In combinatorics, probability, information theory, and elsewhere, one often needs to have estimates of, or bounds on, the factorial function

\[ n! = \prod_{i=1}^{n} i, \]

particularly for large \( n \). When dealing with \( n! \) it always occurs that it may be easier to take logarithms and deal with \( \ln(n!) = \sum_{i=1}^{n} \ln(i) \) instead. Since \( \ln(x) \) is an increasing function of \( x > 0 \), we have, for any \( i \geq 1 \),

\[
\int_{i-1}^{i} \ln(x)dx < \ln(i) < \int_{i}^{i+1} \ln(x)dx.
\]

Adding these inequalities with \( i = 1, 2, \ldots, n \), we get

\[
\int_{0}^{n} \ln(x)dx < \ln(n!) < \int_{1}^{n+1} \ln(x)dx,
\]

Since, for \( 0 < a \leq b \), we have

\[
\int_{a}^{b} \ln(x)dx = (x \ln(x) - x)|_{a}^{b} = b \ln b - b - a \ln a + a
\]

and (remembering that \( \lim_{a \to 0} a \ln a = 0 \)) we get

\[
n \ln n - n < \ln(n!) < (n + 1) \ln(n + 1) - (n + 1) + 1
\]

or

\[
\left(\frac{n}{e}\right)^{n} < n! < e\left(\frac{n + 1}{e}\right)^{n+1}
\]

Thus \( n! \) grows more quickly than \( (n/e)^{n} \), but not as quickly as \( e((n + 1)/e)^{n+1} \), i.e., \( n! \) lies somewhere “in between”.

This betweeness is captured in Stirling’s Formula, which gives

\[
n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^{n} = \sqrt{2\pi}n^{n+1/2}e^{-n},
\]
where $\sim$ means that the ratio of the two sides approaches unity in the limit as $n \to \infty$, i.e.,

$$\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n} = 1. \quad (1)$$

True to Stigler’s Law,\(^1\) Stirling’s formula was first discovered by Abraham de Moivre, who understood that

$$n! \sim K(n/e)^{n+1/2}$$

for some constant $K$. It was Stirling who first realized that $K = \sqrt{2\pi e}$, and both Stirling and de Moivre published proofs in 1730; see [1, Ch. 24] for a history.

In this brief note, inspired by [2], we equip ourselves with (1), and seek an exact expression for $n!$ in the form

$$n! = C(n/A)^{n+B} \prod_{i=n}^{\infty} f(i) \quad (2)$$

for some constants $A$, $B$, and $C$ (to be determined), and a suitably-defined function $f$. The infinite product expresses an $n$-dependent correction factor that provides the precise adjustment that must be made to $C(n/A)^{n+B}$ to get $n!$.

The convenience of (2) becomes evident when one considers the ratio $(n+1)!/n! = n + 1$. We then get

$$n + 1 = \frac{((n+1)/A)^{n+1+B}}{(n/A)^{n+B} f(n)}$$

from which we determine that

$$f(n) = \frac{1}{A} \left(1 + \frac{1}{n}\right)^{n+B}.$$

Plugging into (2), the right-hand side becomes

$$g(n) = C(n/A)^{n+B} \prod_{i=n}^{\infty} \left(\frac{1 + \frac{1}{i}}{A}\right)^{i+B}.$$

Of course to be useful, we need that the infinite product converges to a positive constant, which can only happen if

$$\lim_{i \to \infty} \frac{1}{A} \left(1 + \frac{1}{i}\right)^{i+B} = 1.$$

The limit is easily computed as $e/A$ (independent of $B$). Thus we find that the only possible choice for the constant $A$ is $A = e$, and we get that

$$g(n) = C(n/e)^{n+B} \prod_{i=n}^{\infty} \left(\frac{1 + \frac{1}{i}}{e}\right)^{i+B}.$$

\(^1\)Stigler’s Law of Eponymy: “no scientific discovery is named after its original discoverer.” This law is attributed to R. K. Merton.
Note that $g(n)$ now has the property that $g(n + 1) = (n + 1)g(n)$. Provided that we can choose constants $B$ and $C$ so that $g(1) = 1$, then we will have the desired equality. Thus we now require

$$1 = \lim_{m \to \infty} C(1/e)^{1+B} \prod_{i=1}^{m} \left( \frac{(1 + \frac{1}{i})^{i+B}}{e} \right).$$

(3)

Let

$$h(m) = \sum_{i=1}^{m} \left( (i + B) \ln \left( \frac{i + 1}{i} \right) - 1 \right),$$

which is the logarithm of the product in (3). We require $h(m)$ to approach a finite limit as $m \to \infty$. Expanding $h(m)$, we get

$$h(m) = -m + B \sum_{i=1}^{m} (-\ln(i) + \ln(i + 1)) + \sum_{i=1}^{m} i (-\ln(i) + \ln(i + 1))$$

$$= -m + B (-\ln(1) + \ln(2) - \ln(2) + \ln(3) + \cdots - \ln(m) + \ln(m + 1))$$

$$+ (-\ln(1) + \ln(2) - 2 \ln(2) + 2 \ln(3) - 3 \ln(3) + 3 \ln(4) + \cdots - m \ln(m) + m \ln(m + 1))$$

$$= -m + B \ln(m + 1) - m \ln(m) + m \ln(m + 1)$$

$$= (m + B) \ln(m + 1) - m - \ln(m!).$$

Now, using the fact from (1) that $\ln(m!) \to \frac{1}{2} \ln m + \frac{1}{2} \ln(2\pi) + m \ln(m) - m$ as $m \to \infty$, we get that

$$\lim_{m \to \infty} h(m) = \lim_{m \to \infty} \left( -m + B \ln(m + 1) + m \ln(m + 1) - \frac{1}{2} \ln m - \frac{1}{2} \ln(2\pi) - m \ln(m) + m \right)$$

$$= -\frac{1}{2} \ln(2\pi) + \lim_{m \to \infty} \left( (m + B) \ln(m + 1) - \left( m + \frac{1}{2} \right) \ln(m) \right)$$

$$= -\frac{1}{2} \ln(2\pi) + \lim_{m \to \infty} \left( \left( m + \frac{1}{2} \right) \ln \left( \frac{m + 1}{m} \right) + \left( B - \frac{1}{2} \right) \ln(m + 1) \right)$$

$$= 1 - \frac{1}{2} \ln(2\pi) + \lim_{m \to \infty} \left( B - \frac{1}{2} \right) \ln(m + 1)$$

$$= \begin{cases} 
\infty, & \text{if } B > 1/2; \\
1 - \frac{1}{2} \ln(2\pi), & \text{if } B = 1/2; \\
-\infty, & \text{if } B < 1/2.
\end{cases}$$

Thus we see that we have no choice but to set $B = 1/2$, and, for this choice of $B$ we can determine the value of $C$ to arrive at Mermin’s equality [2]

$$n! = (2\pi n)^{1/2} \left( \frac{n}{e} \right)^n \prod_{i=n}^{\infty} \left( \frac{(1 + 1/i)^{i+1/2}}{e} \right).$$

(4)
Now to find bounds on $n!$, one approach is to simply find bounds on

$$\prod_{i=n}^{\infty} \left( \frac{(1 + 1/i)^{i+1} / e}{e} \right).$$

Since $\frac{(1 + 1/i)^{i+1} / e}{e} > 1$ for all $i \geq 1$, an obvious lower bound is

$$n! > (2\pi n)^{1/2} \left( \frac{n}{e} \right)^n.$$

An upper bound can be obtained from the Taylor series expansion for $\frac{1}{2} \ln((1 + x)/(1 - x))$, namely

$$\frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

Substituting $x = 1/(2i + 1)$, we get

$$\frac{1}{2} \ln \left( \frac{1 + i}{i} \right) = \frac{1}{2i + 1} + \frac{1}{3(2i + 1)^3} + \frac{1}{5(2i + 1)^5} + \cdots,$$

from which it follows that

$$\left( \frac{2i + 1}{2} \right) \ln \left( \frac{1 + i}{i} \right) = 1 + \frac{1}{3(2i + 1)^2} + \frac{1}{5(2i + 1)^4} + \cdots.$$

Subtracting one from both sides we get the upper bound

$$\left( \frac{2i + 1}{2} \right) \ln \left( \frac{1 + i}{i} \right) - 1 < \frac{1}{3} \left( \frac{1}{(2i + 1)^2} + \frac{1}{(2i + 1)^4} + \cdots \right) = \frac{1}{12} \left( \frac{1}{i} - \frac{1}{i + 1} \right).$$

Exponentiating and substituting into (4) we get a telescoping exponent that yields the bound

$$n! < (2\pi n)^{1/2} \left( \frac{n}{e} \right)^n e^{1/(12n)}.$$

More elaborate bounds are possible. Following the same Taylor series approach, we see that

$$n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \prod_{i=n}^{\infty} \exp \left( \frac{1}{3(2i + 1)^2} + \frac{1}{5(2i + 1)^4} + \cdots \right).$$

References
