ECE 1511F

Higher-Order Spectral (H.O.S.) Analysis

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I Introduction

In recent years, digital signal processing methods with higher-order statistics or spectra (H.O.S.) have been widely investigated. Numerous algorithms of this nature have been proposed for applications in areas such as communication systems, spectral analysis and system identification , time delay estimation, array processing, and image processing. It is now well established that H.O.S. based techniques possess many attractive features compared to the traditional second-order statistics (autocorrelation) based methods. Among such features are: higher degree of observation noise reduction, ability to preserve nonminimum phase information, and ability to detect/ identify nonlinearities.

Signal processing approaches that utilize second-order statistics draw their justification from the Gaussian assumption. When the observed signal is Gaussian distributed all the information is contained in its first and second order statistics and nothing can be gained by calculating the H.O.S. of the signal. However, when the observed signal is non-Gaussian, then there is a lot of information that is not conveyed by its second order statistics and can be extracted from its H.O.S.

The following chapter serves as an overview of the definitions and properties of the H.O.S. of discrete stochastic processes and their numerous signal processing applications.

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II H.O.S. in 1-D Signal Processing

A Definitions

Below the definition and some important properties of moments and cumulants of a zero-mean stationary discrete random process are presented.

Let $\{y_k\}_{k=1}^{n}$ be a set of n random variables. The nth-order moment of these variables is defined as, [1]

$$Mom[y_1,\ldots,y_n] = (-j)^n \frac{\partial^n E\{exp(j\mathbf{v}^T\mathbf{y})\}}{\partial v_1 \partial v_2 \ldots \partial v_n}|_{v_1=v_2=\ldots=v_n=0} \quad (1)$$

where $\mathbf{v} = [v_1, v_2, \ldots v_n]^T$, $\mathbf{y} = [y_1, y_2, \ldots y_n]^T$. On the other hand, the nth-order cumulant of the random variables is defined as, [1]

$$Cum[y_1,\ldots,y_n] = (-j)^n \frac{\partial^n lnE\{exp(j\mathbf{v}^T\mathbf{y})\}}{\partial v_1 \partial v_2 \ldots \partial v_n}|_{v_1=v_2=\ldots=v_n=0} \quad (2)$$

Now, let $\{y(i)\}$, $i = 1, 2, \ldots$ be a zero-mean, nth-order stationary random process. Then the nth-order moment and cumulant sequences of $\{y(i)\}$ are defined as

$$M_{y,n}(\tau_1,\tau_2,\ldots,\tau_{n-1}) = E\{y(i)y(i+\tau_1)\ldots y(i+\tau_{n-1})\} \quad (3)$$
$$C_{y,n}(\tau_1,\tau_2,\ldots,\tau_{n-1}) = Cum[y(i), y(i+\tau_1),\ldots,y(i+\tau_{n-1})] \quad (4)$$
$$\tau_k = 0, \pm 1, \pm 2, \ldots$$

The n-th order cumulant and moment sequences are functions of $(n-1)$ lags due to stationarity assumption. Also, the n-th order cumulant is a nonlinear function of moments of order less or equal to n, that is [1]

$$C_{y,n}(\tau_1,\tau_2,\ldots,\tau_{n-1}) = \sum (-1)^{p-1}(p-1)! \cdot E\left\{\prod_{i\in I_1} y_i\right\} \cdots E\left\{\prod_{i\in I_p} y_i\right\} \quad (5)$$

where the summation covers all partitions $(I_1,\ldots,I_p)$, $p = 1, 2, \ldots, n$ of the set $(1, 2, \ldots, n)$ and $y_1 = y(i)$, $y_2 = y(i+\tau_1), \ldots, y_n = y(i+\tau_{n-1})$. The 2-nd, 3-rd and 4-th order cumulants of zero-mean processes are utilized often in practice. They take the form

$$C_{y,2}(\tau) = M_{y,2}(\tau) \quad (6)$$
$$C_{y,3}(\tau_1,\tau_2) = M_{y,3}(\tau_1,\tau_2) \quad (7)$$
$$C_{y,4}(\tau_1,\tau_2,\tau_3) = M_{y,4}(\tau_1,\tau_2,\tau_3) - M_{y,2}(\tau_1)M_{y,2}(\tau_2-\tau_3) - M_{y,2}(\tau_2)M_{y,2}(\tau_3-\tau_1) - M_{y,2}(\tau_3)M_{y,2}(\tau_1-\tau_2) \quad (8)$$

The relations between $\gamma_{y,n} = C_{y,n}(0,\ldots,0)$ and $m_{y,n} = M_{y,n}(0,\ldots,0)$ for $n \leq 8$ of a zero mean process are depicted in Tables I and II. The quantities, $\gamma_{y,2} = C_{y,2}(0)$, $\gamma_{y,3} = C_{y,3}(0,0)$, and $\gamma_{y,4} = C_{y,4}(0,0,0)$ are known as the variance, skewness, and kurtosis of $\{y(i)\}$, respectively. General relations between moments and cumulants can be found in [2,3].

Table I: n-th order moment of a zero-mean process in terms of cumulants

$$m_{y,1} = \gamma_{y,1} = 0$$
$$m_{y,2} = \gamma_{y,2}$$
$$m_{y,3} = \gamma_{y,3}$$
$$m_{y,4} = \gamma_{y,4} + 3\gamma_{y,2}^2$$
$$m_{y,5} = \gamma_{y,5} + 10\gamma_{y,2}\cdot\gamma_{y,3}$$
$$m_{y,6} = \gamma_{y,6} + 15\gamma_{y,2}\cdot\gamma_{y,4} + 10\gamma_{y,3}^2 + 15\gamma_{y,2}^3$$
$$m_{y,7} = \gamma_{y,7} + 21\gamma_{y,2}\cdot\gamma_{y,5} + 105\gamma_{y,2}^2\cdot\gamma_{y,3} + 35\gamma_{y,3}\cdot\gamma_{y,4}$$
$$m_{y,8} = \gamma_{y,8} + 28\gamma_{y,2}\cdot\gamma_{y,6} + 210\gamma_{y,2}^2\cdot\gamma_{y,4} + 105\gamma_{y,2}^4 + 280\gamma_{y,2}\cdot\gamma_{y,3}^2 + 35\gamma_{y,4}^2 + 56\gamma_{y,3}\cdot\gamma_{y,5}$$

Table II: n-th order cumulant of a zero-mean process in terms of moments

$$\gamma_{y,1} = m_{y,1} = 0$$
$$\gamma_{y,2} = m_{y,2}$$
$$\gamma_{y,3} = m_{y,3}$$
$$\gamma_{y,4} = m_{y,4} - 3m_{y,2}^2(0)$$
$$\gamma_{y,5} = m_{y,5} - 10m_{y,2}\cdot m_{y,3}$$
$$\gamma_{y,6} = m_{y,6} - 15m_{y,2}\cdot m_{y,4} - 10m_{y,3}^2 + 30m_{y,2}^3$$
$$\gamma_{y,7} = m_{y,7} - 21m_{y,2}\cdot m_{y,5} + 210m_{y,2}^2\cdot m_{y,3} - 35m_{y,3}\cdot m_{y,4}$$
$$\gamma_{y,8} = m_{y,8} - 28m_{y,2}\cdot m_{y,6} + 420m_{y,2}^2\cdot m_{y,4} - 630m_{y,2}^4 + 560m_{y,2}\cdot m_{y,3}^2 - 35m_{y,4}^2 - 56m_{y,3}\cdot m_{y,5}$$

Assuming that the process $\{y(i)\}$ is complex, then we may define moments or cumulants by introducing conjugation in one or more of the terms in the product. For example, we may define the 3-rd order moment as

$$M_{y^*yy,3}(\tau_1,\tau_2) = E\{y^*(i)y(i+\tau_1)y(i+\tau_2)\} \quad (9)$$

or

$$M_{y^*yy^*,3}(\tau_1,\tau_2) = E\{y^*(i)y(i+\tau_1)y^*(i+\tau_2)\} \tag{10}$$

and so on. For the n-th order moment or cumulant there are 2^n complex definitions. Depending on the application one of the definitions might be more appropriate or desirable than the others, however, in general all definitions provide the same degree of information.

Assuming that the n-th order cumulant of the stationary process $\{y(i)\}$ is summable, i.e., $\sum_{\tau_1=-\infty}^{\infty}\cdots\sum_{\tau_{n-1}=-\infty}^{\infty}|C_{y,n}(\tau_1,\tau_2,\ldots,\tau_{n-1})| < \infty$, then, the n-th order polyspectrum or higher-order spectrum is defined as the (n-1)-dimensional discrete Fourier transform of the n-th order cumulants, that is [1]

$$S_{y,n}(e^{j\omega_1},\ldots,e^{j\omega_{n-1}}) = \sum_{\tau_1=-\infty}^{\infty}\cdots\sum_{\tau_{n-1}=-\infty}^{\infty} C_{y,n}(\tau_1,\ldots,\tau_{n-1})\cdot\prod_{l=1}^{n-1}e^{-j\omega_l\tau_l}$$

$$|\omega_l| \le \pi, \quad l=1,\ldots,n-1, \quad \left|\sum_{l=1}^{n-1}\omega_l\right| \le \pi. \tag{11}$$

The $S_{y,2}(e^{j\omega})$ is the power spectrum, the $S_{y,3}(e^{j\omega_1},e^{j\omega_2})$ is the bispectrum, and the $S_{y,4}(e^{j\omega_1},e^{j\omega_2},e^{j\omega_3})$ is the trispectrum of $\{y(i)\}$.

A few more definitions which will be useful in the sequel are the following.

The n-th order coherence function of $\{y(i)\}$ is the normalized n-th order polyspectrum which is defined as follows [1,2]

$$R_{y,n}(e^{j\omega_1},\ldots,e^{j\omega_{n-1}}) = \frac{S_{y,n}(e^{j\omega_1},\ldots,e^{j\omega_{n-1}})}{\left[S_{y,2}(e^{j\omega_1})\cdots S_{y,2}(e^{j\omega_{n-1}})S_{y,2}(e^{-j(\omega_1+\cdots+\omega_{n-1})})\right]^{\frac{1}{2}}} \tag{12}$$

The $R_{y,3}(e^{j\omega_1},e^{j\omega_2})$ is the bicoherence and the $R_{y,4}(e^{j\omega_1},e^{j\omega_2},e^{j\omega_3})$ the tricoherence of $\{y(i)\}$.

The n-th order polycepstrum of $\{y(i)\}$ is defined as the (n-1)-dimensional inverse discrete Fourier transform of the logarithm of the n-th order polyspectrum. If we define

$$s_{y,n}(e^{j\omega_1},\ldots,e^{j\omega_{n-1}}) = ln\left[S_{y,n}(e^{j\omega_1},\ldots,e^{j\omega_{n-1}})\right] \tag{13}$$

the polycepstrum of $\{y(i)\}$ is given by

$$c_{y,n}(\tau_1,\tau_2,\ldots,\tau_{n-1}) = \mathcal{F}^{-1}\left[s_{y,n}(e^{j\omega_1},\ldots,e^{j\omega_{n-1}})\right]. \tag{14}$$

The $c_{y,2}(\tau)$ is the power cepstrum, the $c_{y,3}(\tau_1,\tau_2)$ the bicepstrum, and the $c_{y,4}(\tau_1,\tau_2,\tau_3)$ the tricepstrum of $\{y(i)\}$. Finally, one can define polycepstrum of the n-th order coherence function in a similar way.

B Properties of Moments and Cumulants

Some very important properties of the higher order moments and cumulants which are useful in applications are listed below [1-4]:

1. Given the set of constants $\{c_1,c_2,\ldots,c_n\}$, and the random variables $\{y_1,y_2,\ldots,y_n\}$, then,

$$Mom[c_1\cdot y_1, c_2\cdot y_2,\ldots,c_n\cdot y_n] = c_1c_2\cdots c_n\cdot Mom[y_1,y_2,\ldots,y_n] \tag{15}$$

$$Cum[c_1\cdot y_1, c_2\cdot y_2,\ldots,c_n\cdot y_n] = c_1c_2\cdots c_n\cdot Cum[y_1,y_2,\ldots,y_n] \tag{16}$$

2. Given the random variables $\{x_1,y_1,y_2,\ldots,y_n\}$, then,

$$Mom[y_1+x_1,y_2,\ldots,y_n] = Mom[y_1,y_2,\ldots,y_n] + Mom[x_1,y_2,\ldots,y_n] \tag{17}$$

$$Cum[y_1+x_1,y_2,\ldots,y_n] = Cum[y_1,y_2,\ldots,y_n] + Cum[x_1,y_2,\ldots,y_n] \tag{18}$$

However, given a constant c,

$$Mom[y_1+c,y_2,\ldots,y_n] = Mom[y_1,y_2,\ldots,y_n] \tag{19}$$

$$Cum[y_1+c,y_2,\ldots,y_n] = Cum[y_1,y_2,\ldots,y_n] \tag{20}$$

3. If the set of random variables $\{y(i),y(i+\tau_1),\ldots,y(i+\tau_{n-1})\}$ are divided into any two or more mutually independent subsets, then,

$$C_{y,n}(\tau_1,\tau_2,\ldots,\tau_{n-1}) = 0 \tag{21}$$

Note that $M_{y,n}(\tau_1,\tau_2,\ldots,\tau_{n-1}) \ne 0$.

4. Let the random process $\{y(i)\}$, be independent identically distributed (i.i.d.). Then,

$$C_{y,n}(\tau_1,\ldots,\tau_{n-1}) = \gamma_{y,n}\delta(\tau_1)\delta(\tau_2)\ldots\delta(\tau_{n-1}) \tag{22}$$

where, $\delta(\tau)=0 \quad \text{for} \quad \tau\ne 0$ is the delta function. It is worth noting that while higher-order cumulants of an i.i.d. process are multidimensional delta functions higher-order moments are not.

5. Let $\{x(i)\}$, $\{z(i)\}$, $i=1,2,\ldots$ be two independent stationary random processes and $y(i)=x(i)+z(i)$. Then,

$$C_{y,n}(\tau_1,\ldots,\tau_{n-1}) = C_{x,n}(\tau_1,\ldots,\tau_{n-1}) + C_{z,n}(\tau_1,\ldots,\tau_{n-1}) \tag{23}$$

However, a similar property does not hold for the $M_{y,n}(\tau_1,\ldots,\tau_{n-1})$.

6. If the random variables $\{y(i), y(i+\tau_1), \ldots, y(i+\tau_{n-1})\}$ are jointly Gaussian , then for all $n \geq 3$ the $C_{y,n}(\tau_1, \ldots, \tau_{n-1}) = 0$. Therefore, for a Gaussian process all information is contained in the first and second-order cumulants. On the other hand, for a non-Gaussian process new information is always obtained from cumulants of order greater than two. Therefore, higher-order cumulants can be utilized as a measure of non-Gaussianity.

7. Moments and cumulants are symmetric with respect to their arguments, i.e., assuming that $\lambda_1, \ldots, \lambda_n$ is a permutation of $0, \tau_1, \ldots, \tau_{n-1}$, then

$$Mom\,[y(i), \ldots, y(i+\tau_{n-1})] = Mom\,[y(i+\lambda_1), \ldots, y(i+\lambda_n)] \quad (24)$$

$$Cum\,[y(i), \ldots, y(i+\tau_{n-1})] = Cum\,[y(i+\lambda_1), \ldots, y(i+\lambda_n)] \quad (25)$$

For the n-th order moment or cumulant there are n ! such permutations. Symmetry properties exist for the polyspectra as well. For example, the bispectrum and trispectrum of a real stationary process have 12 and 96 symmetry regions, respectively.

8. For two random processes, $\{y_D(i)\}$ and $\{y(i))\}$, let $y_D(i) = y(i-D)$, where D is a constant integer. Then,

$$M_{y_D,n}(\tau_1, \ldots, \tau_{n-1}) = M_{y,n}(\tau_1, \ldots, \tau_{n-1}) \quad (26)$$
$$C_{y_D,n}(\tau_1, \ldots, \tau_{n-1}) = C_{y,n}(\tau_1, \ldots, \tau_{n-1}) \quad (27)$$

i.e., linear phase is suppressed by higher-order moments and cumulants and therefore by cumulant spectra.

It is properties 4 to 6 that make cumulants more attractive than moments in practice. That is why polyspectra of stochastic processes are defined in terms of cumulants and not moments.

Example 1.
According to property 7, the third-order cumulant has 6 symmetry regions that can be found from all permutations of the set $(0, \tau_1, \tau_2)$. Thus,

$$C_{y,3}(\tau_1,\tau_2) = C_{y,3}(\tau_2,\tau_1) = C_{y,3}(-\tau_2, \tau_1-\tau_2) =$$
$$C_{y,3}(\tau_2-\tau_1, -\tau_1) = C_{y,3}(-\tau_1, \tau_2-\tau_1) = C_{y,3}(\tau_1-\tau_2, -\tau_2)$$

The 6 symmetry regions of the third order cumulant and the 12 symmetry regions of the bispectrum are shown in Figure 1.

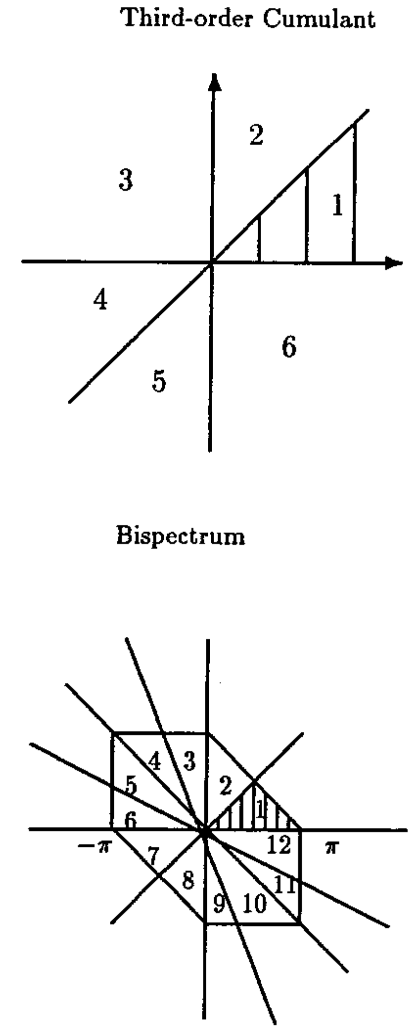


Figure 1: Symmetry regions of third-order cumulant and bispectrum

C Polyspectra and Linear Filtering

Consider the linear filtering problem depicted in Figure 2.

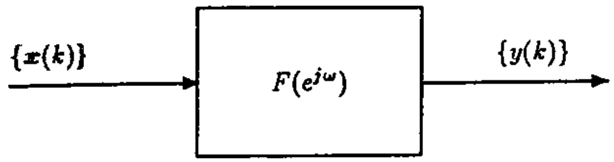


Figure 2: Discrete LTI system

The linear time invariant (LTI) system with impulse response $\{f(k)\}$ is driven by the stationary random sequence $\{x(k)\}_{k=1,2,\dots}$. We assume that the system is stable, that is $\sum_{k=-\infty}^{\infty}|f(k)| < \infty$. Then, it can be shown that the following relation holds true between the higher-order cumulants of the system output $\{y(k)\}$ and those of the system input $\{x(k)\}$ [1,3].

$$C_{y,n}(\tau_1,\dots,\tau_{n-1}) = C_{x,n}(\tau_1,\dots,\tau_{n-1}) * \left[\sum_{k=-\infty}^{\infty} f(k)f(k+\tau_1)\cdots f(k+\tau_{n-1})\right] \tag{28}$$

where, * denotes $(n-1)$-dimensional linear convolution. Relation (28) is a generalization of the well known relation with second-order statistics. By taking the (n-1)-dimensional Fourier transform of Eq. (28),

$$S_{y,n}(e^{j\omega_1},\dots,e^{j\omega_{n-1}}) = S_{x,n}(e^{j\omega_1},\dots,e^{j\omega_{n-1}}) \cdot$$
$$\left[\sum_{\tau_1=-\infty}^{\infty}\cdots\sum_{\tau_{n-1}=-\infty}^{\infty}\sum_k f(k)\cdots f(k+\tau_{n-1})e^{-j(\omega_1\tau_1+\cdots+\omega_{n-1}\tau_{n-1})}\right] =$$
$$S_{x,n}(e^{j\omega_1},\dots,e^{j\omega_{n-1}}) \cdot \left[\sum_k f(k)\prod_{l=1}^{n-1}\left[\sum_{\tau_l=-\infty}^{\infty} f(k+\tau_l)e^{-j\omega_l\tau_l}\right]\right] =$$
$$S_{x,n}(e^{j\omega_1},\dots,e^{j\omega_{n-1}}) \cdot \prod_{l=1}^{n-1}F(e^{j\omega_l}) \cdot \left[\sum_k f(k)\left[\prod_{l=1}^{n-1}e^{j\omega_l k}\right]\right] \Longrightarrow$$

$$S_{y,n}(e^{j\omega_1},\dots,e^{j\omega_{n-1}}) = S_{x,n}(e^{j\omega_1},\dots,e^{j\omega_{n-1}}) \cdot F(e^{j\omega_1})F(e^{j\omega_2})\cdots F(e^{j\omega_{n-1}})F(e^{-j(\omega_1+\cdots+\omega_{n-1})}). \tag{29}$$

Of special interest is the case where the input sequence $\{x(i)\}$ is i.i.d..

Then,

$$C_{y,n}(\tau_1,\dots,\tau_{n-1}) = \gamma_{x,n} \cdot \left[\sum_{k=-\infty}^{\infty} f(k)f(k+\tau_1)\cdots f(k+\tau_{n-1})\right] \tag{30}$$

$$S_{y,n}(e^{j\omega_1},\dots,e^{j\omega_{n-1}}) = \gamma_{x,n} \cdot F(e^{j\omega_1})\cdots F(e^{j\omega_{n-1}})F(e^{-j(\omega_1+\cdots+\omega_{n-1})}) \tag{31}$$

Example 2.
Let $\{x(k)\}$ be an i.i.d., non-Gaussian random process and $F(e^{j\omega}) = (1+a_1e^{-j\omega})\cdot(1+a_2e^{-j\omega})$. Then, from Eq. (31)

$$S_{2,y}(\omega_1) = \gamma_{2,x} \cdot F(e^{j\omega_1})F(e^{-j\omega_1}) \tag{32}$$
$$S_{3,y}(\omega_1,\omega_2) = \gamma_{3,x} \cdot F(e^{j\omega_1})F(e^{j\omega_2})F(e^{-j(\omega_1+\omega_2)}) \tag{33}$$

Let us consider the following four cases for $F(e^{j\omega})$:

1. $a_1 = a,\ a_2 = b,$ $\quad |a|<1, |b|<1$
2. $a_1 = a,\ a_2 = \frac{1}{b^*},$ $\quad |a|<1, |b|<1$
3. $a_1 = \frac{1}{a^*},\ a_2 = b,$ $\quad |a|<1, |b|<1$
4. $a_1 = \frac{1}{a^*},\ a_2 = \frac{1}{b^*},$ $\quad |a|<1, |b|<1$

where, $\{*\}$ denotes conjugation. The first system is minimum phase (i.e, all its' zeros are inside the unit circle) , the second and third systems are mixed phase, and the fourth system is maximum phase (i.e., all its zeros are outside the unit circle) as it is illustrated in Figure 3. Any one of the systems is obtained from the others by replacing one or two zeros with their conjugate mirror images with respect to the unit circle. Using Eqs. (32) and (33), it can be easily shown that the outputs of the above systems have identical power-spectrum, however different bispectrum. The power spectrum domain does not recognize the different phase character of the above systems and thus, unique identification of any of these systems is not possible. Otherwise stated, the power spectrum views all systems as being minimum phase. On the other hand, polyspectra of order greater than two preserve the true phase character of the system and thus are able to identify the system correctly up to a sign and possibly a linear phase term [1,5]. In general, if the system has n zeros, then, there are $2n-1$ other systems with the same power spectrum, but with different polyspectra.

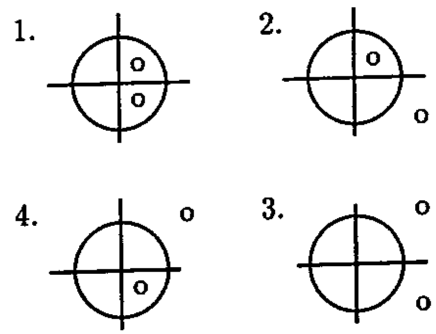


Figure 3: Zero diagram of one minimum phase, two mixed phase, and one maximum phase systems with equivalent power spectrum.

Note that if the input sequence $\{x(k)\}$ is Gaussian and the LTI system is nonminimum phase, then the $F(e^{j\omega})$ cannot be identified correctly by either second or higher-order cumulants of the system output [5,6]. This is a result of property 6 of cumulants.

Example 3.
The model depicted in Figure 4 represents a typical communication link subject to linear distortion and additive noise.

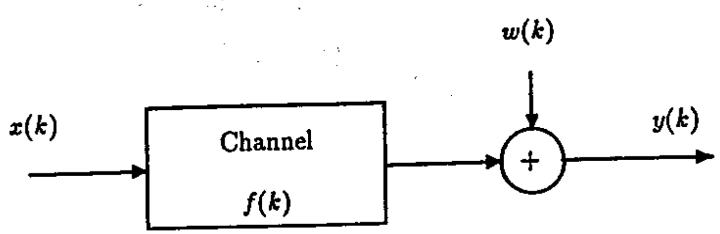


Figure 4: Linear communication channel with additive noise

The data sequence $\{x(k)\}$ is i.i.d. with a symmetric zero-mean non-Gaussian probability density function (p.d.f.), the $\{f(k)\}$ is the impulse response of a stable nonminimum phase LTI system that represents the communication channel, and $\{w(k)\}$ is additive stationary zero-mean Gaussian noise. It is assumed that the noise is statistically independent from the input data. Then, by applying the cumulant properties presented previously, the following relations can be written:

$$y(k) = \sum_n f(n) \cdot x(k-n) + w(k) \quad (34)$$

$$C_{y,1} = 0 \quad (35)$$

$$C_{y,2}(\tau) = \gamma_{x,2} \cdot \sum_k f(k)f(k+\tau) + C_{w,2}(\tau) \quad (36)$$

$$C_{y,3}(\tau_1, \tau_2) = 0 \quad (37)$$

$$C_{y,4}(\tau_1, \tau_2, \tau_3) = \gamma_{x,4} \cdot \sum_k f(k)f(k+\tau_1)f(k+\tau_2)f(k+\tau_3) \quad (38)$$

and so on. We observe that the second-order cumulants of the output are corrupted by noise, and that the third (all odd) order cumulants of the output are zero due to the symmetric p.d.f. of the data and the noise. On the other hand, the fourth order cumulants of the output are noise free and proportional to the fourth-order correlation (or deterministic moment) of the channel coefficients. Similar observations can be made for the cumulants of order greater than four.

Example 4.
The trispectrum of the output sequence $\{y(k)\}$ in Example 3 is written as:

$$S_{y,4}(e^{j\omega_1}, e^{j\omega_2}, e^{j\omega_3}) = \gamma_{x,4} \cdot F(e^{j\omega_1})F(e^{j\omega_2})F(e^{j\omega_3})F(e^{-j(\omega_1+\omega_2+\omega_3)}) \quad (39)$$

Then, it is easy to show that by placing $\omega_2 = \omega_3 = 0$, we find

$$S_{y,2}(e^{j\omega_1}) = \frac{\gamma_{x,2}}{\gamma_{x,4} \cdot F^2(1)} S_{y,4}(e^{j\omega_1}, 1, 1) \quad (40)$$

Thus, assuming that $F(1) \neq 0$, a "noise free" power spectrum can be obtained via Eq. (40).

D Polycepstra and Linear Filtering

Let us consider a nonminimum phase LTI system driven by a non-Gaussian i.i.d. process $\{x(k)\}$, $k = 1, 2, \ldots$, and let us assume that the transfer function $F(e^{j\omega})$ of the system takes the form depicted in Figure 5.

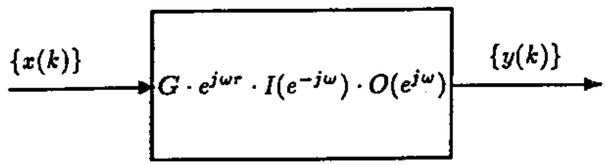


Figure 5: Discrete nonminimum phase LTI system

The $I(e^{-j\omega}) = \frac{\prod_i (1-a_i e^{-j\omega})}{\prod_i (1-c_i e^{-j\omega})}$, $|a_i| < 1$, $|c_i| < 1$ is a minimum phase term and the $O(e^{j\omega}) = \prod_i (1 - b_i e^{j\omega})$, $|b_i| < 1$ is a maximum phase term. The G is a constant gain and r is constant time delay. Thus, the $F(e^{j\omega})$ is non-zero for all frequencies $|\omega| \leq \pi$.

The minimum phase differential cepstrum parameters, $A(k)$, $k = 1, 2, \ldots$, and the maximum phase differential cepstrum parameters $B(k)$, $k = 1, 2, \ldots$, are functions of the zeros and poles of the terms $I(e^{-j\omega})$ and $O(e^{j\omega})$, respectively. They are defined as follows: [1]

$$A(k) \stackrel{def}{=} \sum_i a_i^k - \sum_i c_i^k \qquad B(k) \stackrel{def}{=} \sum_i b_i^k \tag{41}$$

From Eq. (14), the n-th order polycepstrum of the system output is defined as

$$c_{y,n}(\tau_1, \ldots, \tau_{n-1}) = \mathcal{F}^{-1}\left[ln(S_{y,n}(e^{j\omega_1}, \ldots, e^{j\omega_{n-1}})\right] \tag{42}$$

$$\tau_k = 0, \pm 1, \pm 2, \ldots$$

By substituting Eq. (31) with the $F(e^{j\omega})$ considered above into Eq. (42) and then using the power series expansion $ln(1-x) = -\sum_{k=1}^{\infty} \frac{x_k}{k}$, $|x| < 1$, we can find the following relations for the power cepstrum, bicepstrum and tricepstrum of the system output.

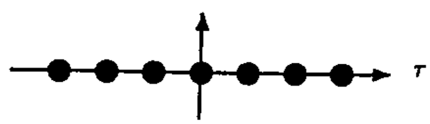
$$c_{y,2}(\tau) = \begin{cases} \ln(\gamma_{x,2} \cdot G^2) & \tau = 0 \\ -\frac{1}{\tau}[A(\tau) + B(\tau)] & \tau > 0 \\ \frac{1}{\tau}[A(-\tau) + B(-\tau)] & \tau < 0 \end{cases} \tag{43}$$

$$c_{y,3}(\tau_1, \tau_2) = \begin{cases} \ln(\gamma_{x,3} \cdot G^3), & \tau_1 = \tau_2 = 0 \\ -\frac{1}{\tau_1} A(\tau_1), & \tau_2 = 0,\ \tau_1 > 0 \\ -\frac{1}{\tau_2} A(\tau_2), & \tau_1 = 0,\ \tau_2 > 0 \\ \frac{1}{\tau_1} B(-\tau_1), & \tau_2 = 0,\ \tau_1 < 0 \\ \frac{1}{\tau_2} B(-\tau_2), & \tau_1 = 0,\ \tau_2 < 0 \\ -\frac{1}{\tau_1} B(\tau_1), & \tau_1 = \tau_2 > 0 \\ \frac{1}{\tau_1} A(-\tau_1), & \tau_1 = \tau_2 < 0 \\ 0, & \text{otherwise} \end{cases} \tag{44}$$

$$c_{y,4}(\tau_1, \tau_2, \tau_3) = \begin{cases} \ln(\gamma_{x,4} \cdot G^4), & \tau_1 = \tau_2 = \tau_3 = 0 \\ -\frac{1}{\tau_1} A(\tau_1), & \tau_2 = \tau_3 = 0,\ \tau_1 > 0 \\ -\frac{1}{\tau_2} A(\tau_2), & \tau_1 = \tau_3 = 0,\ \tau_2 > 0 \\ -\frac{1}{\tau_3} A(\tau_3), & \tau_1 = \tau_2 = 0,\ \tau_3 > 0 \\ \frac{1}{\tau_1} B(-\tau_1), & \tau_2 = \tau_3 = 0,\ \tau_1 < 0 \\ \frac{1}{\tau_2} B(-\tau_2), & \tau_1 = \tau_3 = 0,\ \tau_2 < 0 \\ \frac{1}{\tau_3} B(-\tau_3), & \tau_1 = \tau_2 = 0,\ \tau_3 < 0 \\ -\frac{1}{\tau_2} B(\tau_2), & \tau_1 = \tau_2 = \tau_3 > 0 \\ \frac{1}{\tau_2} A(-\tau_2), & \tau_1 = \tau_2 = \tau_3 < 0 \\ 0, & \text{otherwise} \end{cases} \tag{45}$$

Similar relations hold for higher order cepstra. It is clear from Eq. (43) that the power cepstrum cannot distinguish between maximum and minimum phase information of the system since it is a function of the sum of the minimum and maximum phase cepstrum parameters. On the other hand, the bicepstrum is non-zero only on three lines in the 2-d space and the tricepstrum is non-zero only on four lines in the 3-d space. The minimum and maximum phase differential cepstrum parameters of the system appear separately in each of these lines. The regions of support for the power cepstrum and bicepstrum are illustrated in Figure 6.

Power Cepstrum



Bicepstrum

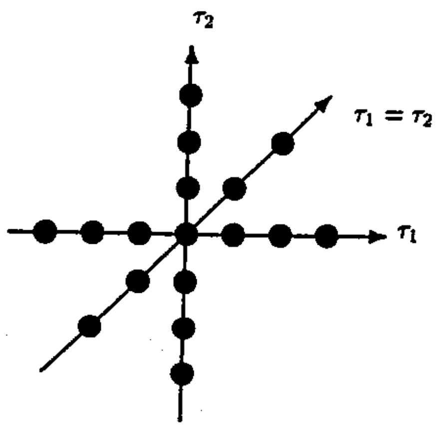


Figure 6: Area of support for power cepstrum and bicepstrum

Properties:

Some important properties of the differential cepstrum parameters $A(k)$, $B(k)$ are the following [1,2]:

1. They are of infinite duration, however, decay exponentially with k because $|a_i| < 1$, $|b_i| < 1$, and $|c_i| < 1$. Therefore, we can always choose two positive integers p, q to be sufficiently large, so that $A(k)$ and $B(k)$ become arbitrarily small for $k > p$ and $k > q$, respectively.
2. Let $i(k) = \mathcal{F}^{-1}[I(e^{-j\omega})]$. Then,

$$i(k) = \begin{cases} 0 & k < 0 \\ 1 & k = 0 \\ -\frac{1}{k}\sum_{n=2}^{k+1} A(n-1)\cdot i(k-n+1) & k = 1, 2, \dots \end{cases} \quad (46)$$

Similarly, let $o(k) = \mathcal{F}^{-1}[O(e^{j\omega})]$. Then,

$$o(k) = \begin{cases} 0 & k > 0 \\ 1 & k = 0 \\ \frac{1}{k}\sum_{n=k+1}^{0} B(1-n)\cdot o(k-n+1) & k = -1, -2, \dots \end{cases} \quad (47)$$

Therefore, the minimum and maximum phase components of the system are functions of the minimum phase and maximum phase differential cepstrum parameters, respectively.

3. Let $F(e^{j\omega}) = |F(e^{j\omega})| \cdot e^{j\phi(\omega)}$. Then, the $|F(e^{j\omega})|$ corresponds to the differential cepstrum parameters $A(k) + B(k)$ and the $e^{j\phi(\omega)}$ corresponds to the differential cepstrum parameters $A(k) - B(k)$ [1]. From Eq. (43) we see that $A(k) + B(k)$ are also the differential cepstrum parameters of the power spectrum. On the other hand, it is easy to show that $A(k) - B(k)$ are the differential cepstrum parameters of the bicoherence function $R_{y,3}(e^{j\omega_1}, e^{j\omega_2})$.

E Higher-Order Cumulants and Nonlinearities

In Figure 7, a LTI system and a nonlinear (N/L) system are driven by the stationary process $\{x(k)\}$.

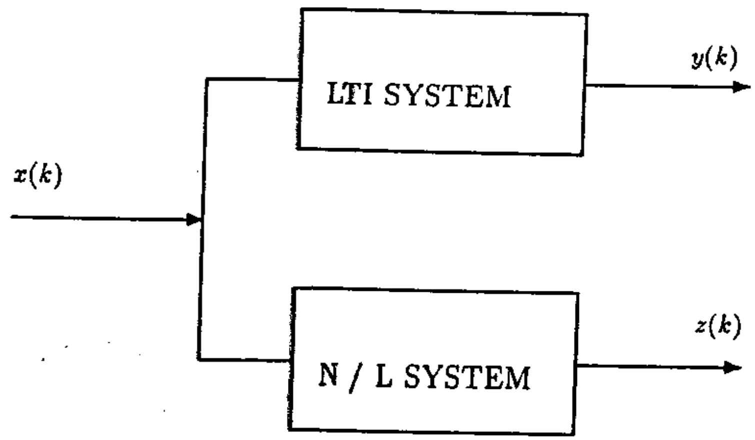


Figure 7: Discrete LTI and N/L systems

Let us consider the following two cases for $\{x(k)\}$.

1. $\{x(k)\}$ is Gaussian process. Then

$$C_{y,2}(\tau_1) \neq 0 \quad (48)$$
$$C_{y,n}(\tau_1,\ldots,\tau_{n-1}) = 0, \qquad n \geq 3 \quad (49)$$

However, in general

$$C_{z,n}(\tau_1,\ldots,\tau_{n-1}) \neq 0 \qquad n \geq 2 \quad (50)$$

2. $x(k) = \sum_n A_n e^{(j\omega_n k + \theta_n)}$ where, θ_n are i.i.d. random variables uniformly distributed in the interval $[-\pi,\pi]$. Thus, $x(k)$ is a harmonic stationary process. The outputs of the two systems are written as

$$y(k) = \sum_n B_n e^{(j\omega_n k + \phi_n)}$$
$$z(k) = \sum_n C_n e^{(j\omega_n k + \psi_n)} + \sum_{m,l} C_k C_l e^{[j(\omega_m+\omega_l)k + \psi_m + \psi_l]} + \sum_{m,l,i} \cdots$$

It can be shown that [1]:

$$C_{y,3}(\tau_1,\tau_2) = 0, \qquad C_{z,3}(\tau_1,\tau_2) \neq 0 \quad (51)$$

From the above, we conclude that non-zero higher-order cumulants or polyspectra at the output of the above systems indicate the presence of nonlinearity. In general, the polyspectra of a system output can be utilized in various ways in detecting as well as characterizing various types of nonlinearities in the system. For example, the bispectrum and trispectrum have been used for detecting quadratic and cubic phase coupling in harmonic processes, respectively [1,3]. In [7] the bispectrum has been used in combination with the power spectrum to identify quadratic Volterra filters driven by Gaussian processes. Similar applications have been proposed in [8].

F Moments and Cumulants of a Mixture of Deterministic and Stochastic Signals

Cumulants are not well defined for deterministic signals. Thus, for deterministic signals higher-order spectra are defined in terms of higher-order moments [1]. Given that a deterministic signal $s(i)$, $i = 1,2,\ldots$ is finite energy, e.g. transient signal, then the n-th order moment and the corresponding n-th order spectrum are defined as follows:

$$M_{s,n}(\tau_1,\ldots,\tau_{n-1}) = \sum_{i=1}^{\infty} s(i)s(i+\tau_1)\cdots s(i+\tau_{n-1}) \quad (52)$$
$$\tau_k = 0, \pm 1, \pm 2, \ldots$$

$$\begin{aligned} S_{s,n}(e^{j\omega_1},\ldots,e^{j\omega_{n-1}}) &= \mathcal{F}[M_{s,n}(\tau_1,\ldots,\tau_{n-1})] = \quad (53) \\ &= S(e^{j\omega_1})\cdots S(e^{j\omega_{n-1}})S(e^{-j(\omega_1+\cdots+\omega_{n-1})}) \end{aligned}$$
$$|\omega_l| \leq \pi, \quad l = 1,\ldots,n-1, \quad \left|\sum_{l=1}^{n-1}\omega_l\right| \leq \pi$$

where, $\mathcal{F}[\cdot]$ denotes $(n-1)$-d discrete Fourier transform, and $S(e^{j\omega})$ is the 1-d discrete Fourier transform of $s(i)$. On the other hand, given that $s(i)$ is power signal, e.g. harmonics, then

$$M_{s,n}(\tau_1,\ldots,\tau_{n-1}) = \lim_{N\to\infty} \frac{1}{N} \cdot \sum_{i=1}^{N} s(i)s(i+\tau_1)\cdots s(i+\tau_{n-1})$$
$$\tau_k = 0, \pm 1, \pm 2, \ldots \quad (54)$$

and for a periodic power signal

$$\begin{aligned} S_{s,n}(e^{j\omega_1},\ldots,e^{j\omega_{n-1}}) &= \mathcal{F}[M_{s,n}(\tau_1,\ldots,\tau_{n-1})] = \qquad (55)\\ &= \frac{1}{N}\cdot S(e^{j\omega_1})\cdots S(e^{j\omega_{n-1}})S(e^{-j(\omega_1+\cdots+\omega_{n-1})})\\ &|\omega_l|\le\pi,\quad l=1,\ldots,n-1,\quad \left|\sum_{l=1}^{n-1}\omega_l\right|\le\pi \end{aligned}$$

where, $\mathcal{F}[\cdot]$ denotes $(n-1)$-d discrete Fourier Series, and $S(e^{j\omega})$ is the 1-d discrete Fourier Series of the periodic $s(i)$. The higher-order moments of deterministic signals have similar properties with those of stochastic signals.

Now let us consider a situation found often in practice where we want to calculate the higher-order cumulants of the process

$$y(k) = s(k) + w(k) \qquad (56)$$

where, the $s(k)$ is a deterministic signal and the $w(k)$ is a zero-mean stationary colour Gaussian noise process independent from $s(k)$. A problem that arises in calculating the n-th order cumulant $C_{y,n}(\tau_1,\ldots,\tau_{n-1})$ is that $y(k)$, in this case, is not stationary due to the presence of the deterministic component $s(k)$ [9]. However, for power signals it can be shown that under certain conditions, such as summability of cumulants and the existence of limits, we can define the n-th order moment $M_{y,n}(\tau_1,\ldots,\tau_{n-1})$ as in Eq. (54). On the other hand, for energy signals it is necessary to assume multiple realizations each of length greater or equal than the length of $s(k)$. Thus, given the realizations $y_j(k)$, $j=1,2,\ldots,J$, $k=1,\ldots,K$ we can define [9, 10]

$$M_{y,n}(\tau_1,\ldots,\tau_{n-1}) = \lim_{J\to\infty}\sum_{j=1}^{J}\sum_{k=1}^{K} y(k)y(k+\tau_1)\cdots y(k+\tau_{n-1}) \qquad (57)$$

Finally, by assuming that $M_{y,1}=0$ (otherwise the mean of $y(k)$ should be removed) [9],

$$\begin{aligned} C_{y,2}(\tau_1) &= M_{y,2}(\tau_1) = M_{s,2}(\tau_1) + M_{w,2}(\tau_1) \qquad (58)\\ C_{y,3}(\tau_1,\tau_2) &= M_{y,3}(\tau_1,\tau_2) = M_{s,3}(\tau_1,\tau_2) \qquad (59)\\ C_{y,4}(\tau_1,\tau_2,\tau_3) &= M_{y,4}(\tau_1,\tau_2,\tau_3) - M_{y,2}(\tau_1)M_{y,2}(\tau_2-\tau_3)\\ &\quad -M_{y,2}(\tau_2)M_{y,2}(\tau_3-\tau_1) - M_{y,2}(\tau_3)M_{y,2}(\tau_1-\tau_2)\\ &= C_{s,4}(\tau_1,\tau_2,\tau_3) \qquad (60) \end{aligned}$$

III H.O.S. Estimation in 1-D Signal Processing

Three categories of H.O.S. estimators are presented: i) the conventional estimators that follow the definition of higher-order cumulants and spectra, ii) parametric estimators which are based on Autoregressive-Moving Average (ARMA) modeling of the observed data process, and iii) the polycepstra estimators that utilize the properties of the complex cepstrum of section II.D. The important problem of reconstructing the magnitude and phase of a signal from its H.O.S., the statistical behaviour of the estimators and practical constraints in their implementation are discussed.

A Estimation of Higher-Order Moments and Cumulants from Data Samples

In practice, the estimation of higher-order moments and cumulants must be based on a finite number of data samples. Assuming that the observed data samples $\{y(i)\}$, $i=1,\ldots,N$ come from a n-th order stationary and ergodic process, then we can obtain an estimate of the n-th order moment by means of time averaging as follows [1,2]:

$$\hat{M}_{y,n}(\tau_1,\ldots,\tau_{n-1}) = \frac{1}{I_2-I_1+1}\sum_{i=I_1}^{I_2} y(i)y(i+\tau_1)\cdots y(i+\tau_{n-1}), \qquad (61)$$

or

$$\hat{M}_{y,n}(\tau_1,\ldots,\tau_{n-1}) = \frac{1}{N}\sum_{i=I_1}^{I_2} y(i)y(i+\tau_1)\cdots y(i+\tau_{n-1}), \qquad (62)$$

$$\tau_l = 0,\pm1,\pm2,\ldots,\pm L$$

where, $I_1 = max(1,1-\tau_2,\ldots,1-\tau_{n-1})$ and $I_2 = min(N,N-\tau_2,\ldots,N-\tau_{n-1})$. Equation (61) describes an unbiased and consistent estimator of the true n-th order moment. Equation (62) describes a biased, nevertheless asymptotically unbiased and consistent estimator. The biased estimator is considered because in practice often behaves better than the unbiased estimator. Note that for large data records or for $N >> L$ the two estimators are practically equivalent.

The sample estimate $\hat{C}_{y,n}(\tau_1, \ldots, \tau_{n-1})$ of the n-th order cumulant [1] is obtained by substituting moments of order less or equal to n by their sample estimates in the definitions of section II.A. Note that the mean of the observed data record should be subtracted if the simplified formulas for the cumulants of a zero mean process are to be used.

An alternative procedure that is widely used in practice is to segment the data record into K segments of length M, estimate the n-th order moment of each segment separately using Eq. (61) or Eq. (62), and then take the average of the estimates from all segments. Thus, by denoting with $\hat{M}_{y,n}^{(k)}(\tau_1, \cdots, \tau_{n-1})$, $k = 1, \ldots, K$ the estimates from K segments,

$$\hat{M}_{y,n}(\tau_1, \ldots, \tau_{n-1}) = \frac{1}{K} \sum_{k=1}^{K} \hat{M}_{y,n}^{(k)}(\tau_1, \cdots, \tau_{n-1}). \qquad (63)$$

The estimate of the n-th order cumulant follows. The segments can be overlapping or not. It has been suggested that for different moment or cumulant lags different segmentation of the data and length of overlapping between segments should be chosen for better results [11]. The segmentation operation improves slightly the asymptotic efficiency of the estimator but does not improve significantly the variance of estimation [9]. However, in digital implementations, segmentation reduces significantly the variance of the quantization noise especially when long data records are involved [12].

It should be mentioned that for reliable estimation of higher-order moments and cumulants longer data records are needed compared to the estimation of second order statistics. The data length requirement increases with the order of moment or cumulant to be estimated. This problem has motivated research towards alternative ways for estimating H.O.S. from finite data records [13]. Also, due to the multi-dimensional and multi-product nature of H.O.S. the computational complexity in the estimation of H.O.S. is high. The complexity can be reduced by exploiting the symmetry properties of H.O.S. and by investigating efficient parallel implementation [14, 15].

Example 5.

Let us consider the following MA(2) model (Moving Average of order two). The finite impulse response LTI system with nonminimum phase transfer function

$$F(e^{j\omega}) = 0.5 + e^{-j\omega} + 0.5 \cdot e^{-j2\omega} \qquad (64)$$

[1] Ergodicity assumptions are met more easily by cumulants rather than moments

is driven by an i.i.d. exponentially distributed process with zero mean and unit variance. The true second order (autocorrelation) and third order cumulants of the output process were obtained from Eq. (30) and are drawn on the top of Figures 8 and 9 respectively. For the two-dimensional third order cumulant a contour diagram is shown as well. In the same figures the estimated second and third order cumulants of the output process obtained via Eq. (63) are illustrated as well. Two cases were considered: i) $K = 1$, $M = 128$,i.e., a single data record of length 128 samples, and ii) $K = 4$, $M = 512$, i.e., four non-overlapping data records each of length 128 data samples. As expected the utilization of more data samples provided better cumulant estimates. In all cases, 11 autocorrelation lags and 121 third order cumulants were calculated.

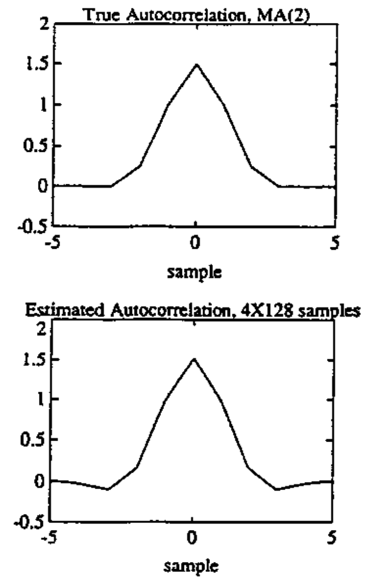


Figure 8: True and estimated autocorrelation of MA(2) model

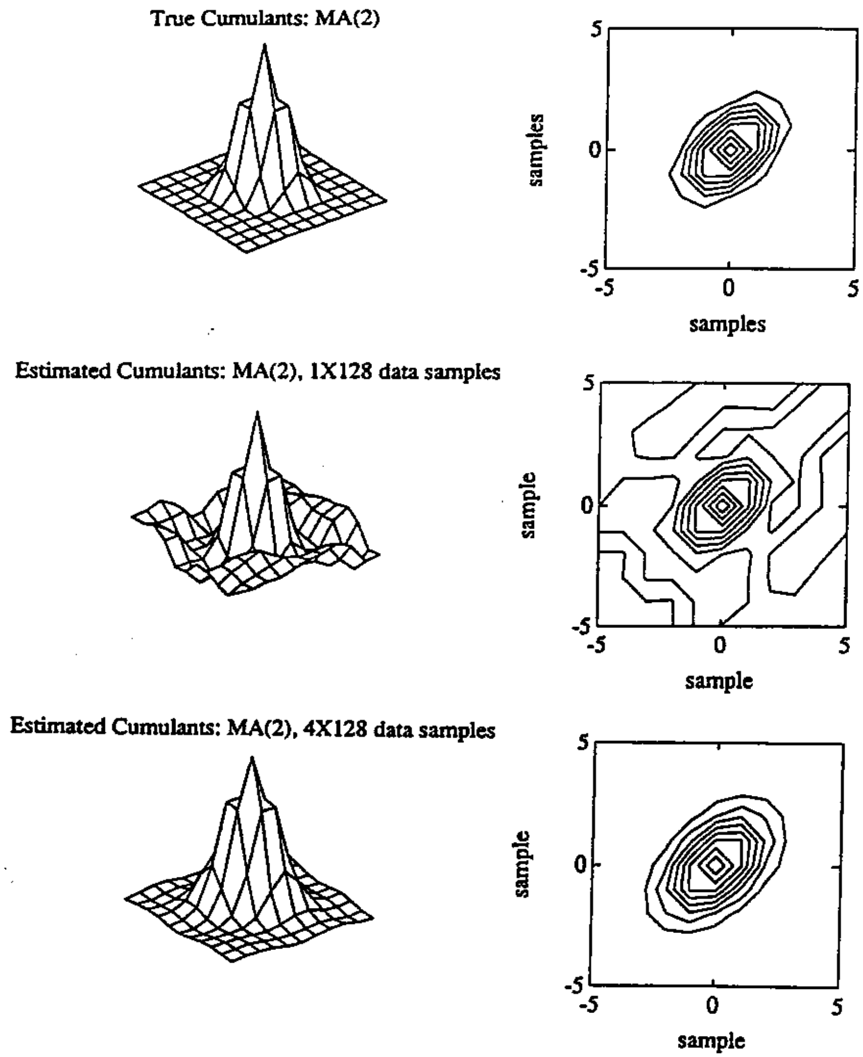


Figure 9: True and estimated third order cumulants of MA(2) model and contour diagrams.

Example 6.

In this case we consider an ARMA(2,2) model where the LTI system with nonminimum phase transfer function

$$F(e^{j\omega}) = \frac{0.5 + e^{-j\omega} + 0.5 \cdot e^{-j2\omega}}{1 - 1.3 \cdot e^{-j\omega} + 0.81 \cdot e^{-j2\omega}} \tag{65}$$

is driven by an i.i.d. exponentially distributed process with zero mean and unit variance. The true and estimated second and third order cumulants are given in Figures 10 and 11, respectively. In all cases, 51 autocorrelation lags and 2601 third order cumulant lags were calculated. Similar conclusions to those of Example 5 can be derived. In addition we observe that for the same number of data samples better estimates were obtained for the second order cumulants compared to the third order cumulants.

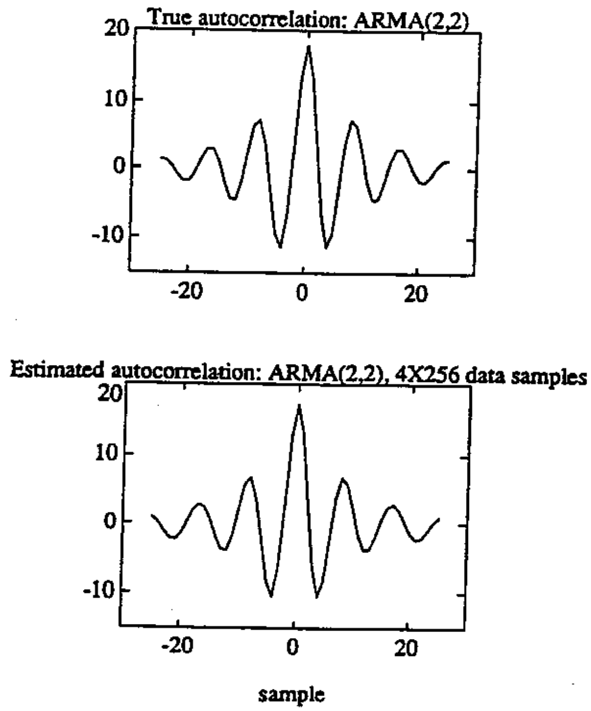


Figure 10: True and estimated autocorrelation of ARMA(2,2) model

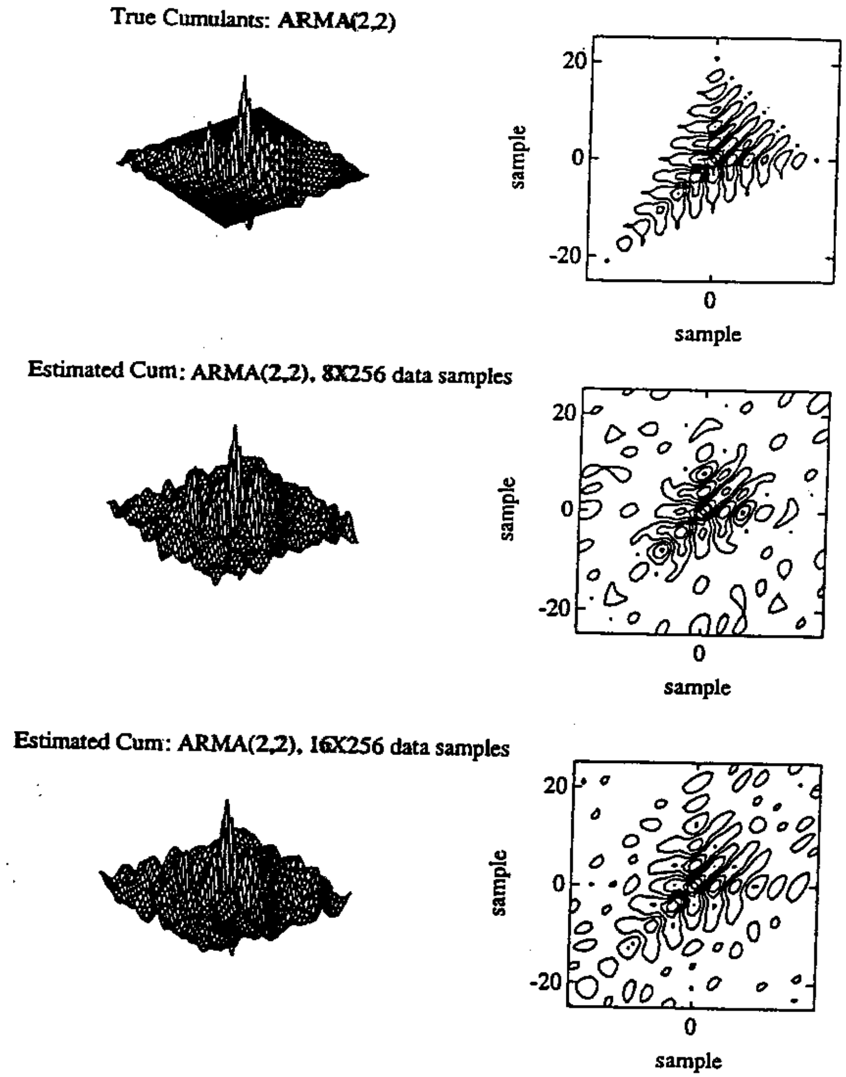


Figure 11: True and estimated third order cumulants of ARMA(2,2) model and contour diagrams.

B Conventional H.O.S. Estimators

Two methods for the estimation of higher-order spectra from a finite number of data samples are described. The indirect method is the generalization of the Blackman-Tukey method in power spectrum estimation to H.O.S. domains. The direct method which can also be called higher-order periodogram is the extension of the Welch method in power spectrum estimation to H.O.S. domains.

1 Indirect Method

Given the data samples $\{y(i)\}$, $i = 1, 2, \ldots, N$ from an n-th order stationary and ergodic process, we follow the following steps:

1. Obtain an estimate of the n-th order cumulant

$$\hat{C}_{y,n}(\tau_1, \ldots, \tau_{n-1}), \qquad \tau_l = 0, \pm 1, \pm 2, \ldots, \pm L \tag{66}$$

   as described in section III.A. As a rule of thumb, if data segments of length M are used then choose $L < \frac{M}{10}$.

2. Obtain the n-th order spectrum estimate

$$\hat{S}_{y,n}(e^{j\omega_1}, \ldots, e^{j\omega_{n-1}}) = \sum_{\tau_1=-L}^{L} \cdots \sum_{\tau_{n-1}=-L}^{L} \hat{C}_{y,n}(\tau_1, \ldots, \tau_{n-1}) \cdot w(\tau_1, \ldots, \tau_{n-1}) \cdot e^{-j(\omega_1\tau_1 + \cdots + \omega_{n-1}\tau_{n-1})},$$

$$|\omega_l| \leq \pi, \quad l = 1, \ldots, n-1, \quad \left|\sum_{l=1}^{n-1} \omega_l\right| \leq \pi \tag{67}$$

   The last relation can be implemented efficiently via $(2L_1 + 1)$-point multidimensional Fast Fourier Transforms (where $L_1 \geq L$) by placing

$$\omega_i = \frac{2\pi k_i}{2L_1 + 1}, \qquad k_i = -L_1, \ldots, 0, \ldots, L_1, \qquad i = 1, \ldots, n-1$$

The $w(\tau_1, \ldots, \tau_{n-1})$ is a multidimensional window function that plays the classical role of smoothing the higher order spectrum estimate. Examples of appropriate windows with the required properties and constraints are given in Table III, [1,16].

Table III: Properties and examples of multidimensional window functions

Properties of window

1. Satisfy symmetry properties of corresponding cumulants
2. Be equal to zero outside the region of support of cumulants
3. Have non-zero Fourier transform
4. $w(0,0,\ldots,0)=1$

n-d windows satisfying above properties

$$w(\tau_1,\ldots,\tau_{n-1}) = p(\tau_1)p(\tau_2)\cdots p(\tau_{n-1})p(\tau_1+\cdots+\tau_{n-1})$$

where, $p(\tau)$ is 1-d window satisfying the properties:
i) $p(\tau)=p(-\tau)$, ii) $p(\tau)=0$ for $|\tau|>L$,
iii) $p(0)=1$, and iv) $P(e^{j\omega})=\mathcal{F}[p(\tau)]\geq 0, \quad |\omega|\leq\pi$.

Examples:

1. UNIT WINDOW

$$p_u(\tau)=1, \qquad |\tau|\leq L$$

2. OPTIMUM BIAS WINDOW

$$p_o(\tau)=\frac{1}{\pi}\left|sin(\frac{\pi\tau}{L})\right| + (1-\frac{|\pi|}{L})cos(\frac{\pi\tau}{L}), \qquad |\tau|\leq 0$$

3. PARZEN WINDOW

$$p_P(\tau)=\begin{cases} 1-6(\frac{|\tau|}{L})^2+6(\frac{|\tau|}{L})^3, & |\tau|\leq\frac{L}{2} \\ 2(1-\frac{|\tau|}{L})^3, & \frac{L}{2}\leq|\tau|\leq L \end{cases}$$

2 Direct Method

Given the data samples $\{y(i)\}$, $i=1,2,\ldots,N$ from an n-th order stationary and ergodic process, we follow the following steps:

1. Divide the data record into K overlapping or non-overlapping zero-mean segments of equal length M.

2. Let $\{y^{(k)}(i)\}$, $i=0,1,\ldots,M-1$ denote the samples of segment k. Then, obtain the discrete Fourier transform of each segment by means of an L-point FFT algorithm ($L\geq M$), that is

$$Y^{(k)}(l) \;=\; \frac{1}{L}\sum_{i=0}^{L-1} y^{(k)}(i)e^{-j\frac{2\pi li}{L}}, \tag{68}$$
$$l=-\frac{L}{2}+1,\ldots,0,\ldots,\frac{L}{2}, \qquad k=1,\ldots,K.$$

It is assumed that L is even no loss of generality.

3. Obtain a "smoothed" estimate of the n-th order moment spectrum for each of the data segments as follows:

$$Q^{(k)}_{y,n}(l_1,l_2,\ldots,l_{n-1}) = W(l_1,l_2,\ldots,l_{n-1}) * \left[Y^{(k)}(l_1)\cdots Y^{(k)}(l_{n-1})Y^{(k)}(-l_1-l_2\cdots-l_{n-1})\right] \tag{69}$$
$$k=1,\ldots,K, \qquad |l_i|<\frac{L}{2}, \qquad |\sum_{i=1}^{n-1} l_i|<\frac{L}{2}$$

where, $\{*\}$ denotes (n-1)-d discrete linear convolution and $W(l_1,\ldots,l_{n-1})$ is a smoothing window in the frequency domain with properties similar to those of of Table III.

4. Average over the n-th order moment spectrum estimates of all segments, that is

$$\hat{Q}_{y,n}(l_1,l_2,\ldots,l_{n-1}) = \frac{1}{K}\sum_{k=1}^{K} Q^{(k)}_{y,n}(l_1,l_2,\ldots,l_{n-1}) \tag{70}$$

5. Obtain the estimate of the n-th order cumulant spectrum by properly combining estimated moment spectra of order less or equal to n [1, 17]. In the case of bispectrum simply place

$$\hat{S}_{y,3}(l_1,l_2)=\hat{Q}_{y,n}(l_1,l_2) \tag{71}$$

3 Behaviour of Conventional Higher Order Spectra Estimators

It has been shown [18, 19, 20] that both the direct and indirect methods provide asymptotically Gaussian distributed and unbiased estimates of the higher order spectra. The variance of estimation is inversely proportional to the number of data samples employed ($N = M \cdot K$), i.e, it can be reduced by increasing either the number of data segments or the length of segments. It is proportional to the size of the window that is applied to cumulants (indirect method), yet, it is inversely proportional to the size of the window that is applied for frequency smoothing (direct method). Segmentation of the data, averaging of higher order spectra estimates from segments, and frequency smoothing are necessary to guarantee consistency of the obtained estimates. However, smoothing usually introduces bias in the estimation.

The conventional higher order spectra estimators exhibit similar characteristics with the conventional methods for power spectrum estimation. They are easy to implement and provide good estimates when long data records are available. Also, analytical results of their performance exist. Nevertheless, they exhibit high variance of estimation and low frequency resolution with short data records. This is an important problem in many applications where stationarity constraints limit significantly the length of data samples that can be utilized. The estimates with the direct method tend to exhibit higher variance similar to the behaviour of the periodogram in power spectrum estimation.

Example 7.

Consider the MA(2) model of Example 5. The true power spectrum of the model output and the estimated power spectrum with the indirect and direct methods are illustrated in Figure 12. The length of data record was 128 samples. In the indirect method 11 autocorrelation lags were estimated and then 128-point Fast Fourier Transform (FFT) was taken. The direct method was implemented with 128-point FFTs and frequency smoothing by means of a rectangular window of length 5 samples. The magnitude, phase, and corresponding contour diagrams of the bispectrum of the MA(2) model output estimated via the indirect and direct methods are illustrated in Figures 13 and 14, respectively. The indirect method was implemented with 121 (11x11) third order cumulant lags obtained from 4 data segments each of length 128 samples and weighted with the Parzen window. Then, a 64x64 FFT algorithm was taken. For the direct method 64-point FFTs were applied to 8 data segments each of length 64 samples. Frequency smoothing with a unit window of length 5 samples was applied.

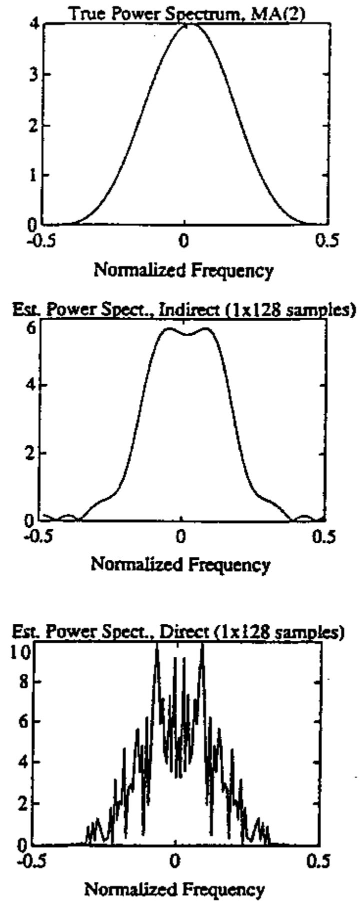


Figure 12: True and Estimated Power Spectrum via the direct and indirect methods

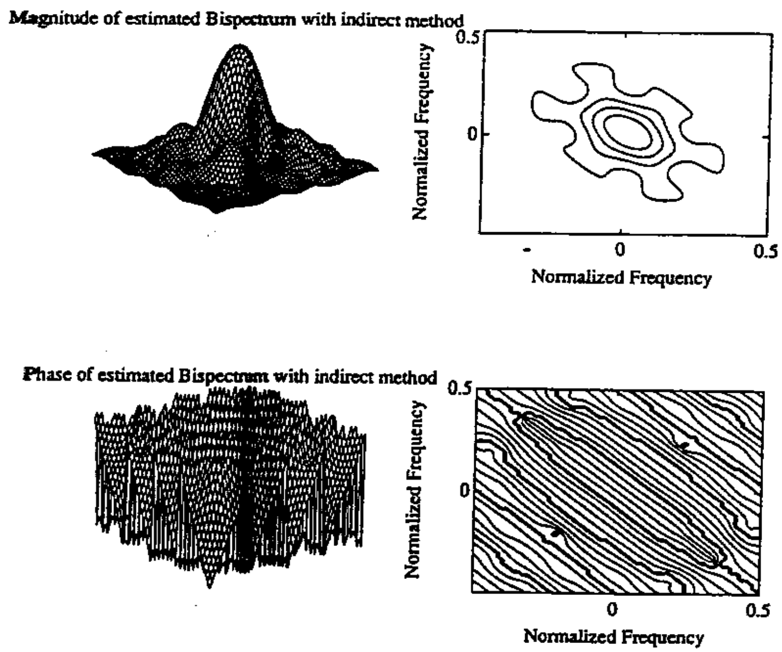


Figure 13: Magnitude, phase and corresponding contour diagrams of the bispectrum of MA(2) model estimated via the indirect method.

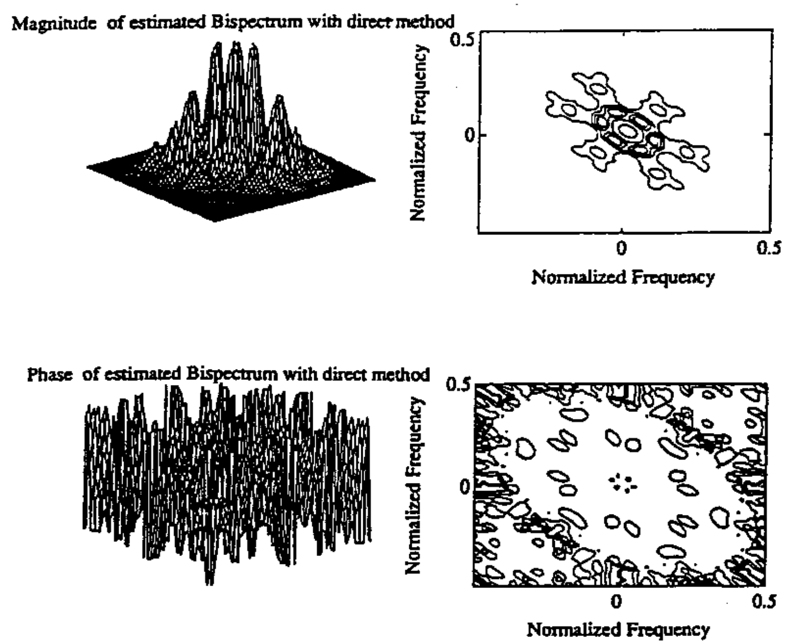


Figure 14: Magnitude, phase and corresponding contour diagrams of the bispectrum of MA(2) model estimated via the direct method.

Example 8.

Consider the ARMA(2,2) model of Example 6. Contour diagrams of the magnitude of the estimated bispectrum via the direct and indirect methods are illustrated in Figure 15. The methods were implemented as in Example 7. However, this time 4 data segments each of length 256 samples were employed and 128 point FFT algorithms were taken.

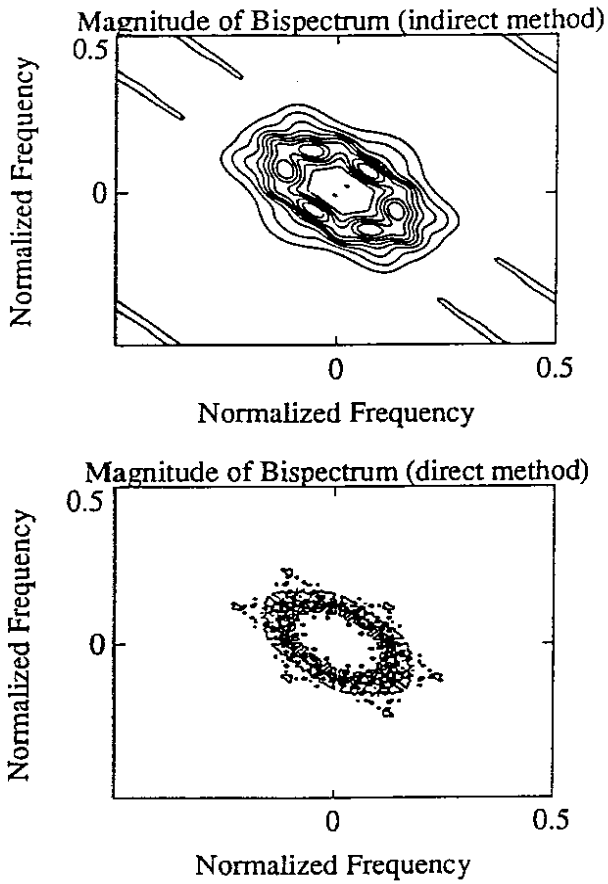


Figure 15: Contour diagrams of the magnitude of the bispectrum of ARMA(2,2) model estimated via the indirect and direct methods.

C Parametric H.O.S. Estimators

In parametric modeling we assume that the observed stationary process $\{y(k)\}$, $k = 1, 2, \ldots$ is the output of a causal stable discrete time and LTI system with transfer function $F(e^{j\omega})$ driven by the non-Gaussian i.i.d. process $\{x(k)\}$, $k = 1, 2, \ldots$. We also assume that the system transfer function takes the form of an Autoregressive - Moving average model (ARMA(p,q)), that is

$$F(e^{j\omega}) = \sum_{k=0}^{\infty} f(k)e^{-j\omega k} = \frac{\sum_{k=0}^{q} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{p} a_k e^{-j\omega k}} = \frac{B(e^{j\omega})}{A(e^{j\omega})} \quad (72)$$

where $\{f(k)\}$ is the impulse response of the system. The $B(e^{j\omega})$ is in general nonminimum phase while $A(e^{j\omega})$ will be minimum phase to satisfy stability requirements (causality assumption). When $q = 0$ the system reduces to an Autoregressive model of order p, i.e., AR(p). When $p = 0$ the system becomes a Moving Average model of order q, i.e., MA(q). The objective is to recover the characteristics of $F(e^{j\omega})$, or equivalently the coefficients $\{b_k\}$, $k = 0, 1, \ldots, q$ and $\{a_k\}$, $k = 1, \ldots, p$, by processing $\{y(k)\}$ only. We know from our discussion in Example 2, section II.C, that if the system is nonminimum phase, then to correctly recover both the magnitude and phase characteristics of $F(e^{j\omega})$ the higher-order statistics of the system output $\{y(k)\}$ must be utilized. Assuming that $F(e^{j\omega})$ has been obtained, then from Eq. (31)

$$S_{y,n}(e^{j\omega_1}, \ldots, e^{j\omega_{n-1}}) = \gamma_{x,n} \cdot F(e^{j\omega_1}) \cdots F(e^{j\omega_{n-1}})F(e^{-j(\omega_1 + \cdots + \omega_{n-1})})$$

When the constant $\gamma_{x,n}$ is unknown a scaled estimate of the n-th order polyspectrum is obtained.

Parametric methods make the assumption that the order of the underlying AR, MA, or ARMA model is known or it can be accurately estimated form the observed data. Therefore, the performance of parametric estimation methods depends on the efficiency of model order selection methods [1-4]. Note that H.O.S. parametric estimators are more sensitive to model order selection compared to the classical autocorrelation based estimators. In what follows we consider only the third-order cumulants domain. However, the obtained relations can be easily extended to higher order cumulant domains.

1 MA Methods For H.O.S. Estimation

Let $\{y(n)\}$ be the output from an MA(q) nonminimum phase model driven by the zero-mean non-Gaussian non-symmetrically[2] distributed i.i.d. pro-

[2] This assumption is non necessary in the even order cumulant domains

cess $\{x(k)\}$. Then,

$$y(n) = \sum_{k=0}^{q} b_k x(n-k) = \sum_{k=0}^{q} f(k)x(n-k) \tag{73}$$

The diagonal third-order cumulants lags of $\{y(n)\}$ can be written according to Eq. (30) as:

$$C_{y,3}(\tau,\tau) = E\{y(n)y^2(n+\tau)\} = \gamma_{x,3}\sum_{k=0}^{q} b_k b_{k+\tau}^2 \qquad \tau = -q,\ldots,0,\ldots,q \tag{74}$$

Given the samples $\{y(n)\}$, $n = 1,2,\ldots,N$ from a third order stationary and ergodic process, we can obtain the biased sample estimates

$$\hat{C}_{y,3}(\tau,\tau) = \frac{1}{N}\sum_{n=I_1}^{I_2} y(n)y^2(n+\tau), \qquad \tau = -q,\ldots,0,\ldots,q \tag{75}$$

$$I_1 = max(1, 1-\tau), \qquad I_2 = min(N, N-\tau).$$

The Non-linear - Least Squares MA Method [21], proposes the minimization of the non-linear function

$$\sum_{\tau=-q}^{q}\left[\hat{C}_{y,3}(\tau,\tau) - \gamma_{x,3}\sum_{k=0}^{q} b_k b_{k+\tau}^2\right]^2 \tag{76}$$

with respect to the unknown parameters $\{\gamma_{x,3}, b_0, b_1, \ldots, b_q\}$. However, minimization of Eq. (76) is a difficult problem which requires tedious searching programming techniques and proper initialization to avoid local equilibria [2]. Furthermore, in practice the unknown MA order q must be estimated by means of model order selection criteria.

The Linear - Least Squares MA Method [22] , proceeds as follows: According to (30) the autocorrelation of $\{y(n)\}$ is

$$C_{y,2}(\tau) = E\{y(n)y(n+\tau)\} = \gamma_{x,2}\sum_{k=0}^{q} b_k b_{k+\tau}, \qquad \tau = -q,\ldots,0,\ldots,q \tag{77}$$

By combining Eqs. (74) and (77) it can be shown that the following relation holds (assuming $b_0 = 1$)

$$\sum_{k=1}^{q} b_k C_{y,3}(\tau-k,\tau-k) - \frac{\gamma_{x,3}}{\gamma_{x,2}}\sum_{k=0}^{q} b_k^2 C_{y,2}(\tau-k) = -C_{y,3}(\tau,\tau), \qquad \tau = -q,\ldots,0,\ldots 2q \tag{78}$$

By replacing $C_{y,3}(\tau,\tau)$ and $C_{y,2}(\tau)$ by their sample estimates $\hat{C}_{y,3}(\tau,\tau)$ and $\hat{C}_{y,2}(\tau)$, respectively, and by repeating (78) for $\tau = -q,\ldots,0,\ldots,2q$ a linear overdetermined system of equations is formed, that is

$$\begin{bmatrix} \textit{matrix of} \\ \textit{third and second} \\ \textit{order cumulant} \\ \textit{estimates} \\ (3q+1)\times(2q+1) \end{bmatrix} \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ b_q \\ -\frac{\gamma_{x,3}}{\gamma_{x,2}} \\ \cdot \\ \cdot \\ -\frac{\gamma_{x,3}}{\gamma_{x,2}} b_q^2 \end{bmatrix} = \begin{bmatrix} -C_{y,3}(-q,-q) \\ \cdot \\ \cdot \\ -C_{y,3}(q,q) \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

or

$$\mathbf{A} \qquad \mathbf{b} \quad = \quad \mathbf{a} \tag{79}$$

The least squares solution to the above system is

$$\mathbf{b} = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{a} \tag{80}$$

where, the H denotes conjugate transpose operation. Given the estimated coefficients $\gamma_{x,3}$, $\{b_k\}$, $k = 1,\ldots,q$, we obtain $F(e^{j\omega}) = 1 + \sum_{k=1}^{q} b_k e^{-j\omega k}$ and then the bispectrum from Eq. (31). Model order selection criteria must be applied to estimate the unknown model order. Usually such criteria are based on rank determination of "special" cumulant matrices similar to matrix $\mathbf{A}$ [1-3]. The linear least-squares method is also known as the Giannakis-Mendel (GM) method. Modifications of this method have been proposed by Tugnait [23, 24], and Porat and Friedlander [25]. Other methods for MA system identification based on forward and backward ARMA approximations or non-causal AR approximation have been proposed by Nikias [26] and Chiang and Nikias [27, 28], and Nikias and Pan [29]. A method based on log-bispectra was proposed by Rangoussi and Giannakis [30]. Recently, another method that utilizes a linear combination of higher-order cumulant slices has been proposed by Fonollosa and Vidal [31].

Note, that the linear least-squares method makes the assumption that both the power spectrum and bispectrum are described by the same MA model. On the other hand the nonlinear least squares method can be applied even when the power spectrum and bispectrum are described by different filters. These comments can be better understood from the following example.

Example 9.
Let us consider the process

$$z(k) = y(k) + w(k)$$

where, the process $y(k)$ is described by a MA(q) model and $w(k)$ is colour Gaussian noise that is described by an ARMA(p,q_1) model . The noise is present in the power spectrum domain while theoretically disappears in the bispectrum domain. Therefore, in general a different order MA model is recognized by the two domains.

2 AR Methods For H.O.S. Estimation

Let $\{y(n)\}$ be the output from an AR(p) stable model driven by the zero-mean non-Gaussian i.i.d. process $\{x(n)\}$. Then,

$$y(n) + \sum_{k=1}^{p} a_k y(n-k) = x(n) \tag{81}$$

By multiplying both sides of the above equation by $y(n-\tau_1)y(n-\tau_2)$ and the taking expectation we find the following Third-Order Recursion (TOR) equation [32, 33]

$$C_{y,3}(-\tau_1,-\tau_2) + \sum_{k=1}^{p} a_k C_{y,3}(k-\tau_1, k-\tau_2) = \gamma_{x,3}\delta(\tau_1,\tau_2) \quad \tau_1,\tau_2 \geq 0 \tag{82}$$

where, $\delta(\tau_1,\tau_2)$ is the 2-d Delta-Dirac function. The last equation is simply the extension of the classical "normal equation" or Yule-Walker (YW) equation of the autocorrelation domain to the third order cumulant domain. It can be easily extended to higher order cumulant domains. By replacing the third order cumulants by their sample estimates $\hat{C}_{y,3}(\tau_1,\tau_2)$ obtained from the observed samples $\{y(n)\}$, $n = 1,\ldots,N$, and by repeating the TOR equation for $\tau_1 = \tau_2 = 1,\ldots p$, we obtain the linear system of equations

$$\begin{bmatrix} \hat{C}_{y,3}(0,0) & \cdot\ \cdot & \hat{C}_{y,3}(p-1,p-1) \\ \hat{C}_{y,3}(-1,-1) & \cdot\ \cdot & \hat{C}_{y,3}(p-2,p-2) \\ \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot\ \cdot & \cdot \\ \hat{C}_{y,3}(-p+1,-p+1) & \cdot\ \cdot & \hat{C}_{y,3}(0,0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_p \end{bmatrix} = - \begin{bmatrix} \hat{C}_{y,3}(-1,-1) \\ 0 \\ 0 \\ 0 \\ \hat{C}_{y,3}(-p,-p) \end{bmatrix}$$

The above system can be solved with respect to the coefficients $\{a_k\}$, $k = 1,\ldots,p$. Then, the $\gamma_{x,3}$ is obtained from the TOR with $\tau_1 = \tau_2 = 0$. Finally, the bispectrum is obtained from Eq. (31) by substituting $F(e^{j\omega}) = \frac{1}{1+\sum_{k=1}^{p} a_k e^{-j\omega k}}$.

It has been shown by Raghuveer and Nikias [32] that as long as the underlying process is truly AR(p), the TOR method provides consistent estimates of the parameters $\{a_k\}$, $k = 1,\ldots,p$. In practice, the order of the AR model needs to be estimated. However, efficient model order selection criteria have not been proposed yet.

The TOR equation was developed from a causal model. Non-causal and anticausal models can be considered as well but then different TOR equations are derived. Other non-causal AR model identification methods based on higher-order statistics have been proposed by Huzii [34] and Tugnait[35].

Example 10.
We observe 1024 samples from the process

$$z(k) = y(k) + w(k).$$

The process $y(k)$ is generated by a stable AR(2) model with transfer function

$$F(e^{j\omega}) = \frac{1}{1 - 1.3\cdot e^{-j\omega} + 0.81\cdot e^{-j2\omega}} \tag{83}$$

which is driven by a real i.i.d. exponentially distributed process with zero mean and unit variance. The $w(k)$ is real zero-mean additive white Gaussian noise independent from $y(k)$. The signal to noise ratio (SNR)is defined as $10log_{10}(P_y/P_w)$, where P_y and P_w denote the power of $y(k)$ and $w(k)$ respectively. The autocorrelation based normal equations and the TOR method are applied with the observed samples to estimate the coefficients a_1 and a_2 of the AR(2) model. The autocorrelation based normal equation (YW) takes the form

$$C_{y,2}(-\tau) + \sum_{k=1}^{p} a_k C_{y,2}(k-\tau) = \gamma_{x,2}\delta(\tau), \qquad \tau \geq 0$$

The model order $p = 2$ was assumed known. The sample estimates of second and third order cumulants were obtained by segmenting the data into records of length 128 samples each. The results for different SNR levels are depicted in Table IV. As expected the performance of the YW method deteriorates faster than the performance of the TOR as the SNR decreases.

Table IV: Estimated AR coefficients via "normal equations"

	Estimated coefficients			
	SNR=28 dB	SNR=20 dB	SNR=8 dB	SNR=0 dB
YW	1.000	1.000	1.000	1.000
	-1.289	-1.200	-1.144	-0.611
	0.805	0.799	0.703	0.297
TOR	1.000	1.000	1.000	1.000
	-1.275	-1.275	-1.237	-0.933
	0.784	0.785	0.786	0.519

3 ARMA Methods For H.O.S. Estimation

Let $\{y(n)\}$ be the output from an ARMA(p,q) stable model driven by the zero-mean non-Gaussian i.i.d. process $\{x(n)\}$. Then,

$$y(n) + \sum_{k=1}^{p} a_k \cdot y(n-k) = \sum_{k=0}^{q} b_k \cdot x(n-k) \tag{84}$$

The modified TOR equation are written as [1-3]:

$$C_{y,3}(-\tau_1, -\tau_2) + \sum_{k=1}^{p} a_k C_{y,3}(k-\tau_1, k-\tau_2), \qquad \tau_1, \tau_2 > q \tag{85}$$

Note that for $q = 0$ the modified TOR equation becomes identical to the TOR equation. Thus, by replacing the third order cumulants by their sample estimates $\hat{C}_{y,3}(\tau_1, \tau_2)$ obtained from the observed samples $\{y(n)\}$, $n = 1, \ldots, N$, and by repeating the modified TOR equation for $\tau_1 = \tau_2 = q+1, \ldots q+p$, we obtain a linear system of equations that can be solved with respect to the unknown AR coefficients $\{a_k\}$, $k = 1, \ldots, p$.

Then, by filtering the observed sequence $\{y(n)\}$, $n = 1, \ldots, N$ with a filter having transfer function

$$\hat{A}(e^{j\omega}) = 1 + \sum_{k=1}^{p} \hat{a}_k e^{-j\omega k}$$

where $\hat{a}_k$ are the estimated AR coefficients, we obtain a sequence $\{z(n)\}$ that is described by a nonminimum phase MA(q) model with coefficients

$\{b_k\}$, $, k = 0, \ldots, q$. This is depicted in Figure 16. The coefficients $\{b_k\}$ can be estimated by applying MA methods for H.O.S. estimation on $\{z(n)\}$.

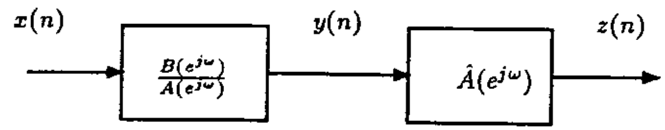


Figure 16: Method for ARMA model identification based on H.O.S.

The method described above was proposed by Giannakis and Mendel [22]. Other methods for ARMA identification have been proposed Giannakis and Swami [36], Swami and Mendel [37,38] and Tugnait [39,40].

D Polycepstra Estimators

Polycepstra methods for H.O.S. estimation are based on the non-causal linear filtering model of section II.D, that is

$$F(e^{j\omega}) = G \cdot e^{j\omega r} \cdot I(e^{-j\omega}) \cdot O(e^{j\omega}) \tag{86}$$

The nonminimum phase system is driven by the i.i.d. non-Gaussian process $\{x(k)\}$. The estimation of the higher-order spectra of the system output $\{y(k)\}$ follows the following steps:

1. Estimate the polycepstrum of the system output $\{y(k)\}$ or equivalently the minimum phase and maximum phase differential cepstrum parameters $\{A(k)\}$ and $\{B(k)\}$.
2. Obtain the maximum and minimum phase components of the channel $\{i(k)\}$ and $\{o(k)\}$ by substituting the $\{A(k)\}$ and $\{B(k)\}$ into Eqs. (46) and (47), respectively.
3. Calculate the discrete Fourier transforms $I(e^{-j\omega}) = \mathcal{F}[i(k)]$ and $O(e^{j\omega}) = \mathcal{F}[o(k)]$ and substitute them in Eq. (86) to obtain an estimate of $F(e^{j\omega})$.
4. Obtain estimates of the bispectrum, trispectrum or any other higher-order spectrum by substituting $F(e^{j\omega})$ into Eq. (31).

In general, the gain G and the time delay r in Eq. (86) are not recoverable. Recall that cumulant spectra are blind to linear phase shifts. Thus, in general a scaled estimate of the higher-order spectra is obtained.

It can be shown that the following relations hold between the bicepstrum $c_{y,3}(\tau_1, \tau_2)$ and the third-order cumulant sequence $C_{y,3}(\tau_1, \tau_2)$ [41].

$$\tau_1 \cdot c_{y,3}(\tau_1, \tau_2) = \mathcal{F}^{-1}\left[\frac{\mathcal{F}[\tau_1 \cdot C_{y,3}(\tau_1, \tau_2)]}{\mathcal{F}[C_{y,3}(\tau_1, \tau_2)]}\right] \tag{87}$$

$$A(k) = -k \cdot c_{y,3}(k, 0), \qquad k = 1, 2, \ldots$$
$$B(-k) = k \cdot c_{y,3}(k, 0), \qquad k = -1, -2, \ldots$$

or

$$\sum_{k=1}^{\infty} A(k)\,[C_{y,3}(\tau_1 - k, \tau_2) - C_{y,3}(\tau_1 + k, \tau_2 + k)] + \sum_{k=1}^{\infty} B(k)\,[C_{y,3}(\tau_1 - k, \tau_2 - k) - C_{y,3}(\tau_1 + k, \tau_2)] = -\tau_1 \cdot C_{y,3}(\tau_1, \tau_2) \tag{88}$$

Similar relations hold true for the power cepstrum and higher-order polycepstra. From the definition of $A(k)$, $B(k)$, $k = 1, 2, \ldots$ in section II.D and the assumption that $|a_i| < 1$, $|b_i| < 1$, and $|c_i| < 1$, we conclude that the differential cepstrum parameters decay exponentially as k increases. Therefore, we can define two truncating parameters $p > 0$ and $q > 0$ and place $A(k) = 0, \quad k > p$ and $B(k) = 0, \quad k > q$ [41]. Hence, some knowledge about the location of the zeros $\{a_i\}$ and $\{b_i\}$ and the poles $\{c_i\}$ of the system is required for choosing the p and q.

Given the sample estimates $\hat{C}_{y,3}(\tau_1, \tau_2)$, $\tau_1, \tau_2 = 0, \pm 1, \ldots, \pm L$, we can choose $Q > max[2L+1, p+q]$ and implement Eq. (87) via 2-d FFT ($Q \times Q$ points) operations. Alternatively, we can rewrite Eq. (88) as follows:

$$\sum_{k=1}^{p} A(k)\,[C_{y,3}(\tau_1 - k, \tau_2) - C_{y,3}(\tau_1 + k, \tau_2 + k)] + \sum_{k=1}^{q} B(k)\,[C_{y,3}(\tau_1 - k, \tau_2 - k) - C_{y,3}(\tau_1 + k, \tau_2)] = -\tau_1 \cdot C_{y,3}(\tau_1, \tau_2) \tag{89}$$

By repeating Eq. (89) N_{pq} times with different values of τ_1 and τ_2, a linear overdetermined system of equation can be obtained , that is

$$\begin{bmatrix} N_{pq} \times (p+q)\ matrix \\ with\ entries \\ of\ the\ form \\ \{C_{y,3}(\lambda_1, \lambda_2)) - C_{y,3}(\nu_1, \nu_2)\} \end{bmatrix} \begin{bmatrix} A(1) \\ \cdot \\ \cdot \\ A(p) \\ B(1) \\ \cdot \\ \cdot \\ B(q) \end{bmatrix} = \begin{bmatrix} N_{pq} \times 1\ vector \\ with\ entries \\ of\ the\ form \\ \{-\lambda \cdot C_{y,3}(\lambda, \tau)\}\} \end{bmatrix}$$

or

$$\mathbf{A} \quad \cdot \quad \mathbf{b} \quad = \quad \mathbf{a} \tag{90}$$

where, $N_{pq} > (p+q)$. The least squares solution to the above system is

$$\mathbf{b} = (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{a} \tag{91}$$

Then, by substituting the estimated $A(k)$, $k = 1, \ldots, p$ and $B(k)$, $k = 1, \ldots, q$ into Eqs. (46) and (47) we obtain $(i(0) = o(0) = 1)$

$$i(k) = -\frac{1}{k} \sum_{n=2}^{k+1} A(n-1) \cdot i(k-n+1), \qquad k = 1, 2, \ldots, L_1 \tag{92}$$

$$o(k) = \frac{1}{k} \sum_{n=k+1}^{0} B(1-n) \cdot o(k-n+1), \qquad k = -1, \ldots, -L_2 \tag{93}$$

Finally,

$$f_{norm}(k) = i(k) * o(k), \qquad k = -L_2, \ldots, 0, \ldots, L_1$$
$$F_{norm}(e^{j\omega}) = \mathcal{F}[f_{norm}(k)]$$

where the $\{*\}$ denotes linear convolution and *norm* refers to gain normalization.

Unique characteristic of the above polycepstrum estimation methods is the ability to identify the minimum and maximum phase components of the observed signals separately. They can deal with AR, MA and ARMA models without requiring model order selection. However, these methods are sensitive to the severe underestimation of the truncating parameters p and q. Their complexity is of the order of $(p+q)$ and therefore increases as the magnitude of $\{a_i\}$, $\{b_i\}$ and $\{c_i\}$ gets closer to one. A detailed description of polycepstra and their properties are given by Pan and Nikias in [41]. Also analytical results for the performance of the bispectrum have been derived by Petropulu and Nikias in [42]. Other results for the polycepstra of non-Gaussian linear processes and their applications can be found in [44-47].

Example 11.
The nonminimum phase system transfer function has the form

$$F(e^{j\omega}) = \frac{(1-0.4e^{j\omega})\cdot(1+0.333e^{-j\omega})}{[1+(0.425+j0.4235)e^{-j\omega}]\cdot[1+(0.425-j0.4235)e^{-j\omega}]}$$

The system is driven by an i.i.d. zero mean exponentially distributed process. 1024 samples of the system output are observed and the bicepstrum estimation approach is applied with $p = q = 6$. The true and estimated impulse responses of the system are illustrated in Figure 17.

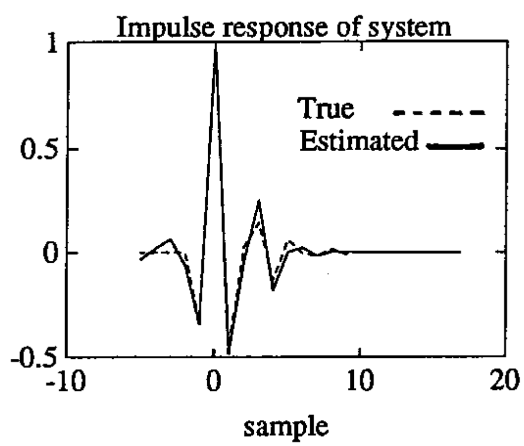


Figure 17: True and estimated impulse responses via bicepstrum method.

IV Applications of H.O.S.

In this section we provide a brief overview of signal processing applications with H.O.S. to demonstrate the potential of H.O.S. in solving real life engineering problems. The majority of applications are in the area of 1-d and 2-d signal processing. The utilization of H.O.S. in higher-dimensional and possibly multichannel applications is severely limited by the high computational complexity involved.

A H.O.S. Applications in 1-d Signal Processing

Applications of H.O.S. to quadratic and cubic phase coupling, blind system identification and deconvolution, time delay estimation, retrieval of harmonics, and array processing are discussed.

1 Quadratic and Cubic Phase Coupling

Frequency and phase coupling relations usually arise when harmonic signals become subject to nonlinear transformations that cause interactions between the harmonic components. Phase coupling occurs when the relation between phases of harmonically related components is the same as the relation between the frequencies of those components. The power spectrum does not distinguish phase coupled components from other frequencies. However, the bispectrum of harmonic processes is nonzero only at quadratically phase coupled components and the trispectrum is nonzero only at cubically phase coupled components. The use of bispectrum for detection and quantification of quadratic phase coupling has been investigated by Ranguveer and Nikias [32, 33] The properties of the bispectrum and trispectrum of cubically phase coupled harmonic signals have been investigated by Swami and Mendel [48].

2 Blind System Identification and Deconvolution

Blind deconvolution refers to the problem of separating two unknown or partially known signal that have been convolved by observing only the result of their convolution. Such a problem arises in seismic signal processing, blind identification and equalization of digital communications channels, speech processing and other applications. A typical situation is that of the linear filtering problem where the input random process is convolved with the impulse response of a linear system. It is assumed that only the statistical properties of the input process are known. For the reasons explained in sections II.C and II.D blind identification of nonminimum phase systems is not possible if we rely on the second order statistics of the system output only. Thus, H.O.S. based approaches are needed. On the other hand, blind equalization deals with the recovery of the unknown input sequence to the channel and in most cases requires to blindly identify the system or its inverse first. The Parametric and Polycepstra H.O.S. estimation methods presented in section III.C and III.D are basically blind identification methods. Other methods for the recovery of the magnitude and phase of the system transfer function from the bispectrum of its output have been proposed by Brillinger [19], Lii and Rosenblatt [21], Matsuoka and Ulrych [49], Jelonnek and Kammeyer [50], and Cheng and Venetsanopoulos [51]. Methods for blind equalization of nonminimum phase communication channels based on fourth-order cumulant techniques have been introduced by Hatzinakos and Nikias [44,45] , Porat and Friedlander [52], Tugnait [53], Bessios and Nikias [54], Ilow, Hatzinakos and Venetsanopoulos [55], Zheng, McLaughlin and Mulgrew [56-58], and Tsatsanis and Giannakis [59].

3 Time Delay Estimation

This is the problem of estimating the difference in arrival time between a signal and its shifted and probably scaled version. When the two signals are corrupted by spatially correlated Gaussian noise sources of unknown cross-correlation function, H.O.S. based approaches outperform the classical second order statistics based methods which utilize the cross-correlation between the signals. Methods for time delay estimation with H.O.S. have been proposed by Nikias and Pan [60], Zhang and Ranguveer [61], Hinich and Wilson [62], and Tugnait [63].

4 Retrieval of Harmonics

The objective of harmonic retrieval is to estimate the number, frequencies and amplitudes of harmonics in in presence of Gaussian noise. This problem has been treated satisfactorily by second-order statistics methods when the Gaussian noise is white. However, when the noise is colour Gaussian, H.O.S. methods provide better results. Fourth-order cumulant based methods for harmonic retrieval have been proposed by Swami and Mendel [64], Moulines and Cardoso [65], and Pan and Nikias [66].

5 Array Processing Applications

Among the problems of array processing where H.O.S. methods have been applied are those of determining the direction of arrival of a signal, determination of the number of sources, beamforming and source classification. H.O.S. methods for array processing were found to perform better than the existing second-order statistics approaches in cases where the background Gaussian noise is colour and the signal sources are coherent. Among the methods proposed are those by Cardoso [67], Forster and Nikias [68], Ruiz [69], Lagunas and Vazquez [70], and Dogan and Mendel [71].

B H.O.S. Applications in 2-d Signal Processing

The extension of the definitions of H.O.S. and their properties to 2-d signals is straightforward. However, in this case the the computational requirements are higher because the dimensionality of H.O.S. increases by a factor of two. Erdem and Tecalp [72] proposed parametric modeling of blur in images and the utilization of the bicepstrum of the blurred image for the identification of the model parameters. They argue that H.O.S. methods exhibit better performance than autocorrelation based approaches because they can deal with non-causal and nonminimum phase models and are less sensitive to Gaussian noise. The existence and uniqueness of higher-order spectrum factorization of a non-Gaussian 2-d process is discussed by Tekalp and Erdem [73], Kang, Lay and Katsaggelos [74] and Petropulu and Nikias [75] proposed algorithms for image restoration which utilize the bispectrum of the noisy and blurred image to recover phase information. Bartelt and Wirnitzer [76] developed a method based on bispectral analysis to avoid motion blur. A recursive algorithm for the estimation of image motion has been introduced by Anderson and Giannakis [77]. Dianat and Raghuveer [78] developed computationally efficient bispectral techniques for magnitude and phase estimation of images. Methods for the detection and classification of texture in images have been introduced by Ramponi and Sicuranza [79]. Hall and Wilson [80] investigated applications of third order statistics in differential pulse code modulation (DPCM) predictive image coding.

C H.O.S. Applications in Multichannel Signal Processing

Multichannel processes can be modeled as vector processes. Then, the definition and properties of the H.O.S. of 1-d processes can be extended to the case of vector processes by exploiting the algebraic properties of Kronecker products [3]. Swami, Giannakis and Mendel [81] developed several higher-order cumulant based techniques for the estimation of the parameters of multichannel ARMA processes. Swami and Mendel [82] proposed recursive algorithms for the computation of the cumulants of multiple-input multiple-output (MIMO) time varying systems. Zhang, Hatzinakos and Venetsanopoulos [83] applied multichannel third-order statistics methods for the restoration of colour image which can be modeled as tri-channel two-dimensional signals.

V Conclusions

An overview of the definitions, properties, and utilization of the H.O.S. of discrete random processes was provided. It was pointed out that this relatively new signal processing framework can be useful in a wide range of applications where the observed signals are not Gaussian and therefore a large degree of information is contained in their H.O.S. A more detailed coverage of signal processing algorithms and applications with H.O.S. can be found in three tutorial papers by Nikias and Ranguveer [2], Mendel [3], and Nikias and Mendel [4] and a book by Nikias and Petropulu [1].

Despite their attractive properties and the emergence of highly sophisticated algorithms for solving real life problems, the utilization of H.O.S. has been limited by the following reasons: i) the high computational complexity and the long data records required in the estimation of H.O.S. compared to the second-order statistics, and ii) the hard assumptions made for stationarity and ergodicity of the observed data records. The introduction of more efficient estimation procedures for obtaining higher-order moments and cumulants from short data records and the availability of fast digital processors and VLSI architectures will open the door for the wider utilization of H.O.S. algorithms.

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