Source Broadcasting over Erasure Channels: Distortion Bounds and Code Design

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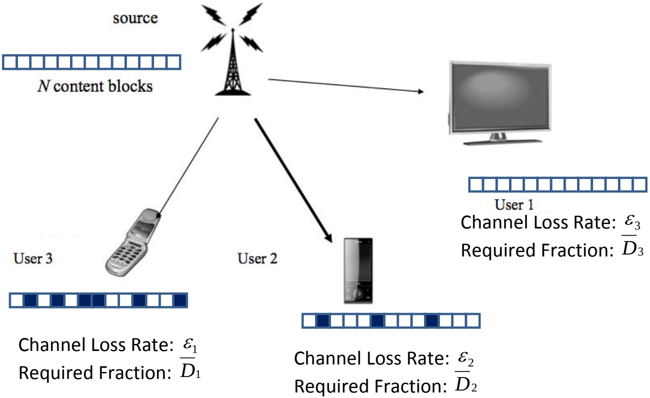
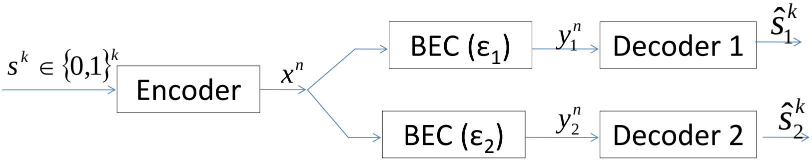
Motivation

Setup: Broadcast to Heterogenous Users



- One Source and Multiple Receivers
- Receiver i: Channel loss rate ε_i
- Receiver i: Required Fraction $\bar{D}_i$

Joint Source-Channel Coding

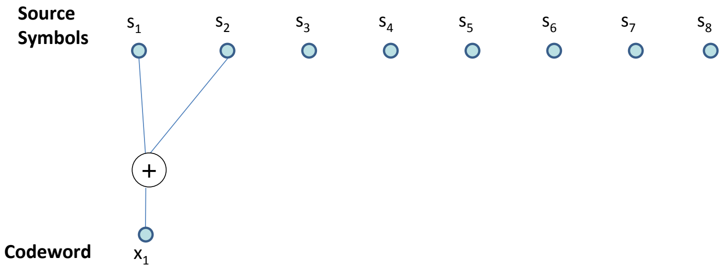


- Binary Source Sequence: $s^k \in \{0,1\}^k$
- Erasure Broadcast Channel: $(\varepsilon_1 < \varepsilon_2)$
- Bandwidth Expansion Factor: $b = \frac{n}{k}$
- Erasure Distortion:

$$d(s_i, \hat{s}_i) = \begin{cases} 0, & s_i = \hat{s}_i, \\ 1, & \hat{s}_i = \star, \\ \infty, & \text{else.} \end{cases}$$

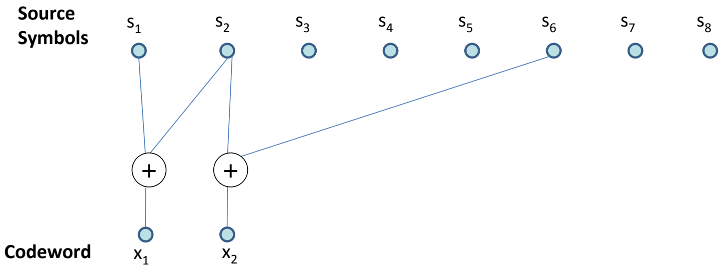
$d(s^k, \hat{s}^k) = \frac{1}{k}\sum_{i=1}^{k} d(s_i, \hat{s}_i) = D$, then $(1-D)$ fraction of source symbols available to the destination.

Code Design



- Given a degree distribution: $P(u) = p_1 u + p_2 u^2 + \ldots + p_L u^L$
- Sample each symbol x_i as follows:
  - Sample $d \in [1, L]$ from the distribution $[p_1, p_2, \ldots, p_L]$
  - Sample d out of k symbols, $s_{i_1}, \ldots, s_{i_d}$ and let $x_i = s_{i_1} \oplus \ldots \oplus s_{i_d}$

Code Design



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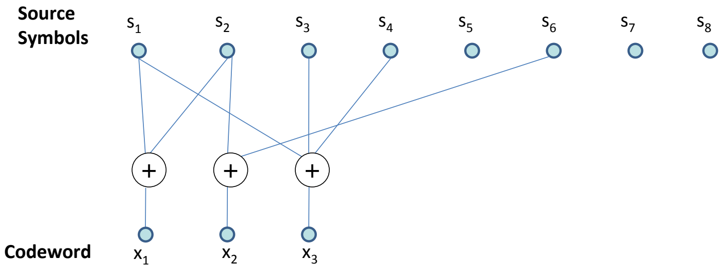

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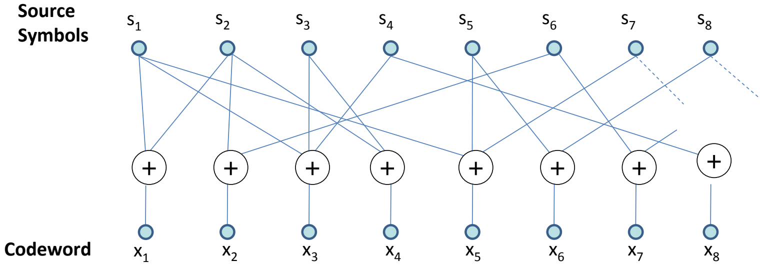

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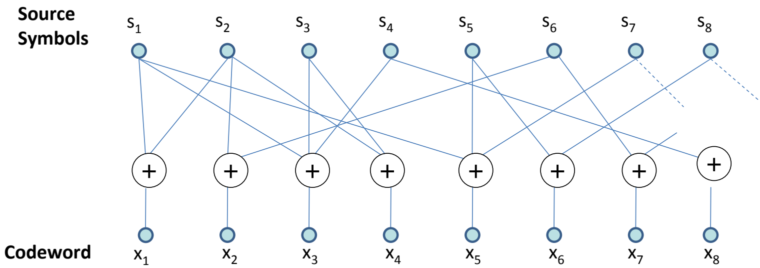
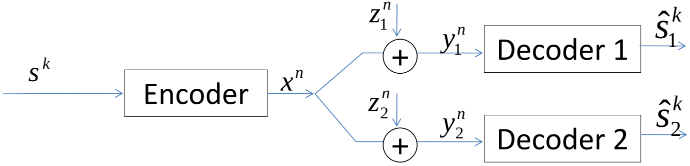
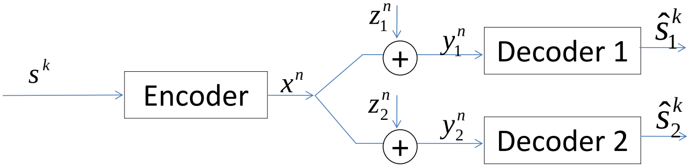


- Robust Soliton Distribution: Near Optimal for Lossless Recovery over all channels
- Partial Recovery: Only a fraction of source symbols need to be recovered by all receivers
- Optimized Degree Distribution



- Gaussian Source: $s^k \overset{\text{i.i.d.}}{\sim} \mathcal{CN}(0, \sigma^2)$
- AWGN Channel: $z_i^n \overset{\text{i.i.d.}}{\sim} \mathcal{CN}(0, N_i)$
- Degradation Order: $N_2 > N_1$
- Power Constraint: $E[x(i)^2] \leq P$
- Quadratic Distortion Measure $d(s, \hat{s}) = (s - \hat{s})^2$,
- Characterize achievable pairs (b, D_1, D_2).



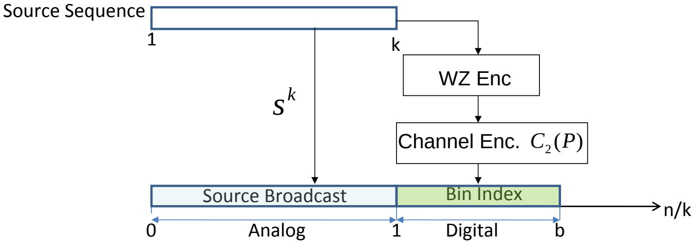
- For $b = 1$, uncoded transmission is optimal.
- Problem Remains Open in General
- Significant Prior Work:
- Shamai-Verdu-Zamir (1998), Mittal and Phamdo (2002), Reznic-Fedar-Zamir (2006), Tian-Diggavi-Shamai (2011) ...
- Unequal Bandwidth Expansion: Tan-Khisti-Soljanin (2012)

Joint Source Channel Coding

Classical Coding Schemes

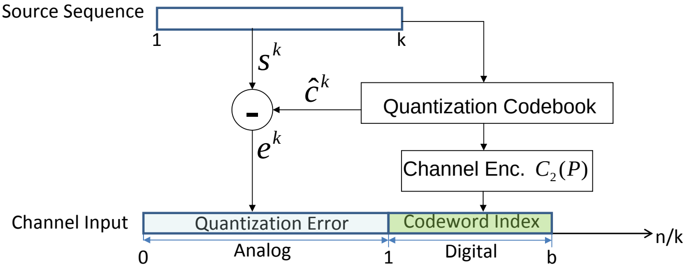
- Systematic Lossy Coding Scheme
- Mittal-Phamdo Coding Scheme

Systematic Lossy Coding (Shamai-Verdu-Zamir '98)



- Analog Phase: $x^k = \sqrt{\frac{P}{\sigma^2}} s^k$
- Digital Phase: $R^{\text{wz}} = (b-1)C_2(P)$
- $D_2 = \frac{\sigma^2}{2^{2bC_2(P)}}$, $D_1 = \frac{\sigma^2}{2^{2bC_2(P)} + \left(\frac{P}{N_1} - \frac{P}{N_2}\right)}$

Mittal and Phamdo (2002)



- Digital Phase: $R_{\mathrm{q}} = \frac{1}{2}\log\frac{\sigma^2}{D_q} = (b-1)C_2(P)$
- Analog Phase: $e^k = s^k - c^k$
- $D_2 = \frac{\sigma^2}{2^{2bC_2(P)}}$, $D_1 = \frac{\sigma^2}{2^{2bC_2(P)}\left(\frac{1+\frac{P}{N_1}}{1+\frac{P}{N_2}}\right)} \leq D_1^{\mathrm{systematic}}$

Point-to-Point Channel

- $s^k \overset{\text{i.i.d.}}{\sim} Ber(1/2)$
- Erasure Distortion: $R(D) = 1 - D$
- i.i.d. Erasure Channel: $C = 1 - \varepsilon$

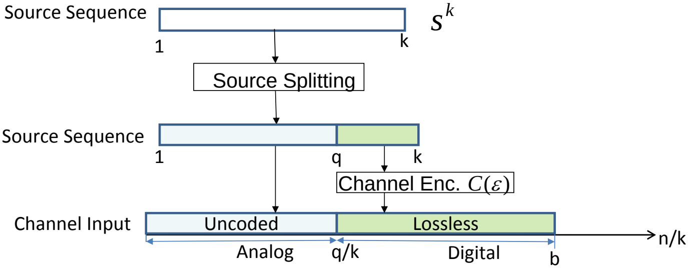
Separation Theorem: $R(D) \leq b \cdot C$

$$D \geq \Delta(b, \varepsilon) = \max\{0, 1 - b(1 - \varepsilon)\}$$

$$b \geq \beta(D, \varepsilon) = \frac{1 - D}{1 - \varepsilon}, \quad 0 \leq D \leq 1$$

Binary Source, Erasure Channel

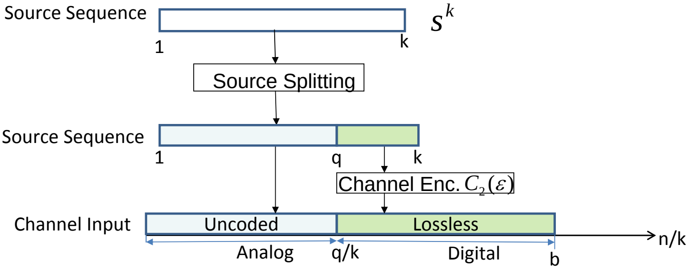
Mittal-Phamdo Coding Scheme



- Split s^k into two subsequences
- Transmit first $q = k\frac{D^\star}{\varepsilon}$ bits uncoded
- Transmit remaining $k\left(1 - \frac{D^\star}{\varepsilon}\right)$ bits at rate $1 - \varepsilon$

$$\frac{D^\star}{\varepsilon} + \frac{1 - \frac{D^\star}{\varepsilon}}{1 - \varepsilon} = \beta(D^\star, \varepsilon)$$

Mittal-Phamdo Coding Scheme

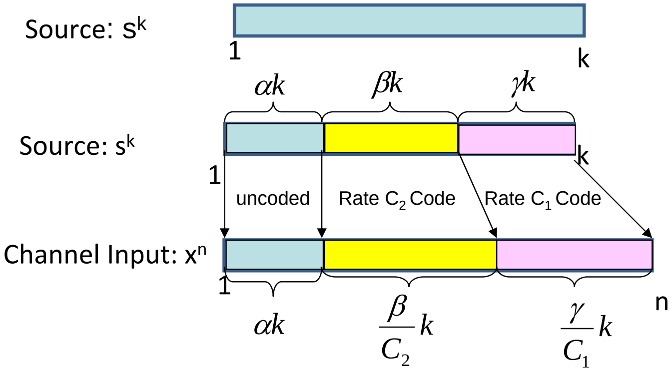


Achievable (D_1, D_2):

$$D_2 = \Delta(b, \varepsilon_2) = 1 - b(1 - \varepsilon_2), \qquad D_1 = \frac{\varepsilon_1}{\varepsilon_2} D_2.$$

Generalized Mittal-Phamdo Scheme

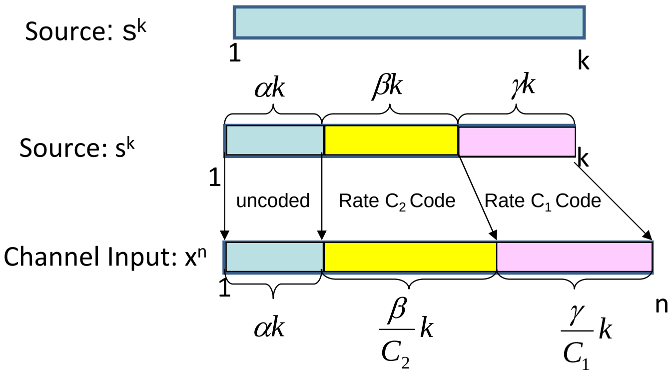
Erasure Setup



- Split s^k into three groups
- First $\alpha \cdot k$ symbols: transmit uncoded
- Next $\beta \cdot k$ symbols: apply rate $C_2 = (1 - \varepsilon_2)$ code
- Last $\gamma \cdot k$ symbols: apply rate $C_1 = (1 - \varepsilon_1)$ code

Generalized Mittal-Phamdo Scheme

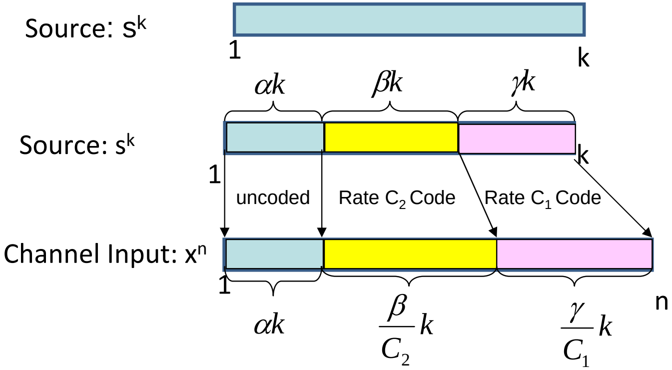
Erasure Setup



- Bandwidth expansion: $b = \alpha + \frac{\beta}{1-\varepsilon_2} + \frac{\gamma}{1-\varepsilon_1}$
- User 1 recovery: $\alpha(1-\varepsilon_1) + \beta + \gamma$
- User 2 recovery: $\alpha(1-\varepsilon_2) + \beta + \gamma(1-\varepsilon_2)$

Generalized Mittal-Phamdo Scheme

Erasure Setup



$$\min_{\alpha,\beta,\gamma} \left\{ \alpha + \frac{\beta}{1-\varepsilon_2} + \frac{\gamma}{1-\varepsilon_1} \right\}$$

$$\text{s.t. } \alpha + \beta + \gamma \leq 1, \alpha, \beta, \gamma \geq 0,$$

$$\alpha(1-\varepsilon_1) + \beta + \gamma \geq 1 - D_1,$$

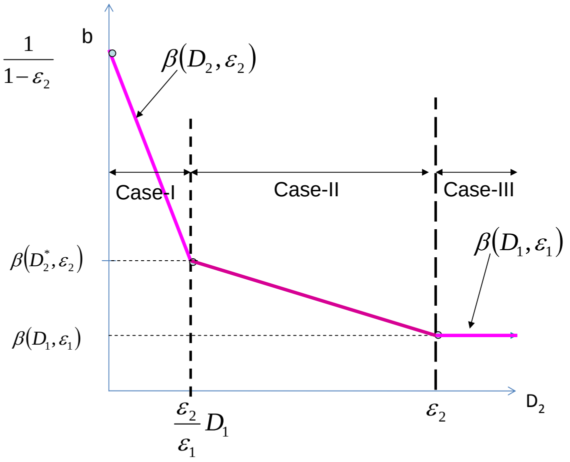
$$\alpha(1-\varepsilon_2) + \beta + \gamma(1-\varepsilon_2) \geq 1 - D_2.$$

Solution to LP Program

- Case 1: $D_2 \in [0, D_1 \frac{\varepsilon_2}{\varepsilon_1}]$
  - $b^\star = \beta(D_2, \varepsilon_2)$
  - $\alpha = \frac{1-D_2-\beta}{1-\varepsilon_2}$, $\beta \in \left[1 - \frac{D_2}{\varepsilon_2}, \left(\frac{1-D_2}{1-\varepsilon_2} - \frac{1-D_1}{1-\varepsilon_1}\right) \frac{(1-\varepsilon_1)(1-\varepsilon_2)}{\varepsilon_2-\varepsilon_1}\right]$, $\gamma = 0$.
- Case 2: $D_2 \in [D_1 \frac{\varepsilon_2}{\varepsilon_1}, \varepsilon_2]$
  - $b^\star = b^{\text{Inner}}$
  - $\alpha = \frac{D_1}{\varepsilon_1}$, $\beta = 1 - \frac{D_2}{\varepsilon_2}$, $\gamma = \frac{D_2}{\varepsilon_2} - \frac{D_1}{\varepsilon_1}$
- Case 3: $D_2 \in [\varepsilon_2, 1]$
  - $b = \beta(D_1, \varepsilon_1)$
  - $\alpha = \frac{1-D_1-\gamma}{1-\varepsilon_1}$, $\beta = 0$, $\gamma = \left[1 - \frac{D_1}{\varepsilon_1}, \left(\frac{1-D_1}{1-\varepsilon_1} - \frac{1-D_2}{1-\varepsilon_2}\right) \frac{(1-\varepsilon_1)}{\varepsilon_1}\right]$

Bandwidth-Distortion Tradeoff

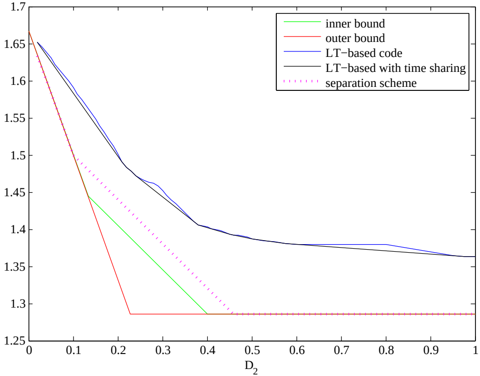
Fix D_1, Find b vs D_2



For Hamming Distortion, Improved Outer Bound: Tan-K-Soljanin ('13)

Bandwidth-Distortion Tradeoff

Li-Soljanin-Spasojevic '11



Optimization Problem for Systematic Rateless Codes

$$\text{subject to}: \min_{b,p_1,\ldots,p_L} b$$

$$-\ln(\varepsilon_i) + \ln(1-u) + (1-\varepsilon_i)(b-1)P'(u) \geq 0,$$

$$\forall u \in [0, 1-D_i], i = 1, 2$$

Conclusions

Summary:

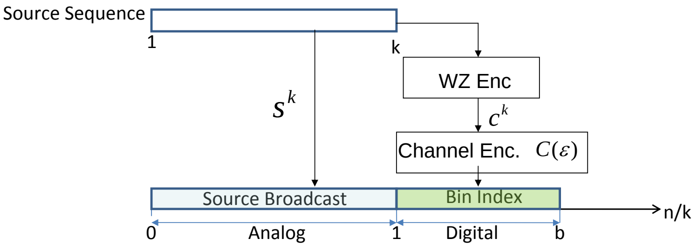
- Lossy Broadcasting to Heterogenous Receivers
- JSCC Perspective involving Erasure Channels
- Generalization of Mittal-Phamdo Scheme
- Practical Code Designs

Future Work:

- Extension to more than two receivers.
- Robust Extensions
- Unequal Bandwidth

Binary Source, Erasure Channel

Systematic Lossy Coding



- Distortion in Analog Phase: ε
- Distortion in W-Z codeword: $d(s^k, c^k) \approx \frac{D^\star}{\varepsilon}$
- Overall Reconstruction Distortion: $D^\star = \Delta(b, \varepsilon)$