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THE SAMPLING THEOREM

Question: How often must I sample an analog signal in order not to lose information from it.

Answer: If the highest frequency contained in a signal $x_a(t)$ is Ω_0 and the signal is uniformly sampled at a rate $\Omega_s \geq 2\Omega_0$, then $x_a(t)$ can be exactly recovered from its sample values using the interpolation function

$$g(t) = \frac{\sin\left[\Omega_s t/2\right]}{\Omega_s t/2}$$

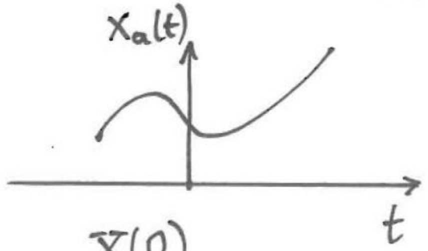
and then, $x_a(t) = \sum_{k=-\infty}^{\infty} x_a(kT)\, g(t-kT)$, where $\{x_a(kT)\}$ are the samples of $x_a(t)$ taken with rate $T = 2\pi/\Omega_s$. see.

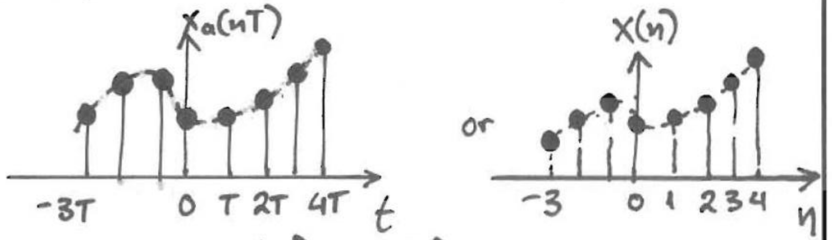
Comments:
- "Aliasing" that is spectral overlapping is caused by inadequate sampling
- Aliasing is not reversable
- Assuming no aliasing has occured $x_a(t)$ can be reconstructed via ideal LP filtering
- In practice: always choose a rate $\Omega > 2\Omega_0$; there is a need for antialiasing filtering; there are no "ideal" LP filters.

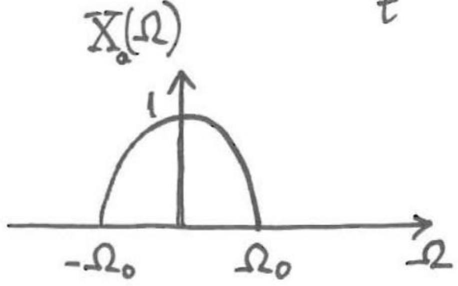
ANALOG TO DIGITAL CONVERSION (A/D)

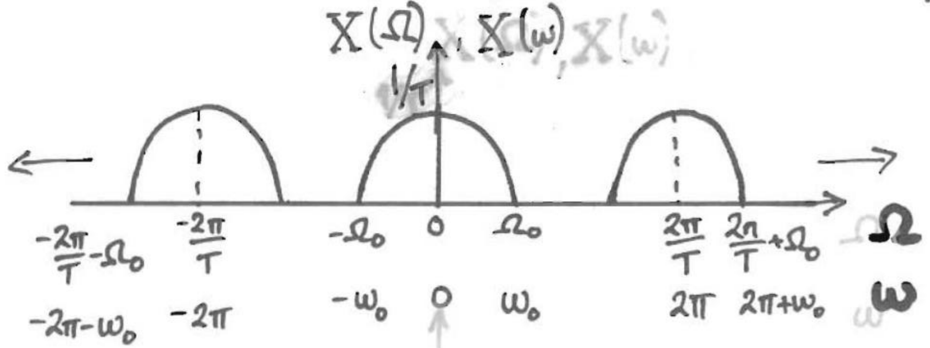
- **Ideal Sampling (uniform)**: take one sample every T seconds (sampling period, $F_s = \frac{1}{T}$ cycles/sec)

Analog signal: $x_a(t) \xrightarrow{F_s}$ Discrete signal: $x(n) = x_a(nT)$, $-\infty < n < \infty$









$$x_a(t) \xleftrightarrow{\text{F.T.}} X_a(\Omega)$$

$$x_a(nT) \xleftrightarrow{\text{DTFT}} X(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(\Omega + k\frac{2\pi}{T}\right)$$

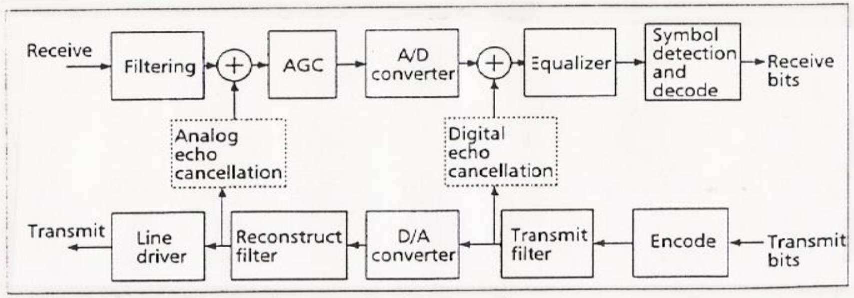
$$x(n) \xleftrightarrow{\text{DTFT}} X(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a(\omega + k2\pi)$$

$$\omega = \Omega T \quad , \quad f = \frac{F}{F_s}$$

ω: Normalized frequency in rads/sample

Ω: Real frequency in rads/sec

ANALOG TO DIGITAL CONVERSION (A/D)



■ **Figure** *A generic digital communication transceiver.*

* Analog and digital signal processors/units are used simultaneously.

* Since 1990 the system design approach places more emphasis on digital processing

* Multiple A/D, D/A conversions may be required

Type	Number of bits	Signal bandwidth	Sampling rate	Latency
Flash	6–8	$<f_s/2$	150–300 ms/s	>1 clock cycle
Two-step	≈ 10–12	$<f_s/2$	10–100 ms/s	>2 clock cycles
Pipeline	10–15	$<f_s/2$	1–100 ms/s	> # of stages
Algorithmic	12–14	$<f_s/2$	10 ks/s to< 1 ms/s	> # of bits
Succ. approx.	10–14	$<f_s/2$	100 ks/s to 5 ms/s	> # of bits
Lowpass Σ–Δ	14–20	20–1000 kHz	2–50 MHz	N/A
Bandpass Σ–Δ	8–15	30–1250 kHz	2–200 MHz	N/A

■ **Table 1.** *Typical state-of-the-art performance of CMOS- and BiCMOS-based A/D converters.*

Nyquist rate converters: quantize input samples every $1/f_s$ sec, ($f_s >$ Nyquist rate); generate one digital sample from a single analog sample (good performance in the 10-12 bit/sample range)

Noise shaping converters: oversample analog signal; generate one digital sample by weighting many input analog samples; (Good performance in the 13-15 bit range)

* Digital converters for signals with bandwidths greater that 100 KHz are commonly used.

"Data Converters for Communication Systems"
From: IEEE Communications Magazine, October 1998, page 113

THE CONCEPT OF FREQUENCY FOR DISCRETE TIME SIGNALS

$x(n) = e^{j\omega n}$, $n = 0, \pm 1, \pm 2, \ldots$

- ω: rads/sample
- periodic only iff $f = \frac{\omega}{2\pi}$ is rational that is $f = \frac{K}{N}$ where K, N co-prime integers then $x(n) = x(n + \ell N)$
- Let $\omega_1 = \omega_2 + 2K\pi$. Then $e^{j\omega_1 n} = e^{j\omega_2 n}$
  Thus distinct discrete exponentials (complex) can be obtained only in intervals of 2π
  i.e., $-\pi \leq \omega \leq \pi$ or $0 \leq \omega \leq 2\pi$ ......
  $-\frac{1}{2} \leq f \leq \frac{1}{2}$ $\quad 0 \leq f \leq 1$
- The highest rate of oscillation for a discrete complex exponential is achieved for $\omega = \pi$ or $f = 1/2$

$x_K(n) = e^{jK\frac{2\pi}{N}n}$, $n = 0, \pm 1, \pm 2, \ldots$ N: integer

N: integer

- periodic in both K and n with period N samples that is
  $$x_{K+N}(n+N) = x_K(n)$$
- Thus, there are only N distinct periodic exponentials (complex) of period N
- Highest oscillation rate for $K = \frac{N}{2}$
- Important relation.
  $$\frac{1}{N}\sum_{n=0}^{N-1} e^{jK\frac{2\pi}{N}n} = \begin{cases} 1 & \text{for } K = 0, \pm N, \pm 2N \ldots \\ 0 & \text{otherwise.} \end{cases}$$

DTFT, DTFS, DFT

Given a discrete signal $x(n)$, $n=0,\pm1,\pm2,\ldots$ form $\tilde{x}(n) = \sum_{\ell=-\infty}^{\infty} x(n+\ell N)$

Ⓘ Discrete time Fourier transform pair (DTFT)

$$\left[\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \end{aligned}\right]$$

$x(n)$ discrete non-periodic

$X(\omega)$ continuous periodic (2π)

Ⓘ Discrete time Fourier Series pair. (DTFS or DFS)

$$\left[\begin{aligned} \tilde{X}(k) &= \sum_{n=0}^{N-1} \tilde{x}(n) e^{-j\frac{2\pi kn}{N}} \\ \tilde{x}(n) &= \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}(k) e^{j\frac{2\pi kn}{N}} \end{aligned}\right]$$

for $n,k = 0,1,\ldots,N-1$ or over $\langle N\rangle$

$\tilde{x}(n)$, $\tilde{X}(k)$ discrete and periodic with period N samples

- N is known a-priori or can be chosen arbitrarily
- Sufficient condition for existence $\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$

N-DFT (Discrete Fourier Transform)

- Given a discrete signal $x(n)$, $n=0,\pm1,\pm2,\dots$ with DTFT $X(\omega)$

1) Form the signal $\tilde{x}(n) = \sum_{\ell=-\infty}^{\infty} x(n+\ell N)$, N: chosen arbitrarily

2) Form the signal $\tilde{X}(k) = X(\omega)\Big|_{\omega=\frac{2\pi k}{N}}$, $k=0,\pm1,\pm2,\dots$

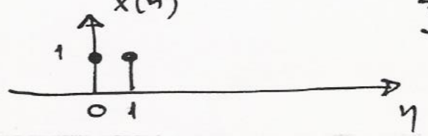
Then,

N-DFT pair

$$\left[\begin{aligned} X(k) &= \sum_{n=\langle N\rangle} \tilde{x}(n)\, e^{-j\frac{2\pi k n}{N}} \\ x(n) &= \frac{1}{N}\sum_{k=\langle N\rangle} \tilde{X}(k)\, e^{j\frac{2\pi k n}{N}} \end{aligned}\right] \xRightarrow[\text{length} \le N]{x(n)\text{ of}} \left[\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k n}{N}} \\ x(n) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi k n}{N}} \end{aligned}\right]$$

for $n,k=0,1,\dots,N-1$ or any period $\langle N\rangle$

- The N-DFT maps one period of $\sum_{\ell=-\infty}^{\infty} x(n+\ell N)$ to one period of $X\left(\omega=\frac{2\pi k}{N}\right)$
- If we do not take sufficient number of (samples per period) in the DTFT domain, then "time aliasing" occurs in time domain (dual of sampling theorem)

Ex. Let 

$$X(\omega) = 2\cos\frac{\omega}{2}\, e^{-j\omega/2} = 1 + e^{-j\omega}$$

Evaluate: 1-DFT, 2-DFT, 3-DFT, 8-DFT

A) Working in frequency domain.

$$|X(k)| = X\left(\omega = \frac{2\pi k}{N}\right), \quad k = 0, \dots, N-1$$

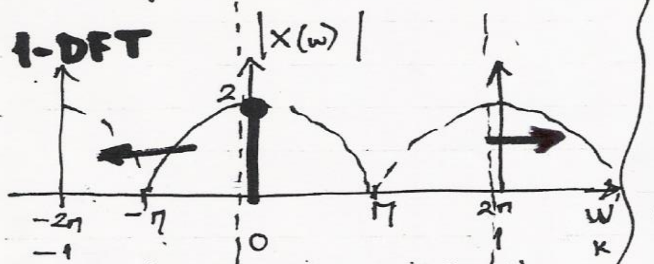
B) Time domain $\xrightarrow{N\text{-DFT}}$ Frequency domain

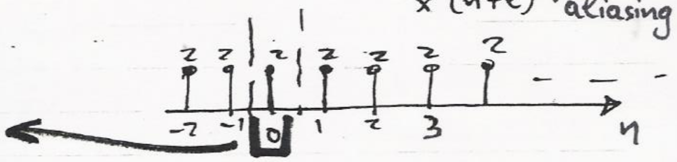
Form $\tilde{x}(n) = \sum_{\ell=-\infty}^{\infty} x(n + \ell N)$

Then

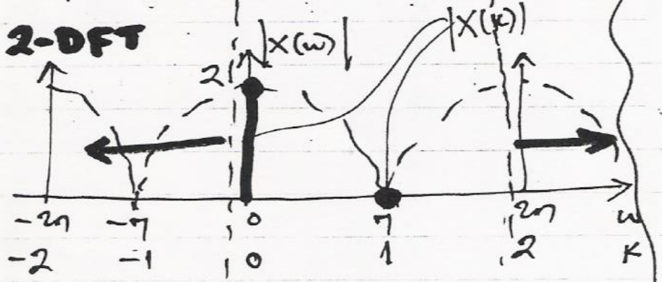
$$X(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j\frac{2\pi k n}{N}}.$$

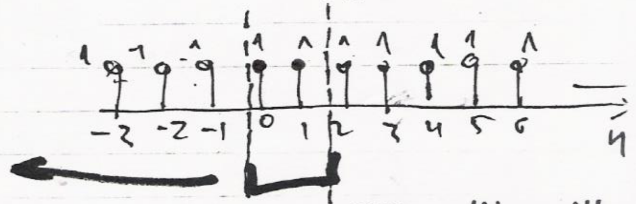
1-DFT



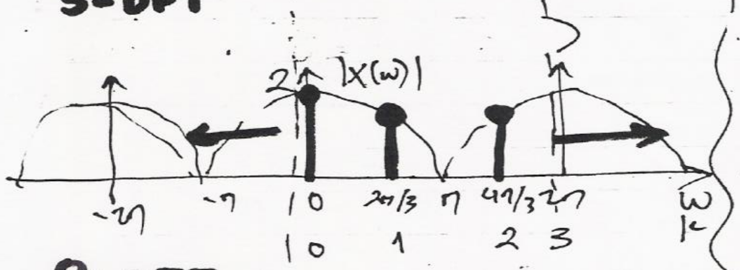


2-DFT

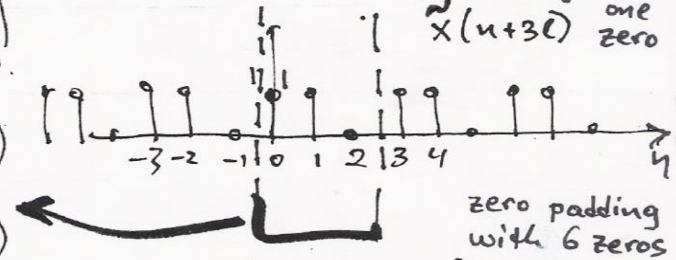




3-DFT

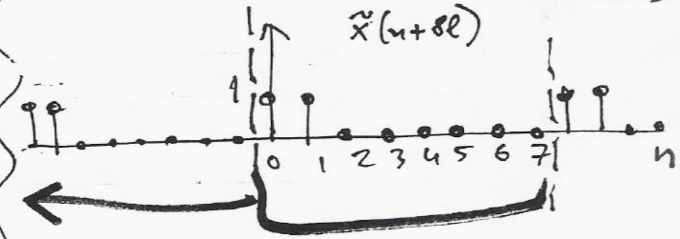


zero padding with one zero



8-DFT

zero padding with 6 zeros



ZERO PADDING – DIGITAL INTERPOLATION

Given $x(n)$ of length less or equal to N and its N-DFT $X(k)$, $k=0,1,\dots,N-1$

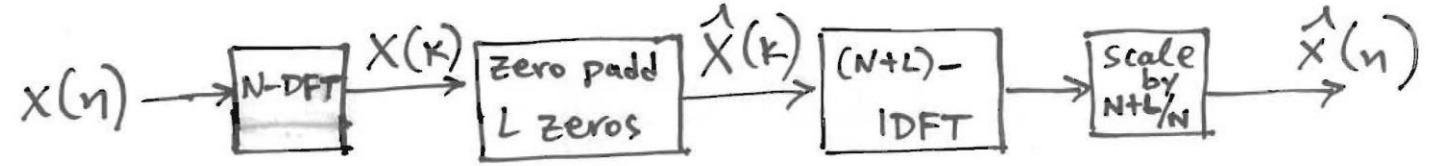
- Zero-padding increases the effective period of the N-DFT by introducing zeros in the "stop-band" of either the signal or the discrete Fourier transform

- By zero padding $x(n)$ with $K \cdot N$ zeros and taking $(K+1)N$-DFT, then K new values are interpolated between (any) two values of the N-DFT

- By zero padding $X(k)$ with $K \cdot N$ zeros (between the samples $k=N/2, N/2+1$) and taking $(K+1)N$-IDFT and multiplying by $(K+1)$, K new values are interpolated between any two values of $x(n)$.

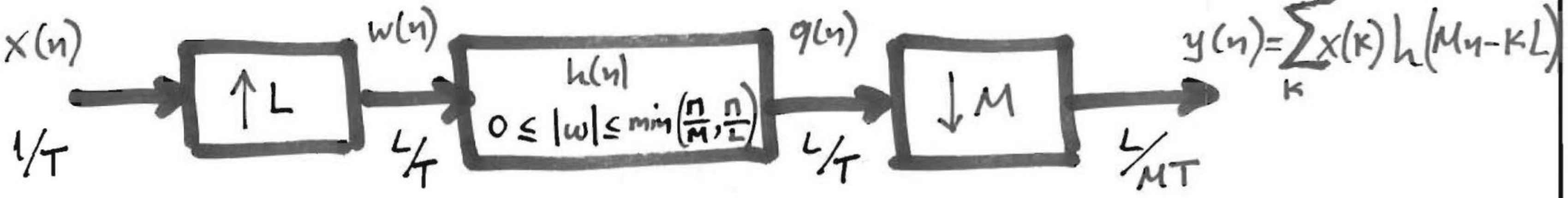
i.e

INTERPOLATION OF L values in $x(n)$:



MULTIRATE SYSTEMS (3)

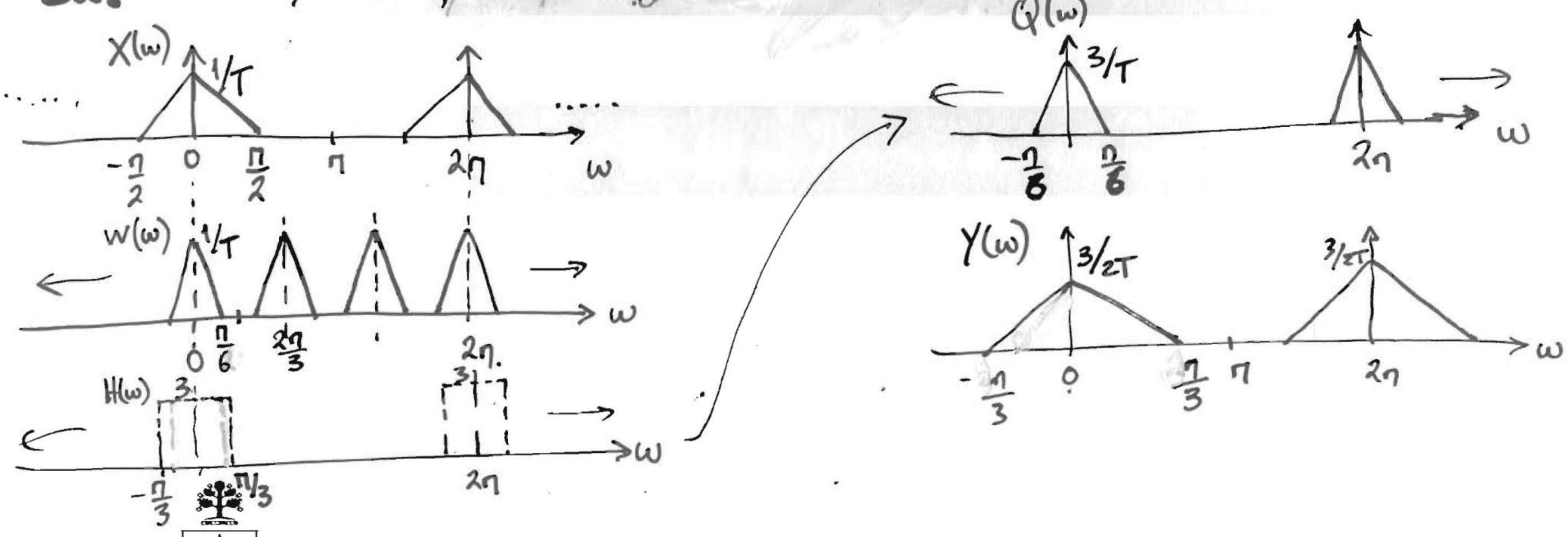
(iii) Change of sampling rate by $\frac{L}{M}$: noninteger



$$Y(\omega) = \frac{1}{M} \sum_{i=0}^{M-1} H\left(\frac{\omega - 2\pi i}{M}\right) X\left(\frac{\omega - 2\pi i}{M} L\right) \xlongequal{\text{no-aliasing}} \frac{1}{M} H\left(\frac{\omega}{M}\right) X\left(\frac{\omega L}{M}\right)$$

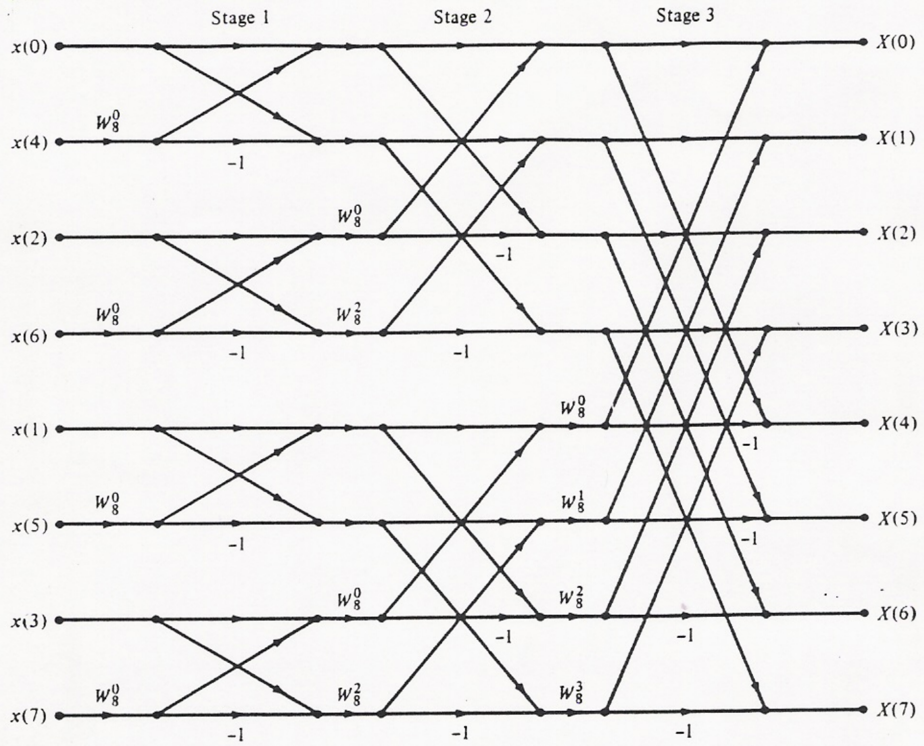
in range $0 \le |\omega| \le \pi$

Ex. $L=3$, $M=2$, no-aliasing



Radix 2 ($N=2^q$) Decimation in time FFT

Input in bit reversed order



Output in normal order

FIGURE Eight-point decimation-in-time FFT algorithm.

Decimation in time FFT is based on the calculation of succesively smaller DFT's by decomposing the signal into successively smaller sequences

<u>In general for $N=2^q$ - FFT.</u> (Decimation in time).

1) $q = \log_2 N$ stages , $N/2$ butterflies / stage

2) m-th stage , 2^{m-1} different butterflies $\left\{ \begin{array}{l} \text{Multiply lower branches of each} \\ \text{butterfly } W_N^{\ell N/2^m} \quad , \ell=0,\dots,2^{m-1}-1 \end{array} \right.$

3) # of $\otimes_c$: $1/\text{butterfly} \times \frac{N}{2}$ butt/stage $\times \log_2 N$ stages $= \frac{N}{2}\log_2 N$

of $\oplus_c$: $2/$ " $\times \frac{N}{2}$ " $\times$ " " $= N\log_2 N$

of storage locations: $2/\text{butterfly} \times \frac{N}{2}$ butt. $= \binom{\text{in place}}{\text{computation}} = N$

4) $2^{\log_2 N-1}$ exponentials stored in look up table

5) $G(N) = \dfrac{\#\text{ of FFT } \otimes}{\#\text{ of DFT } \otimes} = \dfrac{1}{2N}\log_2 N \ll 1$. For ex. $G(8) = \frac{3}{16}$

$G(4096) \simeq 0.0015$

Linear Algebra Relations

Note Title 02/09/2005

→ Vector $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $(n \times 1)$ real or complex

→ Transpose: $\underline{x}^T = [x_1, \dots, x_n]$ $(1 \times n)$

→ Hermitian transpose: $\underline{x}^H = (\underline{x}^T)^* = [x_1^*, \dots, x_n^*]$

→ Magnitude of a vector (norm or distance)

Euclidean L_2 norm: $\|\underline{x}\|_2 = \sqrt{\sum_{i=1}^{N} |x_i|^2}$

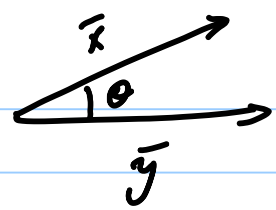
L_1 norm: $\|\underline{x}\|_1 = \sum_{i=1}^{N} |x_i|$

L_∞ norm: $\|\underline{x}\|_\infty = \max_i |x_i|$

→ distance between vectors $d(\underline{x}, \underline{y}) = \|\underline{x} - \underline{y}\|$

→ Inner product: $\langle \underline{x}, \underline{y} \rangle = \underline{x}^H \cdot \underline{y} = \sum_{i=1}^{N} x_i^* \cdot y_i$

for Euclidean space: $\langle \underline{x}, \underline{y} \rangle = \|x\| \|y\| \cos\theta$



unit norm vectors $\dfrac{\underline{x}}{\|\underline{x}\|}, \dfrac{\underline{y}}{\|\underline{y}\|}$

→ $\underline{x}, \underline{y}$: orthogonal, $\langle \underline{x}, \underline{y} \rangle = 0$

→ $\underline{x}, \underline{y}$: linearly independed if

$$a_1 \underline{x} + a_2 \underline{y} = 0 \longrightarrow a_1 = a_2 = 0$$

(in this case $\underline{x}, \underline{y}$ can be seen as the generating vectors of a 2-D space where each vector can be obtained by linear combination of $\underline{x}$ and $\underline{y}$).

Ex: Let $x[n] \longrightarrow \boxed{\text{FIR, } h[n]} \longrightarrow y[n]$

$$y[n] = \sum_{i=0}^{N-1} h[i]\, x[n-i] = \underline{h}^T \underline{x}[n]$$

where: $\underline{h}^T = [h_1, \dots, h_N]$, $\underline{x}^T[n] = [x(n), \dots, x[n-N+1]]$

Now, given a matrix $\underline{A}$: $n \times m$

→ square matrix $\underline{A}$: $n \times n$ elements

→ symmetric square matrix: $\underline{A} = \underline{A}^T$

→ Hermitian square matrix: $\underline{A} = \underline{A}^H$

Also: $(A+B)^H = \underline{A}^H + \underline{B}^H$

$(\underline{A}^H)^H = \underline{A}$

$(\underline{A}\underline{B})^H = \underline{B}^H \underline{A}^H$

→ Rank of a matrix: $\rho(\underline{A}) \leq \min(m,n)$ is the number of linearly independent columns!

Note: $\rho(\underline{A}) = \rho(\underline{A}^H\underline{A}) = \rho(\underline{A}\underline{A}^H)$

→ Inverse A^{-1} of a square matrix $\underline{A}$: $\underline{A}^{-1}\underline{A} = \underline{I}$

where $\underline{I} = \begin{bmatrix} 1 & & \underline{0} \\ & 1 & \\ \underline{0} & & \ddots 1 \end{bmatrix}$, $n \times n$ identity matrix

→ Determinant of a matrix $\underline{A}$ $(n \times n)$

$$\det(\underline{A}) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(\underline{A}_{ij})$$

where A_{ij} is the $(n-1)\times(n-1)$ matrix obtained by deleting the i-th row and j-th column of $\underline{A}$

→ An $n \times n$ matrix $\underline{A}$ is invertible iff $\det(\underline{A}) \neq 0$

Also,
$$\det(\underline{A}\underline{B}) = \det(\underline{A})\det(\underline{B})$$
$$\det(\underline{A}^T) = \det(\underline{A})$$
$$\det(a\underline{A}) = a^n \det(\underline{A})$$
$$\det(\underline{A}^{-1}) = 1/\det(\underline{A})$$

→ trace: $\mathrm{tr}(\underline{A}) = \sum_{i=1}^{n} a_{ii}$

→ Linear equations: $\underline{A}\,\underline{x} = \underline{b} \Rightarrow \underline{x} = \underline{A}^{-1}\underline{b}$

(if $\underline{A}$: $n \times m$, $n < m$ $\quad \underline{x} = \underline{A}^H(\underline{A}\underline{A}^H)^{-1}\underline{b}$: minimum norm solution

* The matrix $\underline{A}^H(\underline{A}\,\underline{A}^H)^{-1}$ is called the "pseudoinverse"

* Triangular matrix (upper or lower)

$$\det(\underline{A}) = \prod_{i=1}^{n} a_{ii}$$

* Symmetric matrix $\underline{A} = \underline{A}^T$

* Toeplitz matrix: All elements along each of the diagonals have the same value

&

etc.

* $\underline{A}$: $n \times n$ is orthogonal if $\underline{A}^T\underline{A} = \underline{I} \Rightarrow \underline{A}^{-1} = \underline{A}^T$

Thus, if $\underline{A} = [\underline{a}_1\ \underline{a}_2 \ldots \underline{a}_n]$ then $\underline{a}_i^T \underline{a}_j = \delta(i-j)$

* $\underline{A}$: $n \times n$ is unitary $\underline{A}^{-1} = \underline{A}^H$
(orthogonal & complex)

Quadratic form of a real symmetric or Hermitian matrix A: $n \times n$

$$Q_A(\underline{x}) = \underline{x}^T \underline{A} \underline{x} = \sum_{i,j=1}^{n} \sum^{n} x_i a_{ij} x_j$$

$$Q_A(\underline{x}) = \underline{x}^H \underline{A} \underline{x} = \sum \sum_{i,j=1}^{n} x_i^* a_{ij}' x_j$$

* If $Q_A(\underline{x}) > 0$ then $\underline{A}$ is positive definite ($\underline{A} > 0$) for all $\underline{x} \neq 0$

If $Q_A(\underline{x}) \geq 0$ then $\underline{A}$ is positive semidefinite

* Eigenvalues & Eigenvectors

Solve: $\underline{A} \underline{v} = \lambda \underline{v} \Rightarrow (\underline{A} - \lambda \underline{I}) \underline{v} = 0$

non-trivial solution: $\det(\underline{A} - \lambda \underline{I}) = 0$

Eigenvalues: $\lambda_1, \lambda_2, \ldots, \lambda_n$

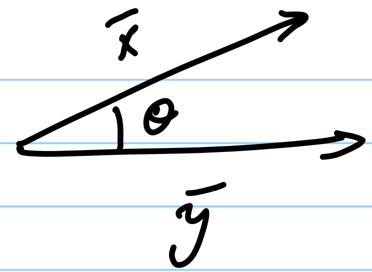
Eigenvectors: $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$

* $\underline{A}$: $n \times n$ singular matrix $\rightarrow$ $\underline{A}\,\underline{v} = 0$ and $\lambda = 0$ is an eigenvalue of $\underline{A}$
* Eigenvectors corr. to distinct eigenvalues are linearly independent
* The eigenvalues of a Hermitian matrix are real
* $\underline{A}$ Hermitian and positive definite $\Leftrightarrow$ all $\lambda_i > 0$
* $\det(\underline{A}) = \prod_{i=1}^{n} \lambda_i$, $tr(\underline{A}) = \sum_{i=1}^{n} \lambda_i$
* $\underline{A}$: Hermitian $\Leftrightarrow$ if $\lambda_i \neq \lambda_j$ then $\langle \underline{v}_i, \underline{v}_j \rangle = 0$
* Eigenvalue decomposition:

Let $\underline{\Lambda} = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$

$\underline{V} = [\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n]$

Then, $\underline{A} = \underline{V}\,\underline{\Lambda}\,\underline{V}^H = \sum_{i=1}^{n} \lambda_i \underbrace{\underline{v}_i \underline{v}_i^H}_{\text{Rank 1 Matrix}}$



unit norm vectors

$$\frac{\underline{x}}{\|\underline{x}\|}, \quad \frac{\underline{y}}{\|\underline{y}\|}$$

$\rightarrow$ $\underline{x}, \underline{y}$: orthogonal, $\langle \underline{x}, \underline{y} \rangle = 0$

$\rightarrow$ $\underline{x}, \underline{y}$: linearly independed if

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where: $\underline{h}^T = [h_1, \ldots, h_N]$, $\underline{x}^T[n] = [x[n], \ldots, x[n-N+1]]$

Optimization Theory / Gradient of a scalar

Note Title 02/09/2005

* Let $f(x)$ be a scalar function of a real variable x. Then,

if $f(x)$ is differentiable local & global minima satisfy the conditions

$$\frac{d f(x)}{dx} = 0 \quad \text{and} \quad \frac{d^2 f(x)}{dx^2} > 0$$

if $f(x)$ convex: global minimum.

* Let $f(z)$ be a scalar function of a complex variable z. Then,

if $f(z)$ is differentiable proceed as before

However, in most practical situations $f(z)$ may not be differentiable, being a function of both z and z^*.

Ex: $f(z) = |z|^2 = z z^*$

Solution:

1. Express $f(z) = f(x+jy)$ in terms of real and imaginary parts. Then, minimize w.r.t. x and y

or

2. Treat z and z^* as independent variables. Then, minimize with r.t. both z and z^*.

Ex: to find global & local minima of $|z|^2$

take $\frac{d}{dz}|z|^2 = z^* = 0$, $\frac{d}{dz^*}|z|^2 = z = 0$

* Note: If $f(z)$ is real function of z and z^* it is sufficient to minimize w.r.t. either z or z^* only. (as in above example)

* Note: The justification of the solution 2 is basically based on the observation that the obtained results from such a treatment provide meaningful solutions!!

* Now let $f(\underline{x})$ be a scalar function of a vector $\underline{x}$

Then,

$$\nabla_{\underline{x}} f(\underline{x}) = \frac{d}{d\underline{x}} f(\underline{x}) = \begin{bmatrix} \frac{\partial}{\partial x_1} f(\underline{x}) \\ \vdots \\ \frac{\partial}{\partial x_n} f(\underline{x}) \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$\underline{x}$: minimum or maximum : $\nabla_{\underline{x}} f(\underline{x}) = 0$
(necessary condition)

$\underline{x}$: minimum : Hessian $\underline{H}_x > 0$ (positive definite)

where: $\{\underline{H}_x\}(i,j) = \dfrac{\partial^2 f(\underline{x})}{\partial x_i \partial x_j}$

* Similarly, if $f(\underline{z})$ is a scalar function of a complex vector z, treat z and z^* as independent and proceed as before

Note: If $f(\underline{z}) = f(\underline{z}, \underline{z}^*)$ is real the the stationary points (max or min) are the solutions of the equation $\nabla_{\underline{z}^*} f(\underline{z}, \underline{z}) = 0$

Example:

Let $\underline{z} = [z_1, \ldots, z_n]$ be a complex vector

$\underline{R}$: be a positive definite Hermitian matrix

$\underline{a}$: be a given complex vector

find $\underline{z}$ that minimizes $\underline{z}^H \underline{R}\,\underline{z}$ s.t.c. $\underline{z}^H \underline{a} = 1$

Solution: (possible approach)

Use the Lagrange multiplier λ and minimize the unconstrained function:

real scalar $\rightarrow$ $$Q(\underline{z}, \lambda) = \frac{1}{2}\underline{z}^H \underline{R}\,\underline{z} + \lambda(1 - \underline{z}^H \underline{a})$$

thus:
$$\nabla_{z^*} Q(\underline{z}, \lambda) = \underline{R}\,\underline{z} - \lambda \underline{a} = 0 \qquad (1)$$

$$\Rightarrow \quad \underline{z} = \lambda \underline{R}^{-1} \underline{a}$$

Also,
$$\frac{\partial Q(\underline{z}, \lambda)}{\partial \lambda} = 1 - \underline{z}^H \underline{a} = 0 \qquad (2)$$

(1) $\rightarrow$ (2) :
$$\lambda = \frac{1}{\underline{a}^H \underline{R}^{-1} \underline{a}} \qquad (3)$$

(3) $\rightarrow$ (1) :
$$\underline{z} = \frac{\underline{R}^{-1} \underline{a}}{\underline{a}^H \underline{R}^{-1} \underline{a}} \qquad (4)$$

Q : scalar, a, b : real vectors, $\underline{B}$: real matrix $(N \times N)$

$$\begin{cases} \nabla_{\underline{a}} (\underline{b}^T \underline{a}) = \nabla_{\underline{a}} (\underline{a}^T \underline{b}) = \underline{b} \\ \nabla_{\underline{a}} (\underline{a}^T \underline{B}\, \underline{a}) = (\underline{B} + \underline{B}^T)\, \underline{a} \end{cases}$$

Q : scalar, a, b : complex vectors, $\underline{B}$: complex Matrix

$$\begin{cases} \nabla_{\underline{a}} Q = \frac{1}{2} \left(\nabla_{\underline{a}_r} Q - j \nabla_{\underline{a}_i} Q \right) \\ \nabla_{\underline{a}^*} Q = \frac{1}{2} \left(\nabla_{\underline{a}_r} Q + j \nabla_{\underline{a}_i} Q \right) \\ \nabla_{\underline{a}} (\underline{b}^H \underline{a}) = 0 \\ \nabla_{\underline{a}^*} (\underline{a}^H \underline{b}) = \underline{b} \\ \nabla_{\underline{a}} (\underline{a}^H \underline{B}\, \underline{a}) = \underline{B}^T \underline{a}^* \\ \nabla_{\underline{a}^*} (\underline{a}^H \underline{B}\, \underline{a}) = \underline{B}\, \underline{a} \end{cases}$$