begin{bmatrix} 0.4859 & 0 & 0 \\ 0 & 0.8701 & 0 \\ 0 & 0 & 0.9422 \end{bmatrix} \tag{D.3}$$

and

$$H^T \left(HS_{xx}H^T + S_{zz}\right)^{-1} H = \begin{bmatrix} 0.4597 & 0 & 0 \\ 0 & 0.4597 & 0 \\ 0 & 0 & 0.4597 \end{bmatrix}. \tag{D.4}$$

The vector channel capacity with the least favorable noise correlation is:

$$C = \frac{1}{2}\log\frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|} = 2.8952. \tag{D.5}$$

The objective is to design a generalized decision-feedback precoder that achieves the vector channel capacity without receiver coordination. This is accomplished by finding an appropriate transmit filter $B = V\Sigma^{\frac{1}{2}}M$ which would induce a diagonal feedforward filter in a GDFE. Following the proof of Lemma 3.3, compute the eigen-decomposition $S_{xx} = V\Sigma V^T$ and $S_{zz} = Q^T\Lambda Q$. Then, compute R as a square root of the following as in (3.78):

$$R^T R = \left(\sqrt{\Sigma}V^T H^T Q^T \Lambda^{-1} Q H V \sqrt{\Sigma} + I\right)^{-1}. \tag{D.6}$$

In particular, R can be found by a Cholesky factorization. In this example, because S_{xx} is the water-filling covariance, the matrix V diagonalizes the channel, so that $R^T R$ is already diagonal. So, finding an R is trivial. Numerically,

$$R = \begin{bmatrix} 0.2191 & 0 & 0 \\ 0 & 0.3451 & 0 \\ 0 & 0 & 0.7312 \end{bmatrix}. \tag{D.7}$$

The next step is to find an orthogonal matrix U, such that $UR\sqrt{\Sigma}V^T H^T Q^T \Lambda^{-1} Q$ is diagonal. The proof of Lemma 3.3 shows that U can be found as follows:

$$U = \Psi^{-\frac{1}{2}} Q^T \Lambda^{-1} Q H V \sqrt{\Sigma} R^T = \begin{bmatrix} 0.0115 & -0.2220 & 0.9750 \\ 0.3147 & 0.9263 & 0.2072 \\ 0.9491 & -0.3045 & -0.0805 \end{bmatrix}. \tag{D.8}$$

The final step is to find an orthogonal matrix M such that URM is lower-triangular. This is done by computing the QR-factorization of $R^T U^T = MT$, where T is upper-triangular, and M is orthogonal. Then, $URM = T^T$ is lower-triangular. In this example,

$$R^T U^T = MT = \begin{bmatrix} -0.0035 & -0.2010 & -0.9796 \\ 0.1069 & -0.9740 & 0.1995 \\ -0.9943 & -0.1040 & 0.0249 \end{bmatrix} \begin{bmatrix} -0.7170 & -0.1167 & 0.0466 \\ 0 & -0.3410 & 0.0666 \\ 0 & 0 & -0.2262 \end{bmatrix}. \tag{D.9}$$

This gives the appropriate M for the desired transmit filter $B = V\Sigma^{\frac{1}{2}}M$.

Now, design a generalized decision-feedback equalizer for the effective channel

$$\widetilde{H} = \frac{1}{\sqrt{\Lambda}} QHV\sqrt{\Sigma}M = \begin{bmatrix} -0.7439 & 2.2489 & 0.0128 \\ 0.1698 & 0.8505 & 4.3105 \\ 0.6027 & 1.4311 & -0.8596 \end{bmatrix}, \tag{D.10}$$

an input covariance $S_{uu} = I$, and a noise covariance $S_{zz} = I$. Compute $G^{-1}\Delta^{-1}G^{-T} = (\widetilde{H}^T\widetilde{H} + I)^{-1}$:

$$G = \begin{bmatrix} 1 & -0.3423 & 0.1051 \\ 0 & 1 & 0.2947 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 1.9454 & 0 & 0 \\ 0 & 8.6009 & 0 \\ 0 & 0 & 19.5512 \end{bmatrix}. \tag{D.11}$$

As expected, the choice of transmit filter makes the feedforward filter a diagonal matrix:

$$F = \Delta^{-1}G^{-T}\widetilde{H}^T \frac{1}{\sqrt{\Lambda}} Q = \begin{bmatrix} -0.4998 & 0 & 0 \\ 0 & -0.3181 & 0 \\ 0 & 0 & -0.2195 \end{bmatrix}. \tag{D.12}$$

First, let's compute the capacities of individual sub-channels in the GDFE feedback configuration. The effective channel is $\mathbf{u}' = FHB\mathbf{u} + (I - G)\mathbf{u} + F\mathbf{z}$:

$$\begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \end{bmatrix} = \begin{bmatrix} 0.4860 & 0 & 0 \\ -0.0398 & 0.8837 & 0 \\ -0.0105 & 0.0151 & 0.9489 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} -0.4998 z_3 \\ -0.3181 z_2 \\ -0.2195 z_1 \end{bmatrix}. \tag{D.13}$$

Thus, the capacities of the three sub-channels are:

$$R_1 = \frac{1}{2}\log\left(1+\frac{0.4860^2}{0.4998^2}\right) = 0.3327 \tag{D.14}$$

$$R_2 = \frac{1}{2}\log\left(1+\frac{0.8837^2}{0.3181^2+0.0398^2}\right) = 1.0759 \tag{D.15}$$

$$R_3 = \frac{1}{2}\log\left(1+\frac{0.9489^2}{0.0105^2+0.0151^2+0.133^2}\right) = 1.4865. \tag{D.16}$$

The sum capacity is $R_1+R_2+R_3 = 2.8952$, which agrees with the vector channel capacity.

Now, compute the capacity of individual sub-channels in the precoding configuration. The effective channel is $\mathbf{y} = HB\mathbf{u}+\mathbf{z}$:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -0.9723 & 0.6847 & -0.2101 \\ 0.1251 & -2.7785 & -0.9265 \\ -0.0480 & -0.0687 & -4.3222 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} z_3 \\ z_2 \\ z_1 \end{bmatrix}. \tag{D.17}$$

Decoding u_3 from y_3, the capacity is:

$$R_3 = \frac{1}{2}\log\left(1+\frac{4.3222^2}{1+0.0480^2+0.0687^2}\right) = 1.4865. \tag{D.18}$$

The signal from u_3 may be pre-subtracted from u_2, leading to:

$$R_2 = \frac{1}{2}\log\left(1+\frac{2.7785^2}{1+0.1251^2}\right) = 1.0759. \tag{D.19}$$

The signals from u_2 and u_3 may be pre-subtracted from u_1, leading to:

$$R_1 = \frac{1}{2}\log(1+0.9723^2) = 0.3327. \tag{D.20}$$

Therefore, without receiver coordination, a sum capacity of $R_1 + R_2 + R_3 = 2.8952$ is also achievable. In fact, it is now possible to identify the appropriate transmit covariance matrices for each user as in Theorem 3.1. Let B_1, B_2 and B_3 be the column vectors of the transmit filter $B = [B_1 B_2 B_3]$. Then information bits u_1, u_2 and u_3 are modulated with covariance matrices $S_1 = B_1 B_1^T$, $S_2 = B_2 B_2^T$ and $S_3 = B_3 B_3^T$. Let H_1, H_2 and H_3 be the row vectors of the channel $H^T = [H_1^T H_2^T H_3^T]$. Then, by Theorem 3.1, the following rates are achievable:

$$R_1 = \frac{1}{2}\log\left(H_1S_1H_1^T+1\right) = 0.3327 \tag{D.21}$$

$$R_2 = \frac{1}{2}\log\left(\frac{H_2S_2H_2^T+H_2S_1H_2^T+1}{H_2S_1H_2^T+1}\right) = 1.0759 \tag{D.22}$$

$$R_3 = \frac{1}{2}\log\left(\frac{H_3S_3H_3^T+H_3S_2H_3^T+H_3S_1H_3^T+1}{H_3S_2H_3^T+H_3S_1H_3^T+1}\right) = 1.4865. \tag{D.23}$$

This again verifies that $R_1 + R_2 + R_3 = 2.8952$ is achievable with no coordination at the receiver side.

Appendix E

Convergence of Iterative Water-filling

The proof for the convergence of iterative water-filling for a two-user Gaussian interference channel is presented here.

Proof of Theorem 4.1: The Nash equilibrium points correspond to power allocations where each user's power spectrum is a water-filling against the combined interference and noise. Call the water-filling level at the Nash equilibrium (L_1, L_2). The first idea in proving the existence of a Nash equilibrium under a power constraint $(\mathbf{P_1}, \mathbf{P_2})$ is to establish the existence of a Nash equilibrium under a fixed water level. Assume $\alpha_1(f)\alpha_2(f) < 1 \ \forall f$, and fix (L_1, L_2). The Nash equilibrium power allocation $(P_1(f), P_2(f))$ can be found by simultaneously solving the water-filling condition at each frequency f: When $P_1(f)$ (or $P_2(f)$) is zero, the combined interference and noise must be greater than or equal to L_1 (or L_2). When $P_1(f)$ and $P_2(f)$ are positive, the following must be true:

$$P_1(f) + \alpha_2(f)P_2(f) + N_1(f) = L_1, \tag{E.1}$$

$$P_2(f) + \alpha_1(f)P_1(f) + N_2(f) = L_2. \tag{E.2}$$

Now, if either $L_1 < N_1(f)$, or $L_2 < N_2(f)$, then trivially $P_1(f) = 0$ or $P_2(f) = 0$ satisfies the water-filling condition. So, without loss of generality, assume that $L_1 > N_1(f)$ and $L_2 > N_2(f)$. If $\alpha_1(f) > \frac{L_2 - N_2(f)}{L_1 - N_1(f)}$, setting $P_1(f) = 0$ and $P_2(f) = L_2 - N_2(f)$ satisfies the condition. Also, if $\alpha_2(f) > \frac{L_1 - N_1(f)}{L_2 - N_2(f)}$, setting $P_2(f) = 0$ and $P_1(f) = L_1 - N_1(f)$ satisfies the condition. The above two conditions on α_1 and α_2 cannot be both true at the same time, because $\alpha_1(f)\alpha_2(f) < 1$. If neither is true, then equations (E.1) and (E.2) have a positive solution:

$$
\begin{aligned}
P_1(f) &= \frac{(L_2 - N_2(f)) - \alpha_1(L_1 - N_1(f))}{1 - \alpha_1\alpha_2} \\
P_2(f) &= \frac{(L_1 - N_1(f)) - \alpha_2(L_2 - N_2(f))}{1 - \alpha_1\alpha_2}
\end{aligned}
\tag{E.3}
$$

Thus, under all cases, the simultaneous water-filling solution exists.

Next, it is established that for a given power constraint $(\mathbf{P_1}, \mathbf{P_2})$, there exists (L_1, L_2) whose Nash equilibrium has exactly this power. For each (L_1, L_2), denote the total power level at the corresponding Nash equilibrium as $(\mathbf{P}_{L_1}, \mathbf{P}_{L_2})$. Observe that when $\alpha_1(f)\alpha_2(f) < 1$, if $L_1 < L_1'$ and $L_2 = L_2'$, then $\mathbf{P}_{L_1} \le \mathbf{P}_{L_1'}$ and $\mathbf{P}_{L_2} \ge \mathbf{P}_{L_2'}$. This can be verified by working through the simultaneous water-filling condition. Now, start with $L_1 = L_2 = 0$. Increase L_1 until $\mathbf{P}_{L_1} = \mathbf{P_1}$, then increase L_2 until $\mathbf{P}_{L_2} = \mathbf{P_2}$. Then, by the previous observation, it must be that $\mathbf{P}_{L_1} \le \mathbf{P_1}$. So, L_1 can be increased again, until $\mathbf{P}_{L_1} = \mathbf{P_1}$, then L_2 increased, and so on. The increasing sequences of L_1 and L_2 cannot go to infinity with finite power constraints, so they must converge. The limit point is a Nash equilibrium corresponding to $(\mathbf{P_1}, \mathbf{P_2})$. This establishes the existence of a Nash equilibrium under a power constraint.

To prove uniqueness, let $(P_1^N(f), P_2^N(f))$ be the power distribution at a Nash equilibrium, whose existence is already established. Start with any power distribution $P_1^{(0)}(f)$ that satisfies the power constraint. Water-fill for $P_2^{(0)}(f)$, assuming $P_1^{(0)}(f)$ as interference. Then, water-fill for $P_1^{(1)}(f)$, assuming $P_2^{(0)}(f)$ as interference. Continue iteratively to obtain $P_2^{(1)}(f) \to P_1^{(2)}(f) \to P_2^{(2)}(f) \to \cdots$. It is shown next that this iterative water-filling process converges in $\mathbf{L}_1$-norm, $||P_1^{(k)}(f) - P_1^N(f)||_1 = \frac{1}{F_s}\int_0^{F_s} |P_1^{(k)}(f) - P_1^N(f)| df$. Define $(\cdot)^+ = \max(0, \cdot)$, and $(\cdot)^- = -\min(0, \cdot)$. Then,

$$
\begin{aligned}
&\max\left\{\int_0^{F_s}\left(P_1^{(k+1)}(f) - P_1^N(f)\right)^+ df, \int_0^{F_s}\left(P_1^{(k+1)}(f) - P_1^N(f)\right)^- df\right\} \\
&\le \sup\alpha_2(f)\max\left\{\int_0^{F_s}\left(P_2^{(k)}(f) - P_2^N(f)\right)^+ df, \int_0^{F_s}\left(P_2^{(k)}(f) - P_2^N(f)\right)^- df\right\} \\
&\le \sup\alpha_2(f)\sup\alpha_1(f)\max\left\{\int_0^{F_s}\left(P_1^{(k)}(f) - P_1^N(f)\right)^+ df, \int_0^{F_s}\left(P_1^{(k)}(f) - P_1^N(f)\right)^- df\right\}.
\end{aligned}
$$

Thus, if $\sup\alpha_1(f)\sup\alpha_2(f) = \epsilon_0 < 1$, $P_1^{(k)}(f) \to P_1^N(f)$ in $\mathbf{L}_1$-norm as $k \to \infty$.

The above condition may be too restrictive in certain cases. To derive the second and third sufficient conditions, let $\Delta^{(k)}P_1(f) = P_1^{(k)}(f) - P_1^N(f)$. Comparing to the interference emitted by $P_1^N(f)$, the power distribution $P_1^{(k)}(f)$ causes a difference in the interference level that is equal to $\alpha_1(f)\Delta^{(k)}P_1(f)$. This difference in interference would cause a difference in $P_2(f)$ by at most $\alpha_1(f)\Delta^{(k)}P_1(f) - \frac{1}{F_s}\int_0^{F_s}\alpha_1(f)\Delta^{(k)}P_1(f)df$. (The mean is subtracted here, because the water-filling process is insensitive to the absolute interference level change and is only affected by the relative interference level change.) This difference in $P_2(f)$ would in turn cause an interference level difference in $P_1(f)$ by: $\alpha_2(f)\alpha_1(f)\Delta^{(k)}P_1(f) - \alpha_2(f)\frac{1}{F_s}\int_0^{F_s}\alpha_1(f)\Delta^{(k)}P_1(f)df$. Now, this difference in interference would cause a difference in $P_1(f)$ by at most an amount:

$$\begin{aligned}\Delta^{(k+1)}P_1(f) \quad \leq \quad & \alpha_2(f)\alpha_1(f)\Delta^{(k)}P_1(f) - \alpha_2(f)\frac{1}{F_s}\int_0^{F_s}\alpha_1(f)\Delta^{(k)}P_1(f)df - \\ & \frac{1}{F_s}\int_0^{F_s}\alpha_2(f)\alpha_1(f)\Delta^{(k)}P_1(f)df - \frac{1}{F_s}\int_0^{F_s}\alpha_2(f)df\frac{1}{F_s}\int_0^{F_s}\alpha_1(f)\Delta^{(k)}P_1(f)df\end{aligned}$$

The $\mathbf{L}_1$-norm of $\Delta^{(k+1)}P_1(f)$ above can be bounded by the triangular inequality as shown below:

$$\begin{aligned}\frac{1}{F_s}\int_0^{F_s}|\Delta^{(k+1)}P_1(f)|df \quad \leq \quad & \frac{1}{F_s}\int_0^{F_s}|\alpha_2(f)\alpha_1(f)\Delta^{(k)}P_1(f)|df + \\ & \frac{1}{F_s}\int_0^{F_s}\alpha_2(f)df\frac{1}{F_s}\int_0^{F_s}|\alpha_1(f)\Delta^{(k)}P_1(f)|df + \\ & \frac{1}{F_s}\int_0^{F_s}|\alpha_2(f)\alpha_1(f)\Delta^{(k)}P_1(f)|df + \\ & \frac{1}{F_s}\int_0^{F_s}\alpha_2(f)df\frac{1}{F_s}\int_0^{F_s}|\alpha_1(f)\Delta^{(k)}P_1(f)|df \\ \leq \quad & \sup\{\alpha_2(f)\alpha_1(f)\}\frac{1}{F_s}\int_0^{F_s}|\Delta^{(k)}P_1(f)|df + \\ & \sup\{\alpha_1(f)\}\frac{1}{F_s}\int_0^{F_s}\alpha_2(f)df\frac{1}{F_s}\int_0^{F_s}|\Delta^{(k)}P_1(f)|df + \\ & \sup\{\alpha_2(f)\alpha_1(f)\}\frac{1}{F_s}\int_0^{F_s}|\Delta^{(k)}P_1(f)|df + \\ & \sup\{\alpha_1(f)\}\frac{1}{F_s}\int_0^{F_s}\alpha_2(f)df\frac{1}{F_s}\int_0^{F_s}|\Delta^{(k)}P_1(f)|df.\end{aligned}$$

Thus, if

$$\sup\{\alpha_2(f)\alpha_1(f)\} + \sup\{\alpha_1(f)\}\frac{1}{F_s}\int_0^{F_s} \alpha_2(f)df = \epsilon_1 + \epsilon_3 < \frac{1}{2}, \tag{E.4}$$

the iterative water-filling algorithm is a contraction, and $P_1^{(k)}(f) \to P_1^N(f)$ in $\mathbf{L}_1$-norm as $k \to \infty$. The same analysis can be applied to $P_2(f)$, which yields the third condition.

The convergence of the iterative water-filling process implies that the Nash equilibrium is unique. This is because the iterative water-filling process converges to the same Nash equilibrium from any starting point. But each Nash equilibrium is its own fixed point. So, there could not have been more than one Nash equilibria. The stability of the Nash equilibrium also follows from the convergence of the iterative procedure. $\square$

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