UNIVERSITY OF TORONTO
Faculty of Applied Science and Engineering
ECE 1528 S — Multiuser Information Theory
Instructor: Wei Yu
Final Examination April 26, 2007.

Take-home Exam. Open Book. Open Notes.
Calculator, Computer and Matlab are allowed.
NO COLLABORATIONS OR DISCUSSIONS DURING THE EXAM.
Duration — 24 hours.

Last Name: ______________________________

First Name: ______________________________

Student Number: |_|_|_|_|_|_|_|_|_|

*You are **not** permitted to communicate via any means with anybody other than the instructor regarding this exam, prior to the end of the examination: 10am, April 27, 2007.*

Answer all questions. Unless otherwise stated, for full credit, solutions must show your reasoning.

Honour Code: The Honour Code is an undertaking of the students, individually and collectively:

(1) that they will not give or receive aid in examinations;

(2) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the sprite and letter of the Honour Code.

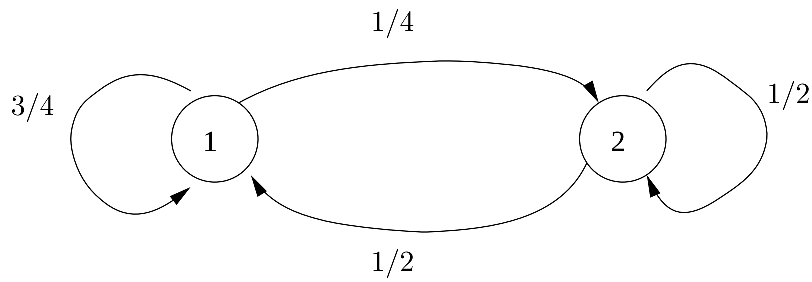
I acknowledge and accept the Honour Code ________________ **(sign)**

# 1:	______ /	13
# 2:	______ /	25
# 3:	______ /	15
# 4:	______ /	10
# 5:	______ /	12
# 6:	______ /	25
TOTAL:	______ /	100

Good Luck!

Question 1. [13 MARKS]

Consider an additive white Gaussian noise channel $Y = X + Z$, where Z is the Gaussian noise with its variance following a Markov Chain with a state-transition diagram shown below:



$Z = Z_i$, whenever the Markov chain is in state i ($i = 1, 2$), where Z_1, Z_2 are Gaussian i.i.d. random processes with variances σ_1^2 and σ_2^2, respectively. The channel has an input power constraint P.

Part (a) [4 MARKS]

Suppose that the state of the Markov chain is known at both the transmitter and the receiver. What is the capacity C_1 of this Gaussian channel? Does your answer depend on whether the state is known causally or noncausally? Please assume $\sigma_1^2 = 0.1$, $\sigma_2^2 = 0.5$, and $P = 10$. Explain how the respective capacities can be achieved in either cases.

Part (b) [3 MARKS]

Suppose that the state information is known only at the receiver, but not at the transmitter, compute the new capacity C_2. Assume $\sigma_1^2 = 0.1$, $\sigma_2^2 = 0.5$, and $P = 10$ as before. Explain why this capacity is achievable.

Part (c) [3 MARKS]

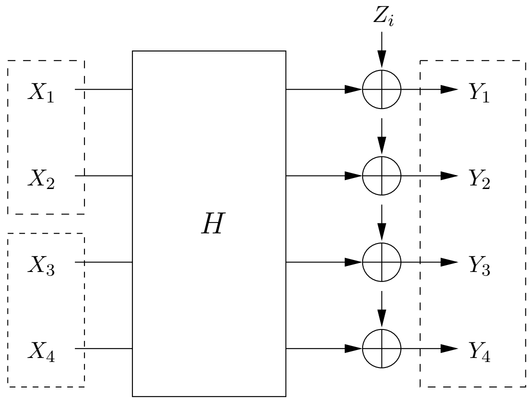
Now, suppose that neither the transmitter nor the receiver knows the state. Further, assume a slightly different set of parameters: $\sigma_1^2 = 0$, $\sigma_2^2 = 0.5$, $P = 10$. What is the capacity of the channel?

Part (d) [3 MARKS]

Design a coding scheme to achieve the capacity found in Part (c).

Question 2. [25 MARKS]

Consider a Gaussian vector channel shown below, where Z_i is i.i.d. $\sim \mathcal{N}(0,1)$.



$$H = \begin{bmatrix} 1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 2 & 0.1 & 0.1 \\ 0.1 & 0.1 & 1.5 & 0.1 \\ 0.1 & 0.1 & 0.1 & 1 \end{bmatrix}$$

Part (a) [3 MARKS]

Numerically compute the channel capacity assuming that $\{X_i\}_{i=1}^4$ cooperate, with a sum-power input constraint $P = 2$. Show your steps.

Part (b) [7 MARKS]

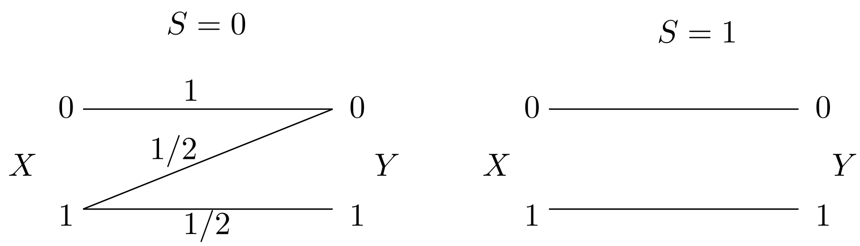
Numerically compute the sum capacity of the multiple-access channel, assuming that X_1 and X_2 cooperate, with a sum-power input constraint $P_1 = 1$ between the two; X_3 and X_4 cooperate, with a sum-power input constraint $P_2 = 1$ between the two; but there is no cooperation between $\{X_i\}_{i=1}^{2}$ and $\{X_i\}_{i=3}^{4}$. Show your steps. Submit your Matlab code.

Part (c) [10 MARKS]

Numerically compute the achievable rate of an OFDM system on an ISI channel $1 - 0.8D - 0.7D^2$ with complex-valued input and output. Assume that the channel noise has a distribution $\mathcal{N}(0, \sigma^2/2)$ in each of its real and complex components with $\sigma^2 = 0.1$. Assume a power constraint of $\bar{P}_x = 2$ per dimension (i.e. in each of the real and imaginary components of the transmitted symbol.) Let the length of cyclic prefix be 10. Please be sure to account for the power expenditure in the cyclic prefix. Let the OFDM blocksize be $N = 512$. Assume an SNR gap of $\Gamma = 10$dB. Show your steps. Submit your Matlab code. Plot the power and bit allocation in each frequency bin. You may assume continuous bit values.

Question 3. [15 MARKS]

A binary-input binary-output channel has two states as shown below:



Part (a) [5 MARKS]

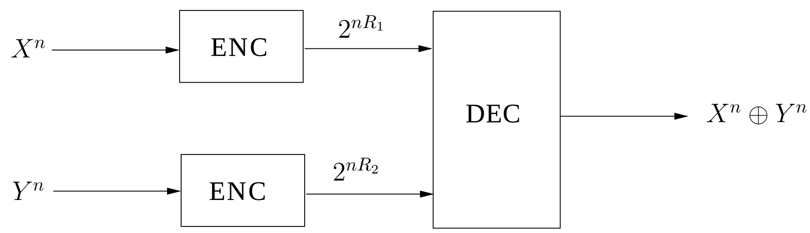
Compute the capacities of the individual channel for state $S = 0$ and $S = 1$.

Part (b) [10 MARKS]

Now, consider the channel as a multiple-access channel with two users: X and S. Find its capacity region, and plot it using Matlab. Compare the maximum sum rate with the cooperative upper bound when (X, S) act as a single user.

Question 4. [10 MARKS]

X^n and Y^n are binary random sequences which agree with each other with probability 0.9 and disagree with probability 0.1. You may think of X^n and Y^n as input and output of a binary symmetric channel with crossover probability 0.1. Two encoders observe X^n and Y^n independently. The decoder wishes to identify locations in which X^n and Y^n disagree.



Find the best achievable rate region (R_1, R_2). (Hint: The following fact may be useful: for a binary symmetric channel with crossover probability ρ, there exists a sequence of *linear* codes of blocklengths n with rates $R_n \to (1 - H(\rho))$ and with probabilities of error $P_e^{(n)} \to 0$ as $n \to \infty$.)

Question 5. [12 MARKS]

Consider a game played with a 3-card deck. In each round of this game, you are randomly given 2 out of the 3 cards. You communicate with your friend by hiding one of them, and showing the other to your friend. Your friend does not see the original 2-card hand.

Part (a) [4 MARKS]

Suppose that you and your friend play this game over n rounds. Think of a simple scheme that allows you to transmit at some average rate R (per round) with *zero* error probability in each round. What is the largest such R?

Part (b) [4 MARKS]

Suppose that we relax the zero-error constraint, and demand only that the probability of error $P_e^n \to 0$ as $n \to \infty$, what is the largest achievable rate?

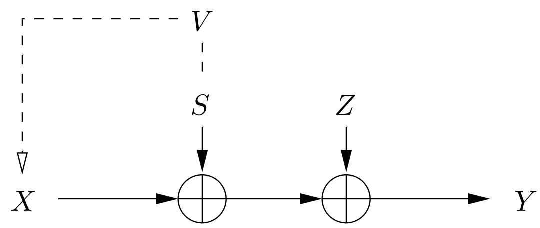
Part (c) [4 MARKS]

Design a coding scheme that achieves the rate in Part (b) (as $n \to \infty$).

Question 6. [25 MARKS]

Part (a) [5 MARKS]

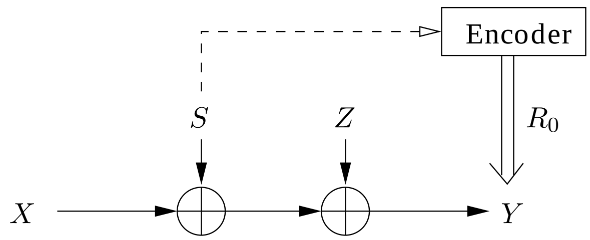
Consider the following channel:



where $Z \sim \mathcal{N}(0, N)$, $S \sim \mathcal{N}(0, Q)$, and X has a power constraint P. (V, S) are jointly Gaussian with a covariance matrix $K_{SV} = Q \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$. Suppose that the value of V is known at the transmitter noncausally. Compute the capacity of this channel as a function of ρ.

Part (b) [10 MARKS]

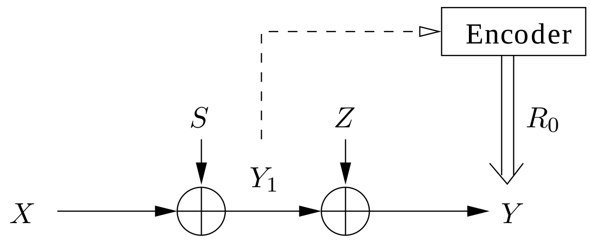
Now, consider the case where S is observed by a third party with a dedicated link of rate R_0 to the receiver.



Can the third party increase the capacity between X and Y? How should he help? Please find the largest achievable rate as a function of R_0.

Part (c) [10 MARKS]

Finally, consider the following scenario where the third-party observes $Y_1 = X + S$ instead of S.



What encoding scheme should he use in order to increase the capacity between X and Y? Please find the largest achievable rate as a function of R_0.

Total Marks = 100