ECE 1511 F

Lecture ~~8~~12,

(D. Hatzinakos)

14 ALPHA-STABLE PROCESSES AND FRACTIONAL LOWER-ORDER MOMENTS (F.L.O.S.)

- Gaussian models are justified by the Central Limit Theorem (C.L.T.)
- Optimization of Non-Gaussian phenomena under the Gaussian assumption may result in significant degradation in performance.
- Stable distributions satisfy the Generalized Central Limit Theorem (G.C.L.T). The Gaussian distribution is only a special case of stable distributions.
- α-Stable distributions (characteristic exponent $0 < \alpha < 2$).
    1. Lack of finite variance or H.O.S.
    2. Finite dispersion
- Least-Squares techniques (Gaussian assumption) are not robust agaist outliers.

GAUSSIAN	NON-GAUSSIAN
1) $\alpha = 2$ 2) Finite variance	1) $\alpha < 2$, ($\alpha = 1$, Cauchy) 2) Infinite variance, finite dispersion
3) H.O. Cumulants $\equiv 0$ 4) MMSE criterion 5) Correlations	3) All H.O.S. of order greater than α dont exist or be finite. However Fractional-lower order moments (F.L.O.M.) of order less than α, greater than 0, exist. 4) Minimum dispersion error criterion (M.D.E.) 5) Covariations
6) Usually linear solutions 7) Any affine tranformation of independent Gaussian r.v.s provides jointly Gaussian r.v.s	6) Nonlinear solutions 7) Not true for non-Gaussian stable processes
8) C.L.T., stability property 9) density function decays exponentially	8) G.C.L.T., stability property 9) density function decays algebraicly (in most cases)

14.1 NON-GAUSSIAN α-STABLE DiSTRIBUTIONS

→ Appropriate to describe Impulsive phenomena (for ex: signals with magnitute exhibiting large deviations from the average value more frequently than Gaussian signals)

- Underwater acoustic signals
- impulsive low frequency atmospheric noise
- clustering of errors in telephone networks

→ Have attracted little attention by researchers in engineering

- no closed form expressions for density and distributions
- F.L.O.S introduce nonlinearities in the calulations
- Variance is associated with power of a process
  Process with infinite variance !!!!

→ $\boxed{\text{The information that these distributions might allow us to obtain for impulsive type phenomena may far outweigh the inconveniences.}}$

- **HISTORICAL PERSPECTIVE**

  P. LEVY (1925)

  S. CAMPANIS (1973, 1981)

  J. MILLER (1978)

  D. MIDDLETON (1977, 1979)

  M. SHAO, C. L. NIKIAS (1993)

- **APPLICATIONS**

  ECONOMICS (price movement)

  PHYSICS, ASTRONOMY
  (Gravitational fields of stars, $\alpha = 1.5$)

  HYDROLOGY, BIOLOGY

  ELECTRICAL ENGINEERING

  Impulsive noise, underwater noise
  , atmospheric noise.

  (Signal detection, parameter estimation and classification,
  Noise filtering, noise cancellation,
  signal image enhancement. - - - .)

α-stable distributions

- **DEFINITIONS**

- The univariate distribution function $F(x)$ of a r.v. X is stable (or α-stable) iff its characteristic function takes the form

$$\boxed{\phi(t) = e^{jat - \gamma|t|^{\alpha}[1 + j\beta \operatorname{sign}(t) \cdot w(t,\alpha)]}}$$

where,
$$w(t,\alpha) = \begin{cases} \tan(\alpha\pi/2) &, \alpha \neq 1 \\ \frac{2}{\pi}\log|t| &, \alpha = 1 \end{cases}$$

$$\operatorname{sign}(t) = \begin{cases} 1 &, t>0 \\ 0 &, t=0 \\ -1 &, t<0 \end{cases}$$

$$-\infty < a < \infty \;,\; \gamma > 0 \;,\; 0 < \alpha \leq 2 \;,\; -1 \leq \beta \leq 1.$$

→ a: location parameter [for $\alpha > 1$, $a = E\{X\}$]

→ γ: scale parameter (dispersion) [for $\alpha = 2$, $\gamma = \frac{\sigma^2}{2}$

→ β: skewness [$\beta = 0$, $f(x)$ symmetric arount a, SαS]

→ α: characteristic exponent (a shape parameters that measures the thickness of tails of $f(x)$)
[$\alpha = 2$: Gaussian, $\alpha = 1$ Cauchy]

["Standard" stable distribution : $a = 0$, $\gamma = 1$

$\frac{X - a}{\gamma^{1/\alpha}}$ is "standard" with α and β

According to that definition the standard Gaussian has variance $= 2$]

"<u>standard</u>" stable density function:

$$\left[\begin{array}{l} f(x;\alpha,\beta) = \frac{1}{\pi}\int_0^\infty e^{-t^\alpha}\cos\left[xt + \beta t^\alpha w(t,\alpha)\right] dt \\ f(x;\alpha,\beta) = f(-x;\alpha,-\beta) \\ f(x;\alpha,\beta) : \text{bounded and differentiable in any order} \end{array}\right]$$

→ **No Closed Form Expression exists except for $\alpha = 2$ (Gaussian), $\alpha = 1, \beta = 0$ (Cauchy), $\alpha = \frac{1}{2}, \beta = -1$ (Pearson)**

→ **Power Series expansions exist.**

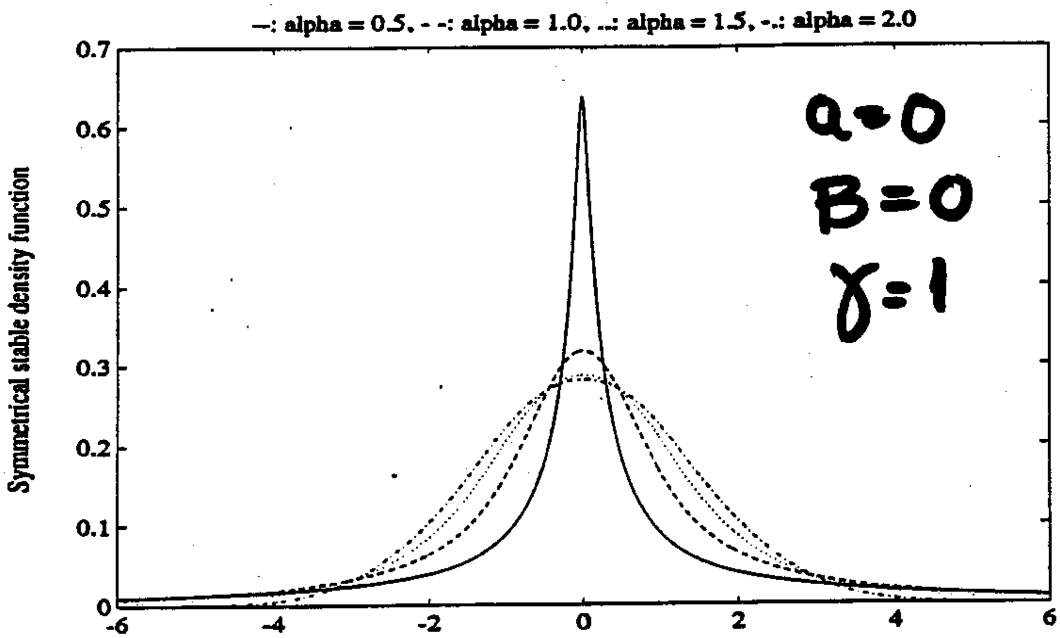
$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\pi x}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\Gamma(\alpha k+1)\left(\frac{x}{r}\right)^{-\alpha k}\sin\left[\frac{k\pi}{2}(\alpha+\gamma)\right], & 0<\alpha<1 \\ \frac{1}{\pi x}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\Gamma\left(\frac{k}{\alpha}+1\right)\left(\frac{x}{r}\right)^{k}\sin\left[\frac{k\pi}{2\alpha}(\alpha+\gamma)\right], & 1<\alpha\le 2 \end{cases}$$

$x > 0$

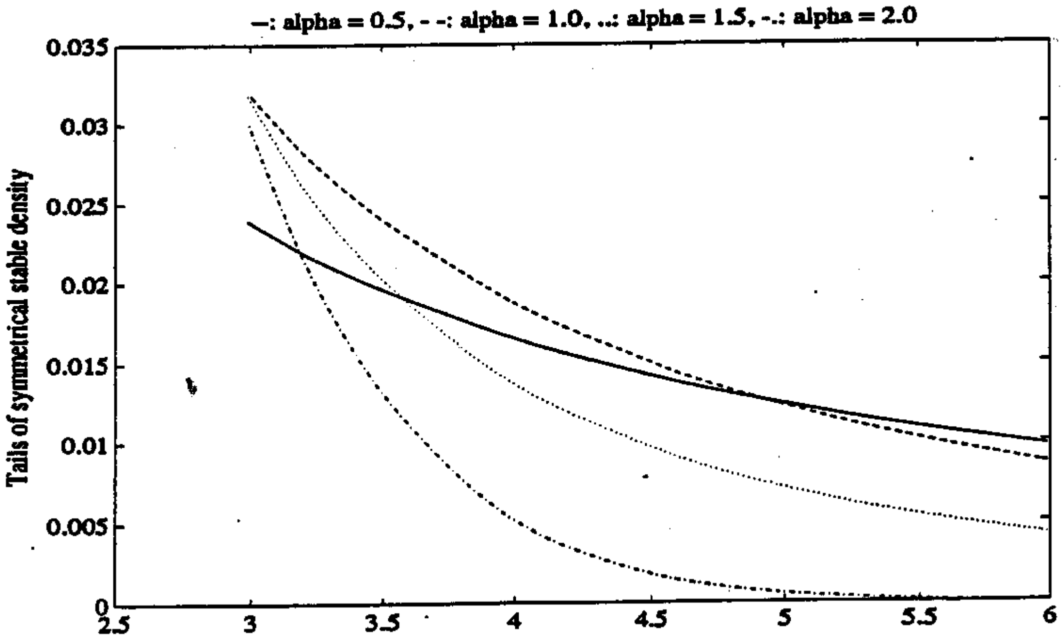
where, $r = (1+\eta^2)^{\frac{1}{2\alpha}}$, $\eta = \beta\tan\left(\frac{\pi\alpha}{2}\right)$

$\gamma = -\left(\frac{2}{\pi}\right)\arctan\eta$, $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$.

- <u>Multivariate</u> stable distributions and characteristic functions assume very complicated expressions in general



(a)



(b)

Density functions of $S\alpha S$ distributions

1) Stability property: A r.v. X has a stable distribution if for all r.v.s X_1, X_2 independent, and X_1, X_2, X identically distributed and arbitrary constants c_1, c_2

$$\boxed{c_1 X_1 + c_2 X_2 \overset{\text{distribution}}{=\!=} bX + d}$$

where, b, d are appropriate constants

Thus, if X_i, $i=1,\dots,n$ are stable independent ident. distr. with the same (α,β), then all linear combinations $\sum_{i=1}^{n} c_i X_i$ are stable with the same parameters (α,β)

Similarly: A multivariate k-dimensional distribution function $F(\underline{x})$, $\underline{x} \in R^k$ is called stable if for all i.i.d. r. vectors $\underline{X}_1$, $\underline{X}_2$ with distrib. $F(\underline{x})$ and arbitrary constants c_1, c_2 there exist $b \in R$ and $d \in R^k$ and a random vector $\underline{X}$ with the same $F(\underline{x})$ so that

$$\boxed{c_1 \underline{X}_1 + c_2 \underline{X}_2 \overset{\text{distr.}}{=\!=} b\underline{X} + \underline{d}}$$

* If $1<\alpha\leq 2$ then, $\{X_i\}$ $i=1,\dots,n$ are jointly stable iff every linear combination of $\{X_i\}$ is also stable with the same α.
* Marginal distributions of stable multivariate r.v.s are also stable not vice ver

Important difference: Jointly stable r.v.s cannot be written as affine transformation of equal number of independent stable r.v.s (with same α,β) unless they are jointly Gaussian

2) The Generalized Central Limit Theorem (G.C.L.T.)

$$S_n = \frac{\sum_{i=1}^{n} X_i}{b_n} - a_n \xrightarrow{\text{distribut.}} X$$

where, $\{X_i\}_{i=1,\dots,n}$ are i.i.d and $\to\infty$, iff X has a stable distribution. If $\{X_i\}$ have finite variance then X is Gaussian.

3) Moments of stable r.v.s

→ Let X be α-stable r.v. If $0<\alpha<2$, then

$$E\{|X|^\delta\} = \infty \quad \text{if } \delta \geq \alpha$$
$$E\{|X|^\delta\} < \infty \quad \text{if } 0 \leq \delta < \alpha$$

[if $\alpha = 2$ then $E|X^\delta| < \infty$ for all $\delta \geq 0$]

→ If $X_1, X_2, \dots, X_n$ are independent and α-stable, then

$$E\{|X_1|^{\delta_1} \dots |X_n|^{\delta_n}\} < \infty \quad \text{if } \delta_i < \alpha, \; i=1,\dots,n$$

→ If $X_1, \dots, X_n$ are dependent and jointly $S\alpha S$, then

$$E\{|X_1|^{\delta_1} \dots |X_n|^{\delta_n}\} < \infty \quad \text{if } 0 < \delta_1 + \dots + \delta_n < \alpha$$

- **Symmetric α-stable (SαS) r.v.s and r. processes**

→ X real r.v. ∼ SαS : $\varphi(t) = e^{jat - \gamma|t|^\alpha}$, $\left\{\begin{matrix} 0<\alpha\le 2 \\ \gamma>0 \\ -\infty<a<\infty \end{matrix}\right\}$

→ $X_1, \ldots, X_n$ real r.v.s ∼ SαS : $\varphi(\underline{t}) = e^{j\underline{t}^T\underline{a} - \int_S |\underline{t}^T\underline{s}|^\alpha \mu(d\underline{s})}$, $\left\{\begin{matrix} \underline{a}, \underline{t} \in R^n \\ S: \text{n-d unit sphere} \\ \mu(d\underline{s}): \text{Spectral measure on } S \end{matrix}\right\}$

→ Stochastic process: $X(t_n)$, $t_n \in T$ if $X(t_1), X(t_2) \ldots$ are SαS with same α.

NOTE: ① The set of r.v.s with finite variance forms a Hilbert space
[i.e. a complete inner product space (linear space with inner product). Inner product generates also a norm i.e distance and includes the concepts of projection and orthogonality. It is the space closest to the three dimens. Euclidean geometry]

② The set of stable r.v.s with $1 \le \alpha < 2$ is a Banach space
[i.e., a complete linear space with a norm (distance) The norm combines the concepts of continuity in metric spaces and linearity in linear spaces Close to Euclidean geometry but not as close as Hilbert spaces]

③ The set of stable r.v.s with $0 < \alpha < 1$ is a metric space
[i.e., a pair of two objects: a set X and a metric $d(x,y)$ Euclidean distance can be one of infinite in general metrics. It is a topological structure that is complete if every Cauchy sequence in (X,d) is also convergent in (X,d) ...]

(4) A Linear space (vector space) over a scalar Field is an nonempty set that is closed under vector addition and scalar multiplication (Recall Lecture 9)
It is an algebraic structure.

* (For the properties of norm and inner product refer to Lecture 9 or any linear operator theory book.)

F.L.O.M. (Fractional Lower-Order Moments)

Let $X: S\alpha S$ real r.v. with $a=0$. Then

$$\boxed{E\{|X|^p\} = C(p,\alpha)\cdot \gamma^{\frac{p}{\alpha}}}\quad , \; 0<p<\alpha$$

where:
$$C(p,\alpha) = \frac{2^{p+1}\,\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(-\frac{p}{\alpha}\right)}{\alpha\sqrt{\pi}\,\Gamma(-p/2)}$$

Also:
$$\boxed{\|X\|_\alpha = \begin{cases} \gamma^{1/\alpha} & , 1\leq \alpha \leq 2 \quad \text{(norm)} \\ \gamma & , 0<\alpha<1 \quad \text{(metric)} \end{cases}}$$

X, Y jointly $S\alpha S$; distance between X, Y is defined as

$$\boxed{d_\alpha(X,Y) = \|X-Y\|_\alpha}$$

Convergence in $d_\alpha(\cdot,\cdot)$ is equivalent to convergence in probability

* **In practice most useful are $S\alpha S$ r.v.s with $1<\alpha<2$**

14.3 COVARIATIONS AND F.L.O.M.

Let X, Y be jointly $S\alpha S$ with $1 < \alpha \leq 2$ and dispersions γ_x, γ_y, respectively. Then,

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Covariation : $[X,Y]_\alpha \triangleq \dfrac{E\{XY^{<p-1>}\}}{E\{|Y|^p\}}\gamma_y = \lambda_{XY}\gamma_y$

Covariation Coefficient : $\lambda_{XY} \triangleq \dfrac{[X,Y]_\alpha}{[Y,Y]_\alpha} = \dfrac{E\{XY^{<p-1>}\}}{E\{|Y|^p\}}$

Dispersion : $[Y,Y]_\alpha = \|Y\|_\alpha^\alpha = \gamma_y$

where, $\left\{ \begin{array}{l} z^{<q>} = |z|^q \operatorname{sign} z \\ z : \text{real} \end{array} \right.$, $\left\{ \begin{array}{l} 1 \leq p < \alpha , \\ \text{or } 1 \leq p \leq 2 , \ \alpha = 2 \end{array} \right\}$

Properties: Given X_1, X_2, Y_1, Y_2 .... jointly $S\alpha S$

1. Linearity w.r.t. X, (c_1, c_2 real constants)
$$[c_1X_1 + c_2X_2, Y]_\alpha = c_1[X_1,Y]_\alpha + c_2[X_2,Y]_\alpha$$

2. For $\alpha = 2$: $[X,Y]_\alpha = E\{XY\}$ (zero mean r.v.s)

3. Pseudolinearity w.r.t. Y, (Y_1, Y_2 independent)
$$[X, c_1Y_1 + c_2Y_2]_\alpha = c_1^{<\alpha-1>}[X,Y_1]_\alpha + c_2^{<\alpha-1>}[X,Y_2]_\alpha$$

4. If X, Y are independent $S\alpha S$ $\underset{\not\Longleftarrow}{\Longrightarrow}$ $[X,Y]_\alpha = 0$

(5) For any r.v.s X, Y jointly $S\alpha S$

$$|[X,Y]_\alpha| \le \|X\|_\alpha \|Y\|_\alpha^{<\alpha-1>} \quad \text{(Cauchy - Schwartz ineq.)}$$

(6) Let U_i, $i=1,\dots,n$ be independent $S\alpha S$ with dispersion γ_i ~~and let~~

$$X = a_1 U_1 + \dots + a_n U_n \qquad \{a_i\} \text{ constants}$$
$$Y = b_1 U_1 + \dots + b_n U_n \qquad \{b_i\} \text{ ''}$$

Then,
$$[X,X]_\alpha = \sum_{i=1}^{n} \gamma_i |a_i|^\alpha \quad , \quad [Y,Y]_\alpha = \sum_{i=1}^{n} \gamma_i |b_i|^\alpha$$
$$[X,Y]_\alpha = \sum_{i=1}^{n} a_i b_i^{<\alpha-1>} \quad , \quad \lambda_{XY} = \frac{[X,Y]_\alpha}{[Y,Y]_\alpha}$$

(7) If X_0, X_1, X_2 are jointly $S\alpha S$ and if $E\{X_0 / X_1, X_2\} = a_1 X_1 + a_2 X_2$

Then,
$$a_1 [X_1, X_1]_\alpha + a_2 [X_2, X_1]_\alpha = [X_0, X_1]_\alpha$$
$$a_1 [X_1, X_2]_\alpha + a_2 [X_2, X_2]_\alpha = [X_0, X_1]_\alpha$$

a_1, a_2 are uniquely determined iff neither X_1 nor X_2 are multiple of each other.

(8) For any two jointly $S\alpha S$ r.v.s X, Y $\boxed{E\{X/Y\} = \lambda_{XY} Y}$

(9) If $X_0, X_1, \dots, X_n$ are jointly $S\alpha S$ r.v.s and $X_1, \dots, X_n$ are indep.

$$E\{X_0 / X_1, \dots, X_n\} = \lambda_{01} X_1 + \dots + \lambda_{0n} X_n$$

(10) Given a $S\alpha S$ stochastic process X_{t_1}, then

$$\lambda_{X_{t_1} X_{t_2}} = \lambda_X(t_1, t_2) \overset{\text{station.}}{=\!=} \lambda_X(t_2 - t_1) \quad \text{and so on....}$$

14.4 TESTS FOR INFINITE VARIANCE

* In many applications is not necessary to know the exact value of α but rather whether the underlying distribution is Gaussian ($\alpha=2$) or stable non-Gaussian ($\alpha<2$)

ASSUMPTION: Distribution is $S\alpha S$

1) GENERALIZED LIKELIHOOD RATIO TEST : Find ML estimate of α
   Computationally intensive method

2) TEST FOR FINITE VARIANCE: Let X_k, $k=1,\dots,N$ be samples from an $S\alpha S$ process (distribution)
   - Form
   $$S_n^2 = \frac{1}{n}\sum_{k=1}^{n}(X_k-\bar{X}_n)^2$$
   $$\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k \qquad , \quad n=1,2,\dots,N$$
   - Plot S_n^2 vs n. If S_n^2 converges then distribution is Gaussian. If it diverges the distribution is $S\alpha S$ non Gaussian.

3) LONG TAIL TEST : Examine the shape of the tails of distrib.
   - For a non-Gaussian α-stable distribution
   $$\lim_{t\to\infty} t^{\alpha} P(|X|>t) = \gamma C(\alpha) \qquad : \text{algebraic decay of tails.}$$
   - Plot log P(X > u) vs log t for large t → straight line with slope

ESTIMATION OF COVARIATION COEFFICIENT (OR THE COVARIATION)

1) FLOM METHOD Given the independent observations $(X_i, Y_i)_{i=1,\dots,N}$

$$\lambda_{XY} = \frac{E\{XY^{<p-1>}\}}{E\{|Y|^p\}} \Rightarrow \boxed{\hat{\lambda}_{FLOM} = \frac{\sum_{i=1}^{N} X_i |Y_i|^{p-1} \operatorname{sign} Y_i}{\sum_{i=1}^{N} |Y_i|^p} \quad \begin{array}{l} 2>\alpha>1 \\ 1\leq p<\alpha. \end{array}}$$

[Consistent estimator even if observations are not independent]

2) SCRENED RATIO ESTIMATOR : Let X, Y be r.v.s that satisfy
(Kanter and Steiger, 1974) $E\{|X|\}<\infty$ and $E\{X/Y\} = \lambda Y$ for some λ

Define the r.v. $x_Y = \begin{cases} 1, & \text{if } c_1 < |Y| < c_2 \\ 0, & \text{otherwise.} \end{cases}$ where $0 < c_1 < c_2 \leq \infty$

Then, $E\{XY^{-1}x_Y\} / P(c_1 < |Y| < c_2) = \lambda$

Thus, given the independent observations (X_i, Y_i) $i=1,\dots,N$

$$\boxed{\hat{\lambda}_{SCR} = \frac{\sum_{i=1}^{N} (X_i Y_i^{-1} x_{Y_i})}{\sum_{i=1}^{N} x_{Y_i}}}$$

[unbiased and consistent estimator (only with independ. observations)]

3) LEAST SQUARES ESTIMATOR

Minimize the error $\sum_{i=1}^{N} (X_i - \lambda Y_i)^2 \Rightarrow \boxed{\hat{\lambda}_{LS} = \frac{\sum_{i=1}^{N} X_i Y_i}{\sum_{i=1}^{N} Y_i^2}}$

Appropriate only for Gaussian random variables

[However, it has been found in practice that $\hat{\lambda}_{LS}$ is an efficient estimator for λ in the case of Parametric Models of stable processes

Example

(Shao, Nikias 1993)

Let U_1, U_2 two independent r.v.s $S\alpha S$ with characteristic function $e^{-|t|^\alpha}$ (i.e. $\gamma=$ / $a=$

Let $X = a_1 U_1 + a_2 U_2$

$Y = b_1 U_1 + b_2 U_2$

Then, $$\lambda_{TRUE} = \frac{a_1 b_1^{\langle\alpha-1\rangle} + a_2 b_2^{\langle\alpha-1\rangle}}{|b_1|^\alpha + |b_2|^\alpha}.$$

Generate 5000 independent samples of U_1 and U_2 and use the estimators

① $$\hat{\lambda}_{FLOM} = \frac{\sum_{i=1}^{N} X_i \operatorname{sign}(Y_i)}{\sum_{i=1}^{N} |Y_i|}$$ (i.e., with $p=1$)

② $$\hat{\lambda}_{SCR} = \sum_{i=1}^{N} (X_i Y_i^{-1} \chi_{Y_i}) \Big/ \sum_{i=1}^{N} \chi_{Y_i}$$ (with $C_1 = 0.1$, $C_2 = \infty$)

③ $$\hat{\lambda}_{LS} = \sum_{i=1}^{N} X_i Y_i \Big/ \sum_{i=1}^{N} Y_i^2$$

RESULTS obtained from 50 realizations of the experiment [Shao, Nikias, 1993]

$\hat{\lambda}$

Model (a_1, a_2, b_1, b_2)	α	LS	SCR	FLOM ($p=1$)	True λ
(-0.75, 0.25, 0.18, 0.78)	1.1	0.3340 (5.0539)	-0.4327 (0.3919)	-0.4707 (3.1266)	-0.4252
	1.3	0.3752 (5.1640)	-0.2591 (0.4991)	-0.2599 (1.0352)	-0.2602
	1.5	0.4059 (4.5142)	-0.1112 (0.3337)	-0.1222 (0.9202)	-0.1273
	1.9	0.1069 (0.8870)	0.0654 (0.3175)	0.06104 (0.4598)	0.0599
	2.0	0.0976 (0.0102)	0.08544 (0.2790)	0.0954 (0.1252)	0.0936

$1 < \alpha \le 2$

MINIMUM DISPERSION CRITERION (MDC)

- Given $\{X(t), t\in T\}$ find the best estimate of of an unknown r.v. Y from the linear space spanned by $\{X(t), t\in T\}$

A criterion based on the Minimum Mean Square Error is not appropriate for $S\alpha S$ r.v.s or processes unless they are Gaussian. (Recall that the concepts of orthogonality and projection exist in the Hilbert space and not the Banach space of $S\alpha S$ non-Gaussian r.v.s)

The MDC: For stable processes minimize the FLOMS of estimation errors. The FLOMS measure the L_p distance between Y and its estimate $\hat{Y}$ generated from $L=\{X(t), t\in T\}$. The MDC is very robust against outliers for $p<2$

highly nonlinear solutions

i.e., $\|Y-\hat{Y}\|_\alpha = \inf_{Z\in L} \|Y-Z\|_\alpha \qquad 0<p<\alpha$

or $E\{|Y-\hat{Y}|^p\} = \inf_{Z\in L} E\{|Y-Z|^p\}$

or $[X(t), Y-\hat{Y}]_\alpha = 0$ for all $t\in T$, $1<\alpha<2$

Note that $L=\{X(t), t\in T\}$ is a subspace of the Banach space. The Banach space is a convex space and thus $\hat{Y}$ always exists and is unique. The above relations represent a metric projection of Y on the Banach space L "similar" to orthogonal projections in the Hilbert space.

14.5 REFERENCES ON α-STABLE DISTRIBUTIONS

1) U. Akgiray and C.G. Lamoureux "**Estimation of stable low parameters: A Comparative study**", J. of B. and Econ. Stat. Vol. 7, pp. 8593, 1989.

2) H. An and Z. Chen, "**On Convergence of LAD Estimates in Autoregression with Infinite Variance**", J. of Multivariate Analysis, Vol. 12, pp. 335-345, 1982.

3) M. Bouwet and S.C. Schwartz, "**Comparission of Adaptive and Robut Recievers for Signal Detection in Ampient Underwater Noise**, IEEE Trans. ASSP, Vol. 37, No. 5, pp. 621-626, 1989.

4) S. Cambanis, C.D. Hardin Jr., and A. Weron, "**Innovations and World Decompositions of Stable Processes**", Probability Theory and Related Fields, Vol. 79, pp. 1-27, 1988.

5) S. Cambanis and A.G. Miamee, "**On Prediction of Harmonizable Stable Processes**", Sanknya A., Vol. 51, pp. 269-294, 1989.

6) S. Cambanis and G. Miller, "**Linear Problems in pth** order and Stable Processes", SIAM Journal of Applied Mathematics, Vol. 41, No. 1, pp. 43-69, 1981.

7) D.B. Cline and P.J. Brockwell, "**Linear Prediction of ARMA Processes with Infinite Variance**", Stochastic Processes and Their Applications, vol. 19, pp 281-286, 1985.

8) W. DuMouchel, "**Estimating the Stable Index α in Order to Measure Tail Thickness: A Critique**", Amals of Statistics, Vol. 11, No. 6, pp. 1019-1051, 1983.

9) W. Feller, "**An Introduction to Probability Theory and Its Applications**, New York: Wiley, 1971.

10) R. Gonin and A.H. Money, "**Nonlinear Lp-Norm Estimations**", New York: Marcel Dekker, Inc. 1989.

11) C. Granger and D. Orr, "**Infinite Variance and Research Strategy in Time Series Analysis**", J. of the American Statistical Society, Vol. 67, No. 338, pp. 275-285 1972.

12) L. Izzo, L. Panico and L. Panra, "**Error Rates for Fading NCFSK Signals in an Additive Mixture of Impulsive and Gaussian Noise**", IEEE Trans. COM, Vol. COM-30 pp. 2434-2438, 1982.

13) S.A. Kassam "**Sinal Detection in NonGaussian Noise**" N.Y: Springer - Verlag, 1988.

14) I. Koutrouvelis, "**An Iterative Procedure for the Estimation of the Parameters of Stable Laws**", Communication, Vol. 10, No. 1, 1981.

15) B. Mandelbrot, "**The variation of Certain Speculative Prices**", Journal of Business, Vol. 36, pp. 394-419, 1963.

16) B. Mandelbrot and J.W. Van Ness, "**Fractional Brownian Motions, Fractional Noises and Applications**" SIAM Review, Vol. 10, pp. 422-437, 1968.

17) W.B. McCain and C.D. McGillen, "**Performance Improvement of DPLL's in Non-Gaussian Noise Using Robust Estimators**", IEEE Trans. COM, Vol. 35, No. 11, 1987.

18) J.H. McGulloch, "**Simple Consistent Estimators of Stable Distribution Parameters**", **Communications Statistics: Simulation and Computation**, Vol. 15, No. 4, 1986.

19) D. Middleton, "**Statistical-Physical Models of Electromagnetic Inerference**", IEEE Transactions on Electromagnetic Compatibility, Vol. EMC-19, No. 3, 1977.

20) - Also, EMC-21, No. 3, 1979.

21) J.H. Miller "**Detections for Discrete-Time Signals in Non-Gaussian Noise**", IEEE Transact. IT., Vol. IT-18, No. 2, 1972.

22) H.I. Pereira, "**Tests for the Characeristic Exponent and the Scale parameter of Symmetric Stable Distributions**" Commun. Stat. - Simula., Vol. 19, pp. 1464-1475.

23) S. Press, "**A Compound Events Model for Security Prices**", Journal of Business, Vol. 40. 40., pp. 317-335, 1968.

24) S. J. Press, "**Estimation in Univariate and Multivariate Stable Distributions**" Journal of the Amerian Statistical Association, Vol. 67, pp. 842-846, 1972.

25) S. J. Press, "**Stable distributions: Probability Inference and Applications in Finance - A Survey, and A Review of Recent Results**", in Statistical Distributions in Scientific Work (G. Patil, Ed.) pp. 87-102, Dordrecht, Holld: D. Reidel Publishing Company, 1975.

26) J. Schroeder and R. Yarlagadda, "**Linear Predictive Spectral Estimation Via the L1 Norm**", Signal Processing, Vol. 17, pp. 19-29, 1989.

27) - Also, Signal Procesing, Vol. 24, pp. 193-216, 1991.

28) J. Seo, S. Cho and K. Feher, "**Impact of Non-Gaussian Impulsive Noise on the Performance of High Level QAM**", IEEE Trans. on Electromagnetic Compatibility, Vol. EMC-31, No. 2, 1989.

29) E. J. Wegman, S.C. Schwartz and J.B. Thomas, eds, "**Topics in Non-Gaussian Signal Processing**", N.Y.: Springer-Verlag, 1989.

30) E. Fama, "**The Behavour ofStock Market Prices**", Journal of Business, Vol. 38, pp. 34-105, 1965.

31) J. Teichnoeller, "**A Note on the Distribution of Stock Price Changes**", J. of American Statistics Association, Vol. 66, pp. 282-284, 1971.