Unstructured Linear Beamforming Design for Interference Alignment in MIMO Cellular Networks

Gokul Sridharan and Wei Yu
The Edward S. Rogers Sr. Department of Electrical and Computer Engineering
University of Toronto, Toronto, ON, M5S 3G4, Canada
Email:gsridharan@comm.utoronto.ca, weiyu@comm.utoronto.ca

***Abstract*—This paper proposes a linear beamforming strategy for interference alignment in multiple-input multiple-output (MIMO) cellular networks. In particular, we consider a network consisting of G mutually interfering cells with K users/cell, having N antennas at each base station (BS) and M antennas at each user — a (G, K, M, N) network. We develop an unstructured approach to designing linear beamformers for interference alignment where transmit beamformers are designed to satisfy conditions for interference alignment without explicitly identifying the underlying structures for alignment. Specifically, the transmit beamformers in the uplink are required to satisfy a certain number of random linear vector equations in order to constrain the number of dimensions occupied by interference at each BS. The conceptual simplicity and the fact that no customization to a given network is needed makes this method applicable to a broad class of cellular networks. The key observation made in this paper is that such an approach appears to be capable of achieving the optimal DoF for MIMO cellular networks in regimes where linear beamforming dominates asymptotic decomposition-based schemes for interference alignment, and a significant portion of the DoF elsewhere. Remarkably, polynomial identity test plays a key role in identifying the scope and limitations of such a technique.**

I. Introduction

Degrees of freedom (DoF) has emerged as a useful metric in characterizing the capacity of multi-cell multi-antenna networks. In this work we focus on the symmetric DoF of multiple-input multiple-output (MIMO) cellular networks. In particular, we consider G mutually interfering cells with K users/cell having N antennas at each base station (BS) and M antennas at each user—denoted in this paper as a (G, K, M, N) network.

Linear beamforming, first analyzed in the context of interference alignment (IA) for the 2×2 X network [1], [2] and the asymptotic scheme for IA over multiple symbol extensions, first developed for the K-user interference channel [3], have emerged as the leading techniques for establishing the optimal DoF of various networks. In this work, we develop a linear beamforming strategy for IA in MIMO cellular networks and study the symmetric DoF achieved using such a scheme. Linear beamforming techniques play a crucial role in establishing the optimal DoF of MIMO networks without requiring the multi-antenna nodes to be decomposed into single-antenna nodes [4], [5]. Although we focus on symmetric DoF in this paper, this strategy can be applied even when the DoF requirements are asymmetric.

Designing linear beamforming strategies that achieve the optimal DoF of MIMO cellular networks is challenging because multiple subspaces can interact and overlap in complicated ways. So far, identifying the underlying structure of IA for each given network (e.g. subspace alignment chains for the three-user MIMO interference channel [4]) has been a prerequisite for developing counting arguments that lead to DoF-optimal linear beamforming strategies.

The need to identify structures for IA can be circumvented by treating the conditions for IA as a system of polynomial equations and studying the feasibility of such a system through techniques in algebraic geometry. While certain sufficient conditions established through such an approach ensure feasibility [6]–[10], such an approach does not provide any insight on designing linear beamformers for IA.

In contrast to structured linear beamformer design for IA and techniques based on algebraic geometry to establish feasibility, this paper proposes a structure-agnostic approach to design linear beamformers for IA while also establishing feasibility. In such an approach, depending on the DoF demand placed on a given MIMO cellular network, we design transmit beamformers in the uplink by solving a requisite number of random linear vector equations that limit the total number of dimensions occupied by interference at each BS.

The crucial element in such an approach is the fact that we construct linear vector equations with random coefficients. This is a significant departure from structured approaches where linear equations that identify the alignment conditions emerge from notions such as subspace alignment chains or packing ratios and are predefined with deterministic coefficients. The flexibility in choosing random coefficients allows us to use this technique for IA in networks of any size, without having to explicitly infer the underlying structure. Such an approach is also discussed in [11]–[13], where it is mostly used to design aligned transmit beamformers when one DoF/user is desired. In this paper, we significantly expand the scope of such an approach by proposing the use of a polynomial identity test to resolve certain linear independence conditions that need to be satisfied when more than one DoF/user are desired. Although multi-stream transmission has also been considered in [13], the need for the polynomial identity test has not been previously stated.

We observe that for a (G, K, M, N) network, in the regime where the proper-improper boundary [6], [10] lies above the decomposition based inner bound [14]–[16], i.e., $\left(\frac{MN}{KM+N} < \frac{M+N}{GK+1}\right)$, the unstructured approach appears to be able to achieve the optimal spatially normalized DoF (sDoF). Remarkably, the polynomial identity test plays a key role in identifying the optimal sDoF in this regime. The DoF obtained numerically from this unstructured approach matches the optimal sDoF characterized in a parallel and independent work [17] using a structured approach. The key advantage of the unstructured approach advocated in this paper is that it is conceptually much simpler and easily adapted to a wide class of networks. For a longer version of this paper that further elaborates on the ideas discussed here, please refer to [16].

II. System Model

Consider a (G, K, M, N) network. The index pair (l, m) is used to denote the mth user in the lth cell. The channel from user (l, m) to the ith BS is assumed to be generic and denoted as the $N \times M$ matrix $\mathbf{H}_{(lm,i)}$. In the uplink, let $\mathbf{x}_{lm}$ denote the $M \times 1$ signal vector transmitted by user (l, m). This transmit signal vector is formed using a $M \times d$ linear transmit beamforming matrix $\mathbf{V}_{lm}$ and received using a $N \times d$ receive beamforming matrix $\mathbf{U}_{lm}$, where d represents the number of data streams transmitted by user (l, m). The received signal after being processed by the receive beamforming matrix $\mathbf{U}_{lm}$ at the lth BS can be written as

$$\mathbf{U}_{lm}^H \mathbf{y}_l = \sum_{i=1}^{G} \sum_{j=1}^{K} \mathbf{U}_{lm}^H \mathbf{H}_{(ij,l)} \mathbf{V}_{ij} \mathbf{s}_{ij} + \mathbf{U}_{lm}^H \mathbf{n}_l. \tag{1}$$

where $\mathbf{s}_{ij}$ is the $d\times 1$ symbol vector transmitted by user (i, j) and $\mathbf{n}_l$ is the $N \times 1$ vector representing circular symmetric additive white Gaussian noise $\sim \mathcal{N}(\mathbf{0}, \mathbf{I})$. The received signal is defined similarly for the downlink.

We denote the space occupied by interference at the lth BS as the column span of a matrix $\mathbf{R}_l$ formed using the column vectors from the set $\{\mathbf{H}_{(ij,l)}\mathbf{v}_{ijk} : i \in \{1, 2, \ldots, G\},\ j \in \{1, 2, \ldots, K\},\ k \in \{1, 2, \ldots, d\},\ l \neq i\}$, where we use the notation $\mathbf{v}_{ijk}$ to denote the kth beamformer associated with user (i, j).

III. Conditions for Interference Alignment

When d data streams/user are desired in a (G, K, M, N) network, the conditions for linear IA can be stated as follows [6]:

$$\mathbf{U}_{ij}^H \mathbf{H}_{lm,i} \mathbf{V}_{lm} = \mathbf{0}\ \forall\ (i, j) \neq (l, m) \tag{2}$$

$$\operatorname{rank}(\mathbf{U}_{ij}^H \mathbf{H}_{ij,i} \mathbf{V}_{ij}) = d\ \forall\ (i, j). \tag{3}$$

A necessary condition for feasibility of the polynomial system of equations in (2) is given by $M + N \geq (GK + 1)d$ [6], [10]. Systems that satisfy this condition are known as proper systems. For proper and feasible systems, solving the system of bilinear equations in (2) typically requires the use of iterative algorithms such as those developed in [18].

When channels are generic, the conditions in (2) and (3) can be stated in an alternate manner. In order to accommodate Kd signal vectors at each BS, the interfering vectors contained in $\mathbf{R}_i$ cannot span anymore than $N - Kd$ dimensions i.e., we require $\operatorname{rank}(\mathbf{R}_i) \leq (N - Kd)\ \forall i$. Next, in order to ensure signals from each user span d dimensions at the intended BS we impose the constraint that $\operatorname{rank}(\mathbf{V}_{ij}) = d\ \forall i, j$. Since channels are generic, and since the rank constraint on $\mathbf{R}_i$ does not involve the direct channels, the received signal $\mathbf{H}_{(ij,i)}\mathbf{V}_{ij}$ will almost surely span d-dimensions at the intended BS and further, the signals from the K users will all be separable. Finally, generic channels also ensure that at each BS, the intersection between useful signal subspace ($\operatorname{span}([\mathbf{H}_{i1,i}\mathbf{V}_{i1}, \mathbf{H}_{i2,i}\mathbf{V}_{i2}, \ldots, \mathbf{H}_{iK,i}\mathbf{V}_{iK}])$ and interference subspace ($\operatorname{span}(\mathbf{R}_i)$) is almost surely zero dimensional whenever the $\operatorname{rank}(\mathbf{R}_i) \leq (N - Kd)\ \forall i$. Thus the requirements for IA can be alternately stated as

$$\operatorname{rank}(\mathbf{R}_i) \leq N - Kd\ \forall\ i, \tag{4}$$

$$\operatorname{rank}(\mathbf{V}_{jl}) = d\ \forall\ j, l. \tag{5}$$

Given a set of transmit precoders $\{\mathbf{V}_{jl}\}$ that satisfy the above conditions, designing the receive filters is then straightforward.

IV. Linear Beamforming Design: The Unstructured Approach

In the uplink of a (G, K, M, N) network, when each user transmits d data streams, each BS observes GKd streams of transmission of which $(G-1)Kd$ streams constitute interference. Setting aside Kd dimensions at each BS for the received signals from the in-cell users, to satisfy (4) the $(G - 1)Kd$ interfering data streams must occupy no more than $N - Kd$ dimensions at each BS. Assuming $(G - 1)Kd > N - Kd$ (no IA is necessary otherwise), we require the $(G - 1)Kd$ transmit beamformers of the interfering signals to satisfy $GKd - N$ distinct linear equations, i.e, at the ith BS, we require

$$\sum_{l=1,l\neq i}^{G} \sum_{m=1}^{K} \sum_{n=1}^{d} \alpha_{lmn,i}^{p} \mathbf{H}_{(lm,i)} \mathbf{v}_{lmn} = \mathbf{0}, \tag{6}$$

where $\alpha_{lmn,i}^{p}$ refers to the coefficient associated with the interfering transmit beamformer $\mathbf{v}_{lmn}$ in the pth linear equation corresponding to the ith BS. Let $L = GKd - N$. Thus, we have GL linear vector equations, each involving a set of $(G - 1)Kd$ transmit beamforming vectors. Concatenating the transmit beamforming vectors $\mathbf{v}_{lmn}$ into a single vector $\mathbf{v} = [\mathbf{v}_{111}, \mathbf{v}_{112}, \ldots, \mathbf{v}_{11d}, \ldots, \mathbf{v}_{GKd}]$ and by appropriately defining the matrix $\mathbf{M}$, the GL linear vector equations can be expressed as the matrix equation $\mathbf{M}\mathbf{v} = \mathbf{0}$. Note that $\mathbf{M}$ is a $GLN \times GKMd$ matrix.

When IA is known to be feasible, it guarantees the existence of a set of coefficients $\{\alpha_{lmn,i}^{p}\}$ such that the system of equations $\mathbf{M}\mathbf{v} = \mathbf{0}$ has a non-trivial solution. Determining the right set of coefficients is non-trivial and highlights a particular difficulty in finding aligned beamformers using the set of equations characterized by $\mathbf{M}\mathbf{v} = \mathbf{0}$. However, in some cases, even a random choice of coefficients permits non-trivial solutions to this system of equations. The ability to choose a random set of coefficients is quite significant as instead of solving a set of bilinear polynomial equations for IA, we now only need to solve a set of linear equations. In this paper, we focus on those networks that permit the choice of random coefficients.

While aligned beamformers satisfy the system of equations $\mathbf{M}\mathbf{v} = \mathbf{0}$ for a set of coefficients, not all solutions to $\mathbf{M}\mathbf{v} = \mathbf{0}$ with a fixed set of coefficients form aligned beamformers. A vector $\hat{\mathbf{v}}$ satisfying $\mathbf{M}\hat{\mathbf{v}} = 0$, can be considered to constitute a set of aligned beamformers provided (a) the set of beamformers corresponding to a user are linearly independent, i.e., $\mathbf{V}_{ij}$ is full rank $\forall i, j$; (b) the signal received from a user at the intended BS

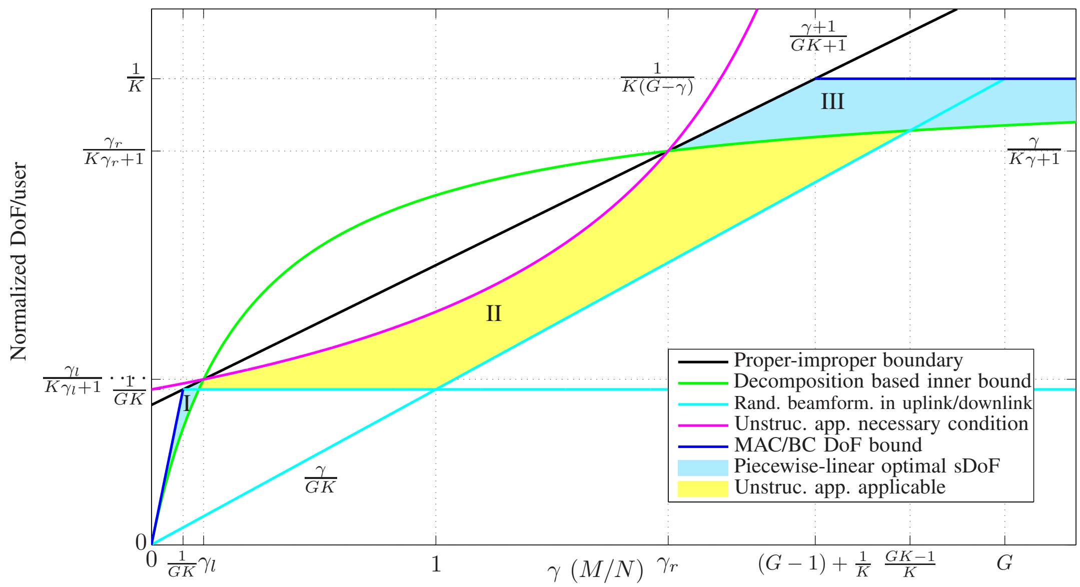

Fig. 1. Inner and outer bounds on the DoF of the G-cell, K-user/cell network. The optimal DoF consists of infinitely many piecewise-linear components for $\gamma < \gamma_l$ and $\gamma > \gamma_r$, while the decomposition based approach determines the optimal DoF when $\gamma_l \leq \gamma \leq \gamma_r$.

is full rank i.e., $\mathbf{H}_{ij,i}\mathbf{V}_{ij}$ is full rank; (c) signal and interference are separable at each BS; and (d) the received signal vectors at each BS span Kd dimensions i.e., signals from two users in a cell do not overlap. Since we assume generic channel coefficients and since direct channels are not used in forming the matrix $\mathbf{M}$, (c) and (d) are satisfied almost surely, while (b) and (d) are true provided (a) is true and the channel coefficients are generic. While the idea of satisfying conditions for IA through random linear equations is also discussed in [11], the presentation in [11] is limited to achieving one DoF/user, thereby avoiding the necessity to check for linear independence of the transmit beamformers.

Since $\mathbf{M}$ is a $GLN \times GKMd$ matrix, whenever $LN < KMd$ the system of equations $\mathbf{Mv} = \mathbf{0}$ permits a non-trivial solution for any random choice of coefficients. This condition, i.e., $LN < KMd$, is a necessary condition for the applicability of unstructured beamformer design. As shown in Fig. 1, it encompasses a significant set of (although not all) feasible networks. When $LN < KMd$, a solution to the equation $\mathbf{Mv} = \mathbf{0}$ can be expressed as $\hat{\mathbf{v}} = \det(\mathbf{MM}^H)(\mathbf{I} - \mathbf{M}^H(\mathbf{MM}^H)^{-1}\mathbf{M})\mathbf{r}$, where $\mathbf{r}$ is a $GKMd \times 1$ vector with randomly chosen entries. For $\hat{\mathbf{v}}$ to qualify as a solution for IA, we need to ensure that the set of transmit beamformers $\hat{\mathbf{v}}_{ij1}, \hat{\mathbf{v}}_{ij2} \ldots \hat{\mathbf{v}}_{ijd}$ obtained from $\hat{\mathbf{v}}$ are linearly independent for any $i \in \{1, 2, \ldots, G\}$, $j \in \{1, 2, \ldots, K\}$. Letting $\hat{\mathbf{V}}_{ij}$ be the $M \times d$ matrix formed using $\hat{\mathbf{v}}_{ij1}, \hat{\mathbf{v}}_{ij2} \ldots \hat{\mathbf{v}}_{ijd}$, checking for linear independence is equivalent to checking if the determinant of the matrix $[\hat{\mathbf{V}}_{ij}\, \mathbf{W}_{ij}]$, where $\mathbf{W}_{ij}$ is a $(M-d)\times d$ matrix of random entries, is non-zero or not.

Since the determinant of $[\hat{\mathbf{V}}_{ij}\, \mathbf{W}_{ij}]$ is a polynomial in the variables $\mathbf{W}_{ij}$, $\mathbf{r}$, $\{\alpha^p_{lmn,i}\}$, and $\{\mathbf{H}_{(lm,i)}\}$, checking for linear independence of the transmit beamformers is equivalent to checking if this polynomial is the zero-polynomial or not. This problem is known as polynomial identity testing (PIT) and is well studied in complexity theory. While a general deterministic algorithm to solve this problem is not known, a randomized algorithm based on the Schwartz-Zippel lemma [19], [20] is available and it involves evaluating this polynomial at a random instance of $\mathbf{W}_{ij}$, $\mathbf{r}$, $\{\alpha^p_{lmn,i}\}$, and $\{\mathbf{H}_{lm,i}\}$. If the value of the polynomial at this point is non-zero, then this polynomial is determined to be not identical to the zero-polynomial. Further, it can be concluded that this polynomial evaluates to a non-zero value for almost all values of $\mathbf{W}_{ij}$, $\mathbf{r}$, $\{\alpha^p_{lmn,i}\}$, and $\{\mathbf{H}_{lm,i}\}$. If on the other hand, the polynomial evaluates to the zero, the polynomial is declared to be identical to the zero-polynomial and this statement is true with a very high probability as a consequence of the Schwartz-Zippel lemma.

Thus, whenever $LN < KMd$, we propose a two-step approach to designing aligned beamformers. We first pick a set of random coefficients, form the linear equations to be satisfied by the transmit beamformers and compute a set of transmit beamformers by solving the system of linear equations. We then perform the numerical test outlined above to ensure that the transmit beamformers are indeed linearly independent. If the transmit beamformers pass the numerical test then they can be considered to be a set of aligned transmit beamformers. Further, if such a procedure works for a (G, K, M, N) network with d DoF/user for a particular generic channel realization, then it works almost surely for all generic channel realizations of this network. This observation allows us to construct a numerical experiment to verify the limits of using such an approach.

V. Scope and Limitations

A. Numerical Experiment

We consider a network with G cells and K users/cell. For this network, we consider all possible pairs of M and N such that $M \leq M_{max}$ and $N \leq N_{max}$, where M_{max} and N_{max} are some fixed positive integers. For a fixed M and N, we then consider the feasibility of constructing aligned beamformers using the method described above in order to achieve d DoF/user where d is such that (a) $LN < KMd$; (b) $L > 0$ and $M < GKd$ (else random beamforming in either uplink or downlink achieves

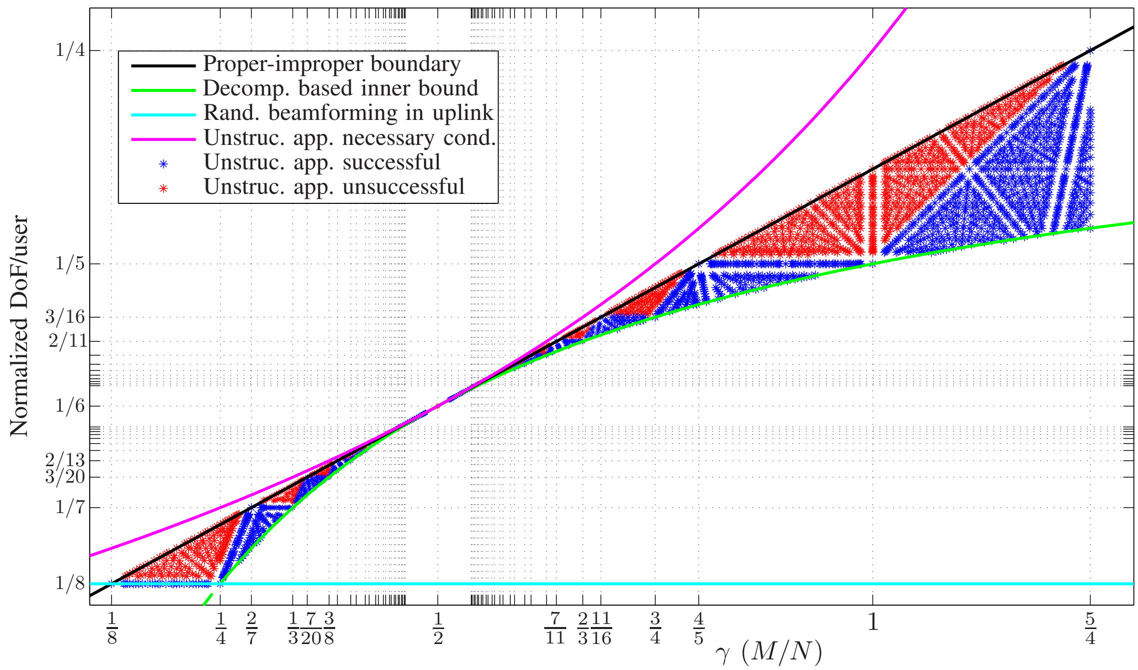

Fig. 2. Results of the numerical experiment for the two-cell, four-user/cell network. Note the clear piecewise-linear boundary that emerges between the successful and unsuccessful trials of the proposed method. The observed boundary matches with the result in [17].

the requisite DoF); (c) $d \leq M$ and $Kd \leq N$ (to ensure sufficient antennas for signal vectors); (d) $\gcd(M, N, d) = 1$ (due to spatial scale invariance[1]); and (e) (G, K, M, N, d) form a proper system. For such a set of M, N, and d, we generate an instance of generic channel matrices and apply the two step procedure outlined earlier. This procedure is said to be successful if the polynomial test returns a non-zero value and unsuccessful otherwise. If successful, we conclude that such a procedure can be reliably used to design transmit beamformers for almost all channel instances of the (G, K, M, N, d) network under consideration. When unsuccessful, we conclude that with a very high probability such a procedure does not yield a set of aligned transmit beamformers for almost all channel instances. Using the results from such an experiment, we discuss the scope and limitations of the unstructured approach in the next section.

B. Applicability of the Unstructured Approach

In Fig. 1 we sketch some well known bounds on the normalized DoF/user (sDoF/user/N) as a function of γ. This figure applies to any MIMO cellular network, with the exception of the two-cell, two-user/cell and the two-cell, three-user/cell networks, where the decomposition based inner bound lies strictly below the proper-improper boundary. Note that the necessary condition for the unstructured approach, the proper-improper boundary and the decomposition based inner bound all intersect at the same two points γ_l and γ_r, given by $\frac{K(G-1) \pm \sqrt{K^2(G-1)^2 - 4K}}{2K}$. The optimal sDoF of a general cellular network is recently investigated in [17].

[1]Spatial scale invariance [21] states that if d DoF/user are feasible for a (G, K, M, N) network, then sd DoF/user are feasible in a (G, K, sM, sN) network where $s \in \mathcal{Z}^+$ denotes the scale factor. While no proof of such a statement is available, no contradictions to this statement exist to the best of our knowledge.

The optimal sDoF as characterized in [17] has a piecewise-linear behavior in regions I ($\gamma < \gamma_l$) and III ($\gamma > \gamma_r$) (see Fig. 1). Based on the results in [5] for the MIMO interference channel, the decomposition-based inner bound is likely to characterize the optimal DoF whenever $\gamma_l \leq \gamma \leq \gamma_r$. A simple DoF bound obtained by letting all the BSs or users cooperate (denoted as MAC/BC DoF bound) is also plotted along with the maximum achievable sDoF using random transmit beamforming in the uplink/downlink.

Focusing on regions I and III, we note that the unstructured approach is applicable to all points in these two regions. To gain insight on the scope of this technique for cellular networks, we perform the numerical experiment outlined earlier for the 2-cell 4-user/cell network. For this network, the proper-improper boundary and the decomposition-based inner bound touch each other at $\gamma = 1/2$, i.e., $\gamma_l = \gamma_r = 1/2$. The results of the numerical experiment are plotted in Fig. 2 and it is easy to see that a clear piecewise linear boundary emerges between the successful and unsuccessful trials. Interestingly, the boundary is completely determined by the PIT.

Remarkably, the boundary of the achievable sDoF determined by our unstructured approach matches with the optimal sDoF claimed in [17]. This leads us to conjecture that for any (G, K, M, N) network whenever the decomposition-based inner bound lies below the proper-improper boundary, the optimal sDoF can be achieved by constructing linear beamformers using the proposed method.

Shifting focus to region II[2], note that this region lies entirely below the decomposition-based inner bound and does not impact

[2]When $G > 4$, region II consists of two separate parts. See [16] for further details.

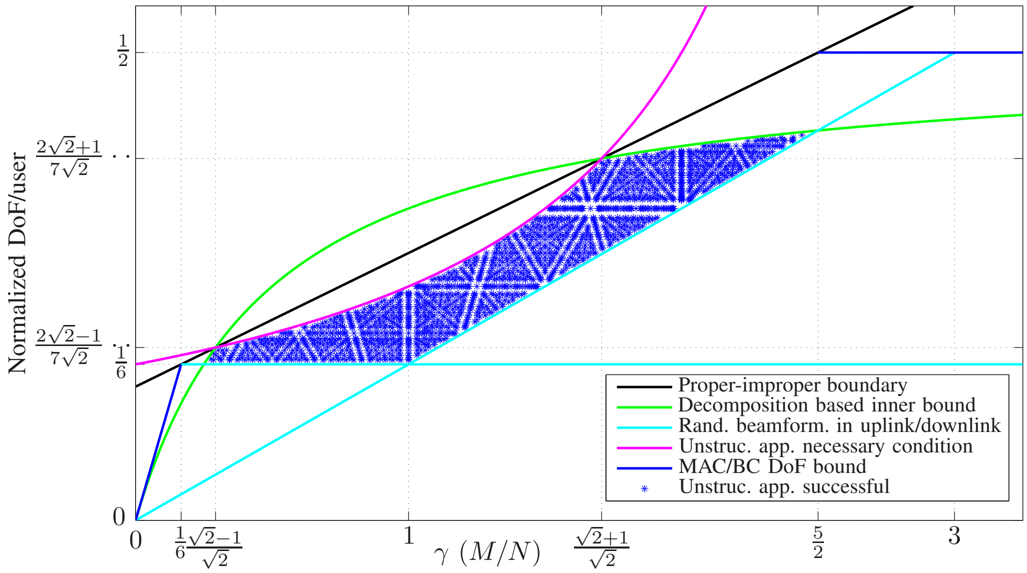

Fig. 3. Results of the numerical experiment in region II of the three-cell, two-user/cell network. Observe that the necessary condition for the unstructured approach completely determines the success of failure of the proposed approach.

the characterization of the optimal sDoF. This region is bounded below by the maximum DoF that can be trivially achieved using random transmit beamforming in the uplink. By running the numerical experiment on the 3-cell, 2-user/cell network for (M, N, d) such that $(M/N, d/N)$ lies in region II, we note from Fig. 3 that the necessary condition $LN < KMd$ also ensures the success of the polynomial identity test. It is thus seen that even in the regime where $\gamma_l \leq \gamma \leq \gamma_r$, a significant portion of the achievable sDoF can be achieved using the unstructured approach.

VI. Conclusion

This paper proposes a new strategy for designing linear beamformers for IA in MIMO cellular networks. The proposed method is agnostic to the underlying structure of alignment. It relies on random linear vector equations and a polynomial identity test to satisfy the conditions for IA. Numerical experiments appear to suggest that this approach can achieve the optimal sDoF of MIMO cellular networks in the regime where the decomposition-based inner bound lies below the proper-improper boundary.

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