Experiment 2: Z-Transform

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Abstract
This experiment will deal with the placement of poles and zeros in the unit circle, and how this placement affects the transfer function of a system (and vice-versa). At the end, you will design an oscillator, which consists of a pole located on the unit circle at a certain angle, which determines the frequency of the oscillation.

Keywords
transfer function — difference equations — LTI systems — poles — zeros — unit circle — stability

Contents

Introduction

In this lab, you will study linear time-invariant (LTI) discrete-time systems characterized by a difference equation that translates into a system function, also known as the transfer function. You will experiment with the location of the poles and zeros of such system function and verify how that affects frequency response, phase response and stability. At the end, it is hoped that you will appreciate the relation between pole/zero location and system response, feedback and instability, and maybe even how numeric precision on a DSP may affect the behaviour of a system.

This will be done in three main parts:

- First, you will understand the *system function*;
- Second, you will look at two simple systems, identify their poles and zeros and derive the system magnitude and phase responses; and
- Finally, you will modify parameters of the *system function* on more complex systems to deliberately cause features to appear or modify features in the system responses.

At the end, you will see how you would make the system do what you want (that is, if it can be done), and you will see that sometimes any small error can cause a very undesireable outcome. That is, in fact, one of the beauties of control theory, but that is another story.

Throughout this document, you will find in general the terminology and flow of standards textbooks in the field [1]. You may use other books such as [3] as well if you so desire, but be careful with the terminology, as one can get lost very fast. In this lab, you will deal *exclusively* with causal systems. Also, to be consistent with the lectures, this document will use the term *transfer function* rather than *system function* as the text uses. Keep this in mind as you go back and forth from this outline to the textbook. In fairness, a great book for studying this particular topic is [2], Chapter 6.

You know that in the time domain a causal LTI system is characterized by its impulse response $h[n]$, and in the frequency domain it is characterized by its frequency response $H(e^{j\omega})$, assuming it exists. For the system to be causal, $h[n] = 0$ for $n < 0$. The function $H(e^{j\omega})$ is the discrete-time Fourier transform (DTFT) of $h[n]$. Related to the frequency response function is the transfer function, $H(z)$, which is the z-transform of $h[n]$. By causality, this is the 1-sided z-transform. Finally, the frequency response and the transfer function are related by

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}\,.$$

The frequency response should be seen as gain and phase changes applied by the system onto the input. This occurs for every individual frequency within the frequency band $[-\pi,\pi)$. For example, suppose that the input is one sinusoid, such as $x[n] = \sin(\pi n/10)$. Assuming stability, in steady state the output $y[n]$ is a sinusoid of exactly the same frequency, $\pi/10$ radians per sample, but of amplitude $|H(e^{j\pi/10})|$ and with a phase difference of $\angle H(e^{j\pi/10})$ from the input. An interesting example is $H(z) = z^{-2}$, for which $\angle H(e^{j\omega})$ is a linear function of ω and $y[n] = x[n-2]$, a delayed version of the input. Linear phase filters are important in audio applications.

Finally, note that a discrete-time system frequently is embedded in a continuous-time system as in Figure 1.

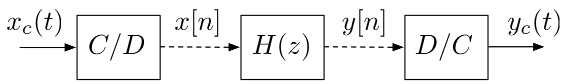


Figure 1. Embedded discrete-time system.

This implements a continuous-time frequency response $H_c(j\Omega)$ from $x_c(t)$ to $y_c(t)$ provided the input is bandlimited to frequencies less than the Nyquist rate $f_s/2$.

Throughout this lab you will see that some changes imposed by the system to the input are desireable, some are not. Your task as the engineer is to identify what are these changes (i.e., measure them) and then determine whether you want to manipulate them further or not.

Note that the lab **answer sheet** is to be done in groups of two students. You should print the answer sheet to bring to the lab. In the lab, then, you will follow the procedure below on the computer and record your answers for the remaining questions on the answer sheet.

1. Experiment

You will first explore the characteristics of two simple systems: a differentiator and an integrator. You will study the location of poles and zeros and how they determine the frequency response in magnitude and phase. Then you will look at two more complex systems and will try to manipulate their parameters in order to achieve a certain result. This should give you a better understanding of your freedoms and limitations in making changes to discrete-time systems.

1.1 The Transfer Function

Your previous course on Signals and Systems should have made you familiar with transitioning from time domain to frequency domain and vice-versa. Discrete time is similar, except for the fact that you are now operating with sampled signals. You should think of signals and system responses in terms of "sequences" of numbers. Some of them will represent voltage, some gain, some phase shift. Below you will find a more appropriate mathematical development, taken from Chapter 5 of [1]. Towards the end of this lab, some notions will come from Chapter 6 of the text.

1.1.1 Problem Definition

In time domain, an LTI system can be characterized by its impulse response $h[n]$. Given an input, the output is produced by the convolution of that input and the system impulse response:

$$y[n] = x[n] * h[n] \tag{1}$$

The impulse response relates to the system frequency response through the Fourier Transform, providing this frequency response exists. You know that the z-transform is a generalization of the Fourier transform. Therefore

$$Y(z) = X(z)H(z) \qquad or \qquad Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) \tag{2}$$

$H(z)$ is called the *transfer function*, and $H(e^{j\omega})$ is the frequency response. Note that the complex exponential yields magnitude and phase, therefore the frequency response will be represented by two plots over frequency: one indicating the variation in gain (or magnitude) and the other indicating phase shift as it occurs over the frequency range.

We assume $H(z)$ is rational, a ratio of polynomials in z. It is convenient to write the numerator and denominator as polynomials in z^{-1}:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

The numerator and denominator can then be factored as monomials:

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})} \tag{3}$$

In the numerator product, if any of the factors results in zero, the whole numerator goes to zero, bringing $H(z)$ to zero. Likewise, in the denominator product, if any of the factors results in zero, the denominator goes to zero, bringing $H(z)$ to *infinity*. Whenever you identify a z that brings the transfer function to zero, you have identified a *zero* (surprised?). Likewise, whenever you identify a z that brings your transfer function to infinity, you have identified a *pole*. Poles and zeros will interact with each other and determine the frequency response (its magnitude and phase), as you will see next with examples.

From this point on, you should start referring to the answer sheet and recording your answers.

1.2 The Differentiator

You will now study one of the simplest systems that serves a variety of purposes. You have already seen it in basic circuit theory, both in passive (i.e., as an RC circuit) and active (i.e., with an Op-Amp) configurations. Now you will see it in discrete time, analyze its poles and zeros and modify things to see what happens. The transfer function for the differentiator you will study is this:

$$H(z) = 1 - bz^{-1}$$

The system has one zero and one pole. Their positions determine the magnitude and phase on the frequency response.

Before you start answering the questions pertaining to the differentiator, you may find it helpful to explore the `FDA Tool` found within `MATLAB`. You can open it by typing `fdatool` at the `MATLAB` propmpt, or you can go a longer way and look for it within `Simulink`. For the latter, do this:

- open `Simulink`, by typing "simulink" at the prompt or by clicking the `Simulink` icon;
- open a blank model, then click on the `Library Browser`;
- open the `DSP System Toolbox` from the blockset tree at the left-hand panel;
- within the `DSP System Toolbox`, open `Filtering` and then `Filter Implementations`;
- drag the block labelled `Digital Filter Design` and drop it into your blank project;
- double-click on the `Digital Filter Design` block from within the project and it will open up the `FDA Tool`.
- now catch your breath.

Within the `FDA Tool`, the part that you will be exploring in this lab is found under `File/Import From Workspace`. When you select that, you will be presented with some options at the bottom half of the window, and a plot at the top half. For the `Filter Structure`, leave it as `Direct-Form II Transposed` (you will study some more of this later in the course). You are interested in manipulating the `Numerator` and `Denominator` vectors in all exercises in this experiment.

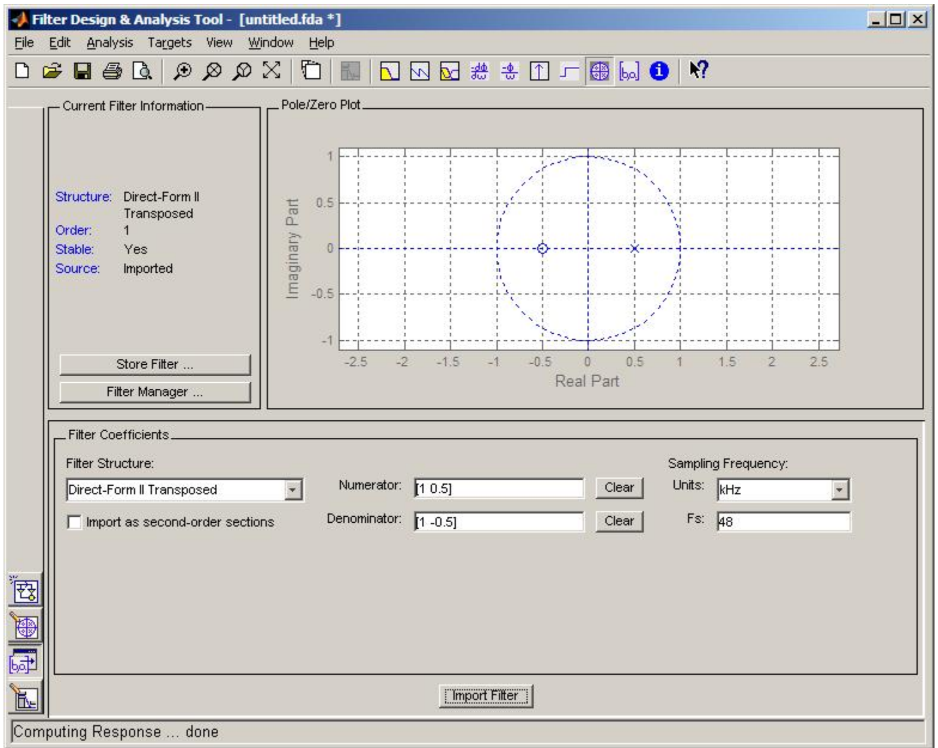


Figure 2. View of the FDA Tool on Import mode

Take the time now to try some numerator and denominator vectors to input to the `FDA Tool`. Figure 2 shows one of the views of the window that you should see, but with different numbers.

Also, explore the buttons below the top menu to see the plots for impulse response, magnitude and phase of frequency response, coefficients and pole-zero plot. It's all there. You must be careful, as the designer, in knowing what you expect to see from the plots that the tool produces. You cannot blindly believe that if it is on the computer screen it *must be* the answer you are looking for. Know what to expect beforehand. If you input your parameters carefully, the tool will provide you with the analysis you are looking for.

1.2.1 Creating a Notch in the Frequency Response

You saw above that by having a zero along the real axis of the unit circle (and a pole at the origin) you could create a notch in the magnitude of the frequency response. Suppose we want the notch to be at the frequency $\pi/2$. Then we should make b close to j. Intuitively, by controlling where this zero is located, you can place that notch in the magnitude anywhere you want.

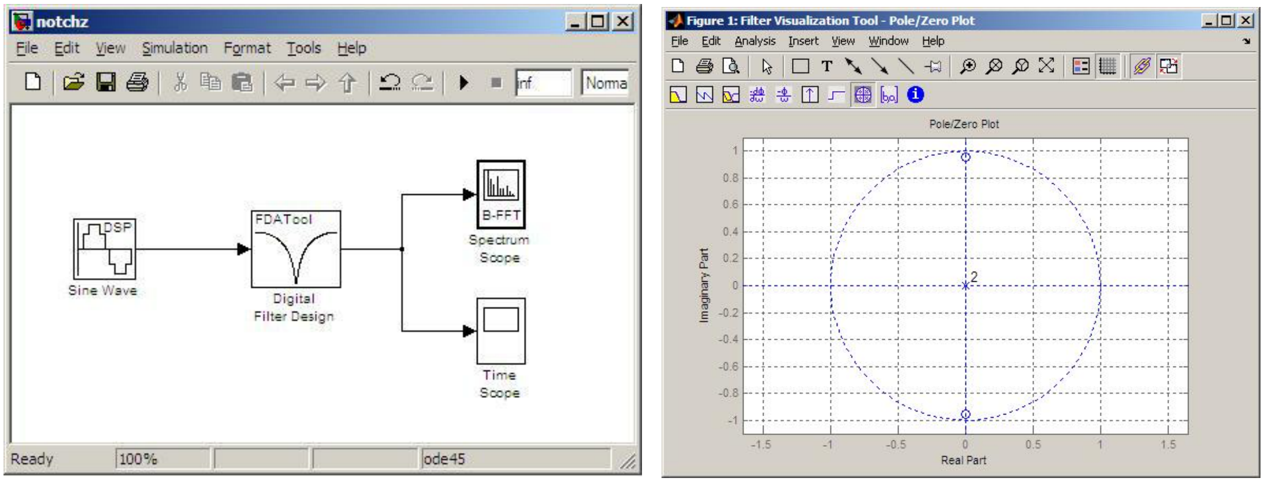
There is a catch, though. If you try to place a single zero at $b = j$, thinking that you will create a notch at $\pi/2$, you will not be able to realize (i.e., implement) your system. That is because you thought only half the way, meaning, going around the unit circle from DC up to π.

You know at this point that in discrete-time what happens between DC to π is mirrorred between π and 2π, so your system at this point is incomplete. The solution is to try to implement the "other half" of the unit circle as well. A notch system that you *can* realize will have two zeros: one at the upper half of your unit circle and one at the bottom half, as well as two poles at the origin.

Now use your `Simulink` skills and try to simulate such system. You already know how to derive the transfer function for such system and how to work with the `FDA Tool`. All you need now is to put a system together. Figure 3 shows you a suggestion for a simulation system, as well as a suggestion for the pole-zero plot to be achieved.

You may think about this pole-zero interaction in the following (rather non-elegant) way: If you imagine the magnitude plot of the transfer function to be a string, a "clothes line", zeros will pull the magnitude of the transfer function down from where it is, and poles will pull it back up. Poles at the centre pull everything equally up, and zeros at the centre pull everything equally down. Poles close to the unit circle with zeros at the centre of the circle create a peak (a "pass-band"), and poles at the centre with zeros close to the unit circle create a notch (a "reject-band"). They counter-act each other.

Now that you know the trick, move on to the answer sheet to answer some questions on the differentiator and make a fancier system work.



(a) Simulation Model

(b) Pole-Zero Plot

Figure 3. Creating a Notch

1.3 The Integrator

The transfer function for the integrator you will study is

$$H(z) = \frac{1}{1 - az^{-1}} \tag{4}$$

You can see that $z = a$ is the pole. If you want your system to work, stay away from placing your pole on or outside the unit circle. Try a couple of these scenarios now, and when you are ready answer the questions pertaining to the integrator on the answer sheet.

1.4 Blowing It All Up (or Not)

Time has come for you to make stranger changes to the transfer function. Since this is a course on Digital Signal Processing, the changes you are about to propose are likely to be changes in numbers, which represent parameters of a system implemented in real-time on a microprocessor. In the analog world, a change intended to place a zero somewhere in the transfer function is likely to be physical, such as "place a damper here." For instance, such change could be the insertion of a capacitor to ground somewhere in a circuit (which equates mechanically to a damper anyway – both absorb higher frequencies or vibrations). Now, in the digital world you will have the freedom to create a mathematical representation of your system and implement it in code using a microprocessor. However, you will have limitations in terms of memory, precision, execution time, etc. You are the Engineer now: would you throw hardware at your problem or would you polish your assembly skills and make your design run lean? It's your call!

1.4.1 Generating a Sine Wave

This is only one method to generate a sine wave using a signal processor. The straightforward method would be to create a table of values and read from (selected) values cyclically. This method is explained in a variety of practical DSP books, and it is best described in [4], Chapter 5. What you seek to implement here is also called a *digital resonator*. You can look at it as an "inverted notch", that produces a sinusoidal output upon being presented with an impulse. Such systems are used, for instance, in parametric equalizers along with notch filters.

The idea is simple. Open reference [1] to page 104, Table 3.1. You will see a number of z-transform pairs. Pairs number 11 and 12 indicate that if you implement a system whose transfer function $H(z)$ is the one found under the *Transform* column, and present such system with an impulse, the output will be a scaled cosine or sine. This is exactly what you will do here. You just need to pick the frequency, and use that frequency and the right scale to make your system work. The relevant z-transform pair is

$$(\sin \omega_0 n)u[n] \quad \leftrightarrow \quad \frac{(\sin \omega_0) z^{-1}}{1 - [2\cos \omega_0] z^{-1} + r^2 z^{-2}} \tag{5}$$

You will implement the transfer function below and give it an impulse. This means that you will present to it an input with a high energy lasting a very short period of time. Yes, think of it as a “bang”.The system will then resonate at the frequency you selected.

$$H(z) = \frac{(\sin\omega_0)z^{-1}}{1-(2\cos\omega_0)z^{-1}+r^2z^{-2}} \quad (6)$$

Now that this is all figured out, move on to the answer sheet and fill it out as to baffle the TAs with brilliance. You can also expand the breadth of your knowledge and look up the very interesting early studies on resonators done by Hermann von Helmholtz, which eventually lead to a theory on how the cochlea works, and to a better understanding on how people hear and perceive sounds.

2. Accomplishments

In this lab, you learned about the role played by the roots of the denominator and the numberator of the system transfer function. The roots of the denomiator are the poles of the transfer function, while the zeros are are roots of its numerator. Their locations with respect to the unit circle determine the most significant features of the system frequency response. You started with single-zero and single-pole systems and escalated to systems with multiple poles and zeros. You have seen how changing their number, their magnitude and angle modifies sometimes drastically the behaviour of a system. It is hoped that at the end of this lab you are able to assess the response of a system and propose improvements and changes by manipulating the position of poles and zeros, thus modifying the system transfer function.

Acknowledgments

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References

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