Lecture 3

The z-transform of the autocorrelation function

* By definition $\left[S_x(z) = \sum_{k=-\infty}^{\infty} R_x(\ell) z^{-\ell} \right]$ complex quantity.
  $\left(R_x(\ell) = \frac{1}{2\pi j} \oint_C S_x(z) z^{\ell-1} dz \right)$ contour integral

  then : $S_x(\omega) = S_x(z)\Big|_{z=e^{j\omega}}$ i.e. on the unit circle.

* $\boxed{S_x(z) = S_x^*(1/z^*)}$ due to the conjugate symmetry of $R_x(\ell)$

  or $S_x(z) = S_x(z^{-1})$ for real processes

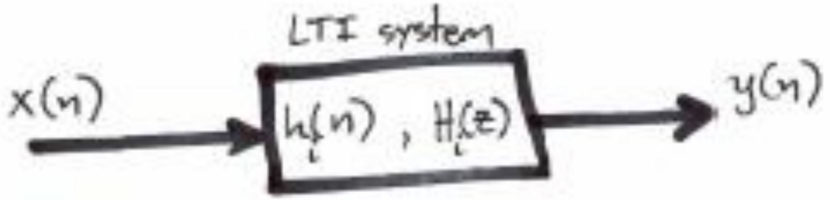
* **Important special case:** $S_x(z) = \dfrac{G \prod_{i=1}^{M} (1 - z_i z^{-1})}{\prod_{i=1}^{N} (1 - p_i z^{-1})}$, rational polynomial if $N \geq M$.
  z_i : zeros , p_i : poles

  [If z_0 is a zero or pole.
  $1/z_0^*$ is also a zero or pole.]

(for real processes poles and zeros appear in groups of 4
$(z_0, 1/z_0, z_0^*, 1/z_0^*)$)

Region of convergence (R.O.C.), $r_x < |z| < \frac{1}{r_x}$

The z-transform of $R_x(\ell)$: Example



$H_i(z)$: minimum phase if all zeros inside or on unit circle
maximum phase if all zeros outside or on unit circle

* Let $x(n)$ be a real s.s.s. process with zero mean, $R_x(\ell) = \delta(\ell)$
* Let $h_i(n)$ be a finite impulse response (FIR) system with two possible z-transforms (two cases): $H_1(z) = (1 - z^{-1})(1 - 0.5 z^{-1})$ (minimum phase) or $H_2(z) = (1 - z^{-1})(1 - 0.5 z)$ (maximum phase)

Then, $y(n) = h_i(n) * x(n) = \sum$ is also a wide sense stationary process (WSS) (to be shown later)

Also, $R_y(\ell) = R_h(\ell) * R_x(\ell)$ where $R_h(\ell) = \sum_k h_i(k) h_i(k+\ell)$

Thus, $R_y(\ell) = R_h(\ell) \Rightarrow S_y(z) = Z[R_h(\ell)] = \sum_\ell \sum_k h_i(k) h_i(k+\ell) z^{-\ell} = \dots = H_i(z) H_i(z^{-1})$

So
- case 1: $S_y(z) = (1-z^{-1})(1-0.5z^{-1})(1-z)(1-0.5z)$
- case 2: $S_y(z) = (1-z^{-1})(1-0.5z)(1-z)(1-0.5z^{-1})$

identical !!!!

* The power spectrum cannot distinguish phase information.

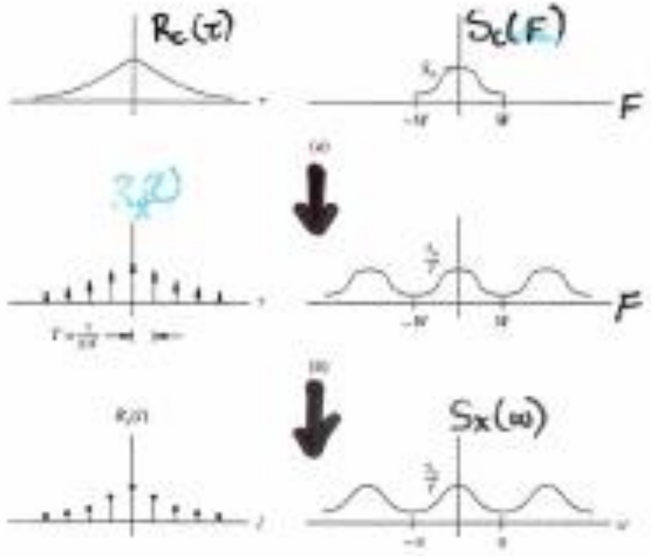
SAMPLING A CONTINUOUS R. PROCESS

→ Let $x_c(t)$ be a continuous-time stationary random process with autocorrelation
$R_c(\tau) = E\{x_c(t)x_c(t+\tau)\}$ and P.S.D. $S_c(\Omega) = \int_{-\infty}^{\infty} R_c(\tau)e^{-j\Omega\tau}d\tau$

→ Let $x(n)$ be a discrete r. process $x(n)$ obtained as follows $x(n) = x_c(nT)$
Then it is straightforward to show that $x(n)$ is also stationary and

$$\underline{R_x(\ell)} = E\{x(n)x(n+\ell)\} = E\{x_c(nT)x_c(nT+\ell T)\} \stackrel{\text{stat.}}{=\!=} E\{x_c(t)x_c(t+\ell T)\} = \underline{R_c(\ell T)}$$

→ Aliasing occurs when $S_c(\Omega)$ is not bandlimited and sampling rate is not chosen high enough.



sampling period $T = \frac{1}{2W}$

SAMPLING THEOREM FOR R. PROCESSES

Let $x_c(t)$ be a random process with power density spectrum that is zero outside an interval $-W < F < W$. Let $x(n) = x_c(nT)$ be a sequence corresponding to samples of $x_c(t)$ taken at the sampling interval $T = 1/2W$. Define the random process $\hat{x}_c(t)$ by

$$\hat{x}_c(t) = \sum_{n=-\infty}^{\infty} x(n)\, \mathrm{sinc}(2\pi W t - n\pi) \qquad \left(\text{where } \mathrm{sinc}(A) = \frac{\sin A}{A}\right)$$

Then,

$$E\{|x_c(t) - \hat{x}_c(t)|^2\} = 0$$

i.e., $x_c(t)$ and $\hat{x}_c(t)$ are equal almost anywhere or in the mean square sense.

WHITE NOISE PROCESSES

- Continuous <u>white noise</u> process: A process with autocovariance $\left[C_x(t_1,t_2) = \underset{\geq 0}{q(t_1)}\, \delta(t_1 - t_2) \right]$

  t_1, t_2 real

  <u>strictly white</u> noise is a noise process with independent values for every time instant

  <u>WSS white noise</u>: A process with $\left[C_x(t_1,t_2) = C_x(t_2 - t_1) = q\, \delta(t_2 - t_1) \right]$

* A white noise process cannot exist in reality since this would imply a process with infinite power (flat power spectrum over all frequencies). However, it is useful to describe a r. process with mean power spectral density flat over all frequencies of interest (there are thermal noise processes with spectrum flat up to 100 GHz!)

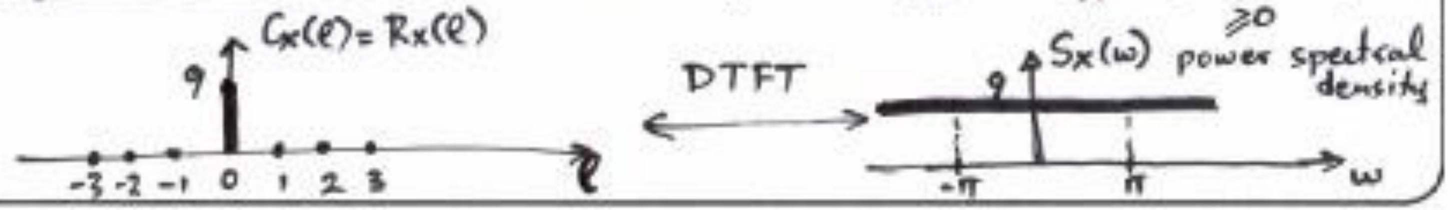
* white noise processes are usually considerer zero mean.

- Discrete <u>white noise</u> process: A zero mean process with autocovariance $C_x(n_1,n_2) = \underset{\geq 0}{q(n_1)}\, \delta(n_2 - n_1)$

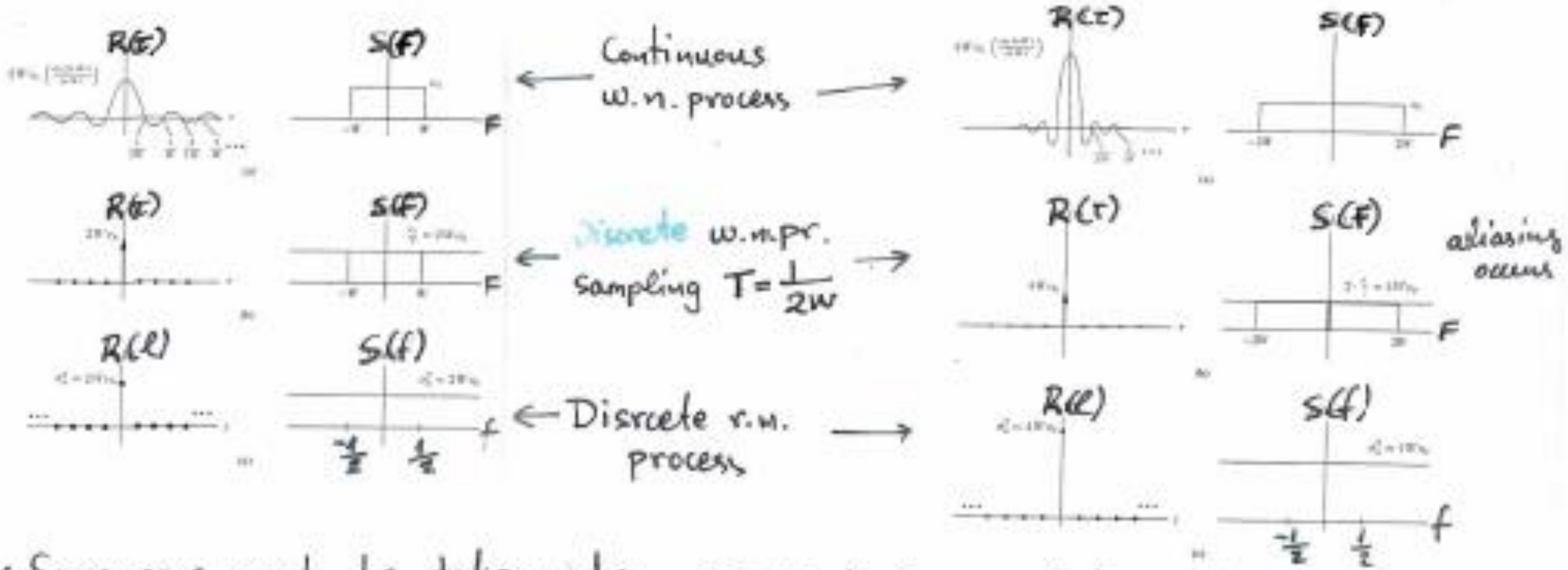
  $n, \ell \ldots$ integers

  <u>strictly white</u> noise if the values of the process are independent for every n

  <u>stationary or WSS white noise</u>: A zero mean process with $C_x(\ell) = q\, \delta(\ell)$



SAMPLING "BANDLIMITED" WHITE NOISE



* Some care must be taken when representing sampled continuous random processes by discrete random processes. Normally the signals are filtered prior to sampling so that out of band noise does not become folded in and degrade the signal to noise ratio

* Many white noise processes are considered to be white mainly because of the C.L.T. In this case, the Gaussian noise process is not only uncorrelated but also independent

[which describes in this case the formation of noise as the superposition of many independent effects]

CYCLOSTATIONARY PROCESSES - CYCLIC AUTOCORRELATION

Given a cyclostationary process $x(n)$ of period L

$$R_x(n,\ell) \triangleq E\{x^*(n)\,x(n+\ell)\} = E\{x^*(n+L)\,x(n+L+\ell)\} = \cdots = R_x(n+\kappa L, \ell)$$

(notice the notation) Also: $S_x(n,\omega) = \sum_{n=-\infty}^{\infty} R_x(n,\ell)e^{-j\omega\ell} = S_x(n+\kappa L,\omega)$

So the autocorrelation is periodic in time with period L. Then one can obtain a Fourier series expansion of this periodic function as follows

$$\left[R_x^{\alpha}(\ell) = \frac{1}{L}\sum_{n=0}^{L-1} R_x(n,\ell)e^{-j2\pi\alpha n}\right], \text{ where } \alpha = \frac{k}{L}, \; k=0,\pm1,\pm2,\ldots$$

Cyclic autocorrelation, Cyclic PSD.

Also,

$$\left[S_x^{\alpha}(\omega) = \sum_{\ell=-\infty}^{\infty} R_x^{\alpha}(\ell)e^{-j\omega\ell} = \frac{1}{L}\sum_{n=0}^{L-1} S_x(n,\omega)e^{-j2\pi\alpha n}\right]$$

* A mapping from a time variant quantity to the frequency domain
* The case for $\alpha=0$ corresponds to time averaging over one period.

THE DISCRETE KARHUNEN-LOEVE TRANSFORM (DKLT)

- Expand a function as a linear combination of orthonormal basis functions with orthogonal (or uncorrelated) coefficients

That is given $x[n]$, $n=0,1,\ldots,N-1$

determine basis functions $\{\varphi_i(n)\}_{i=1,\ldots,N}$, $n=0,\ldots,N-1$, and $\boxed{\sum_{n=0}^{N-1} \varphi_i^*(n)\varphi_j(n) = \begin{Bmatrix} 1 & i=j \\ 0 & i\neq j \end{Bmatrix} = \delta(i-j)}$

So that

$$\boxed{x(n) = k_1\varphi_1(n) + k_2\varphi_2(n) + \ldots + k_N\varphi_N(n)}, \quad n=0,1,\ldots,N-1$$

easy to show that the coefficient k_i is obtained as follows →

$$\boxed{k_i = \sum_{n=0}^{N-1} \varphi_i^*(n)\, x(n)}$$

We also want that $\{k_i\}$ is an orthogonal set, i.e., $\boxed{E\{k_i k_j^*\} = S_i^2\,\delta(i-j)}$

The DKLT – DFT

Ex: Consider the functions $\varphi_k(n) = e^{jk\frac{2\pi}{N}n}$, $n = 0, 1, \ldots, N-1$, $k = 0, 1, \ldots, N-1$

Then,
$$\sum_{n=0}^{N-1} e^{-jk_1\frac{2\pi}{N}n}\, e^{jk_2\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k_2-k_1)n} = \begin{cases} N, & k_2-k_1=0 \\ 0, & \text{otherwise} \end{cases}$$

So the set $\frac{1}{\sqrt{N}}\varphi_k(n)$ is an orthonormal set of basis functions

Then, $\underbrace{X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k n}{N}}}$. the is nothing else but the DFT of $x(n)$

- It can be shown that for stationary $x(n)$ and N long enough compared to the duration of the autocorrelation function $R_x(m)$ the DFT is close to a DKLT of the process $x(n)$. For periodic ~~and~~ r.v. processes the DKLT $\equiv$ DFT.
- Fourier transforms are expansions of signals in terms of orthogonal basis functions but in general they are different from the KLT.

THE DKLT – Solution

Let $\underline{K} = \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_N \end{bmatrix}$, $\underline{\varphi_i} = \begin{bmatrix} \varphi_i(0) \\ \varphi_i(1) \\ \vdots \\ \varphi_i(N-1) \end{bmatrix}$, $\underline{\Phi} = \underbrace{\begin{bmatrix} | & | & & | \\ \varphi_1 & \varphi_2 & \dots & \varphi_N \\ | & | & & | \end{bmatrix}}$: $N \times N$ matrix

$\underline{X} = [x(1) \dots x(N-1)]^T$,

unitary matrix : $\underline{\Phi}^H \underline{\Phi} = \underline{I}$ intentity matrix

Then, DKLT pair

$$\boxed{\begin{aligned} \underline{X} &= \underline{\Phi}^H \underline{K} \\ \underline{K} &= \underline{\Phi}^H \underline{X} \end{aligned}}$$

[Fundamendal property of autocorrelation matrix]

We want $\underline{\varphi_i}$ to be orthonormal and $\{K_i\}$ to be uncorrelated (zero mean process)

Such a relation exist between the eigenvectors and eigenvalues of the autocorrelation function of $x(n)$

$$\sum_{k=0}^{N-1} R_x(l,k)\varphi_i(k) = \lambda_i \varphi_i(l) \leftarrow \boxed{\underline{R_x}\, \underline{\varphi_i} = \lambda_i \underline{\varphi_i}} \quad \text{and} \quad \boxed{\lambda_i = E\{K_i K_j^*\} = E\{|K_i|^2\}}$$

$i = 1, \dots, N$

eigenvector eigenvalue

* While $\underline{x}$ may have a non-diagonal $\underline{R_x}$, $\underline{K}$ has a diagonal correl. matrix for zero-mean processes.

THE DKLT – Optimality – Properties

* Consider approximating $\underset{n=0,1,\dots,N-1}{x(n)}$ as $\hat{x}(n) = \sum_{i=1}^{M} \hat{k}_i \hat{\varphi}_i(n)$, $M<N$

Then minimizing the Mean Square error (MSE)

$$\mathcal{E} = E\left\{ \sum_{n=0}^{N-1} |x(n) - \hat{x}(n)|^2 \right\} \longrightarrow \left\{ \begin{array}{ll} \hat{\varphi}_i(n) = \varphi_i(n) & , i=0,1,\dots,M-1 \\ \hat{k}_i = k_i & , i=0,1,\dots,M-1 \\ \mathcal{E} = \sum_{i=M+1}^{N} \lambda_i = \sum_{i=M+1}^{N} E\{|k_i|^2\} & \end{array} \right]$$

Thus, the best approximation of $x(n)$ (in the MMSE sense) based on M term expansion is $\hat{x}(n) = \sum_{i=1}^{M} k_i \varphi_i(n)$ where $k_i, \varphi_i(n)$ are the first M coefficients and basis functions of the DKLT.

* So the DKLT provides an "optimal" representation of a random process over a finite time interval ("optimal" in the second-order or second-moment analysis framework). In the case of Gaussian processes the DKLT provides a complete representation of the process. In this case the coefficients $\{k_i\}$ are also independent r.v.s.

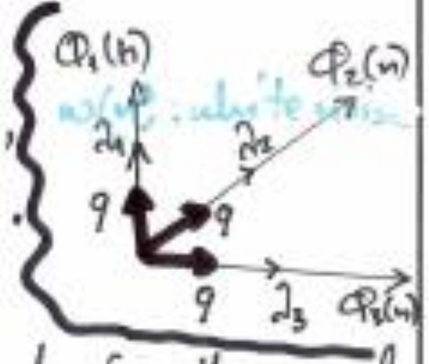
The DKLT — Application

✱ Consider the white noise process $x(n)$, $n=0,\dots,N-1$, $R_x(m)=q\,\delta(m)$
or $R_x(\ell,k)=q\,\delta(k-\ell)$

Then, $\underline{R_x}\,\underline{\Phi_i}=\lambda_i\underline{\Phi_i}$, has a solution $\lambda_i=q$, $i=1,\dots,N$ for **any set** of orthonormal functions $\{\Phi_i(n)\}$, $i=1,\dots,N$

So any set of orthonormal functions is suitable for decomposing white noise.

✱ Consider now a process $y(n)=x(n)+w(n)$, $n=0,\dots,N-1$, and let us assume that $x(n)$ and $w(n)$ are independent. Then, $\lambda_{y,i}=\lambda_{x,i}+\lambda_{w,i}$.



Case 1: $w(n)$ is white noise. Then $\lambda_{y,i}=\lambda_{x,i}+q$ no matter what set of orthonormal basis function we choose. "In every direction the noise has the same power."

Case 2: $w(n)$ is colour noise. Then by properly designing our signal we can place its power along these basis functions where the noise is weak.

LINEAR TRANSFORMATIONS OF R.PR.

* Linear system

$$x(n) \longrightarrow \boxed{h(n,k)} \longrightarrow y(n) = \sum_{k=-\infty}^{\infty} h(n,k)\,x(k)$$

$$E\{y(n)\} = \sum_{k=-\infty}^{\infty} h(n,k)\, E\{x(k)\} \Rightarrow m_y(n) = \sum_k h(n,k)\, m_x(k)$$

$$E\{y^*(n_1)\,y(n_2)\} = \sum_{k_1}\sum_{k_2} h^*(n_1,k_1)\,h(n_2,k_2)\, E\{x^*(k_1)\,x(k_2)\} \Rightarrow R_y(n_1,n_2) = \sum_{k_1,k_2}\sum h^*(n_1,k_1)\,h(n_2,k_2)\,R_x(k$$

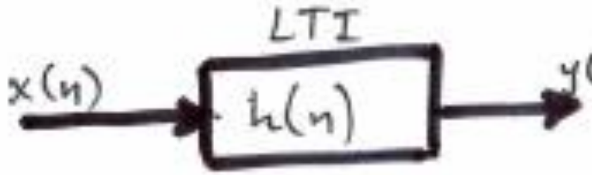
Also,

$$R_y(n_1,n_2) = \sum_k h(n_1,k)\,R_{xy}(k,n_2)$$

$$R_{xy}(n_1,n_2) = \sum_k h(n_2,k)\,R_x(k,n_2)$$

* If $x(n)$ is Gaussian process then $y(n)$ is also Gaussian process and $x(n)$, $y(n)$ are jointly Gaussian

* If $x(n)$ is stationary (SSS) then $y(n)$ may not be (SSS)

* 

$$y(n) = \sum_{k=-\infty}^{\infty} h(n-k)\, x(k) = h(n) * x(n)$$

Linear convolution

* If $x(n)$ is SSS then $y(n)$ is also SSS

* furthermore: $m_y(n) = \sum_k h(n-k)\, m_x(n) = h(n) * m_x(n)$

$$R_y(n_1,n_2) = \sum_{k_1,k_2}\sum h^*(n_1-k_1)\, h(n_2-k_2)\, R_x(k_1,k_2) = \left[h^*(n_1)\cdot h(n_2)\right] * R_x(n_1,n_2)$$

2-D linear convolution

If $x(n)$ is stationary then $m_x(n) = m_x$, $R_x(n_1,n_2) = R_x(n_2-n_1)$

Then it can be easily shown that: $m_y(n) = m_y = \underbrace{\left[\sum_n h(n)\right]}_{m_h} \cdot m_x = \text{ct.}$

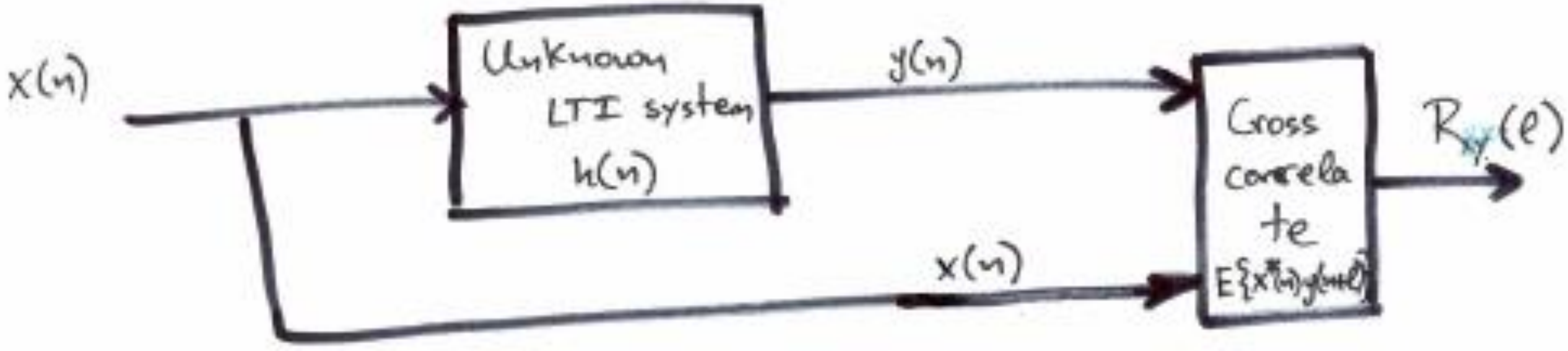
$$R_y(n_1,n_2) = R_y(n_2-n_1) = R_y(\ell) = \underbrace{\left[\sum_m h^*(m)\, h(m+\ell)\right]}_{R_h(\ell)} * R_x(\ell)$$

= time invariant.

Thus, if $x(n)$ is WSS then $y(n)$ is also WSS. They are also jointly WSS, i.e.

$$R_{xy}(\ell) = \sum_k h(\ell-k)\, R_x(k) = E\{x^*(n)\, y(n+\ell)\}$$

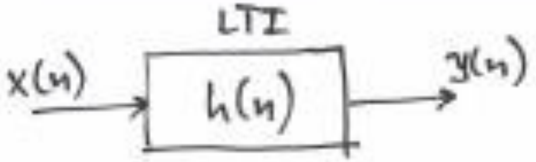
LTI System identification



- Let $x(n)$ be white noise (WSS): Then $R_x(\ell) = q\,\delta(\ell)$
- From theory $R_{xy}(\ell) = \sum_k h(\ell-k)R_x(k) = h(\ell) * R_x(\ell) = h(\ell) * q\,\delta(\ell) = q \cdot h(\ell)$

Thus, $\boxed{h(\ell) = \frac{1}{q} R_{xy}(\ell)}$

LTI System. Input-output relations summary



GENERAL	**x(n) SSS or WSS**
$y(n) = h(n) * x(n)$	$y(n) = h(n) * x(n)$
$m_y(n) = h(n) * m_x(n)$	$m_y(n) = m_h \cdot m_x$
$R_y(n_1,n_2) = E\{y^*(n_1)y(n_2)\} = [h^*(n_1)\,h(n_2)] * R_x(n_1,n_2)$	$R_y(\ell) = \left[\sum_k h^*(k)h(k+\ell)\right] * R_x(\ell) = R_h(\ell) * R_x(\ell)$
$L_y(n_1,n_2,n_3) = E\{y(n_1)y(n_2)y(n_3)\} = [h(n_1)h(n_2)h(n_3)] * L_x(n_1,n_2,n_3)$	$L_y(\ell_1,\ell_2) = \left[\sum_k h(k)h(k+\ell_1)h(k+\ell_2)\right] * L_x(\ell_1,\ell_2) = L_h(\ell_1,\ell_2) * L_x(\ell_1,\ell_2)$
$R_{xy}(n_1,n_2) = E\{x^*(n_1)y(n_2)\} = h(n_2) * R_x(n_1,n_2)$	$R_{x,y}(\ell) = E\{x^*(n)y(n+\ell)\} = h(\ell) * R_x(\ell)$
$L_{xyy}(n_1,n_2,n_3) = E\{x(n_1)y(n_2)y(n_3)\} = [h(n_2)h(n_3)] * L_x(n_1,n_2,n_3)$	$L_{xyy}(\ell_1,\ell_2) = [h(\ell_1)h(\ell_2)] * L_x(\ell_1,\ell_2)$

LTI system with cyclostationary input.

In this case $m_x(n) = m_x(n+N)$, $R_x(n_1,n_2) = R_x(n_1+N, n_2+N)$ where N is the period of the cyclostationary input $x(n)$

It is straightforward to show that

$$m_y(n) = m_y(n+N) \quad \text{and} \quad R_y(n_1,n_2) = R_y(n_1+N, n_2+N) \; \ldots\ldots$$

[In general if the input $x(n)$ to a LTI system is SSCS (WSCS) then the output is also SSCS (WSCS) with the same period]

Furthermore:

$$R_y^{\alpha}(\ell) = \underbrace{\left[\sum_k h^*(k)\, h(k+\ell)\, e^{-j2\pi\alpha k}\right]}\, R * R_x^{\alpha}(\ell)$$

$$= \quad R_h^{\alpha}(\ell) \; * R_x^{\alpha}(\ell)$$

$\alpha = \frac{i}{N}$, $i = 0, \pm 1, \ldots$

SPECTRA OF STATIONARY PROCESSES AND LTI SYSTEMS

Let $y(n) = h(n) * x(n)$ where $x(n)$ is a SSS (WSS) process

Then, $R_y(\ell) = R_h(\ell) * R_x(\ell) \Rightarrow \underset{\text{PSD of } y(n)}{S_y(\omega)} = \underset{\text{F.T. of } R_h(\ell)}{F[R_h(\ell)]} \cdot \underset{\text{PSD of } x(n)}{S_x(\omega)}$

where,

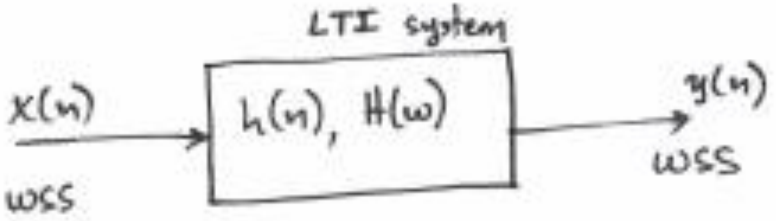
$$R_h(\ell) = \sum_k h^*(k)\, h(k+\ell) = h^*(-\ell) * h(\ell) \Rightarrow F[R_h(\ell)] = H^*(\omega) \cdot H(\omega)$$

where $H(\omega) = F[h(n)]$

Thus $\boxed{S_y(\omega) = |H(\omega)|^2 \cdot S_x(\omega)}$ input - output PSD relation.

Also, similarly $S_{xy}(\omega) = F[R_{xy}(\ell)] = H(\omega) \cdot S_x(\omega)$

Ex:

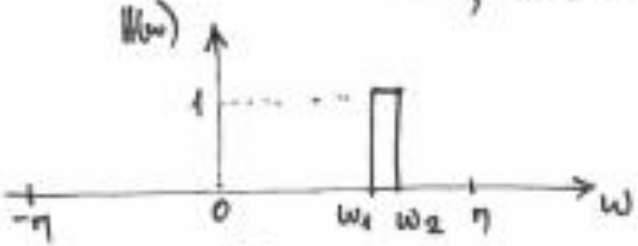


Average power of $y(n)$: $E\{|y(n)|^2\} = R_y(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\cdot|H(\omega)|^2 d\omega$

Since $E\{|y(n)|^2\} \geq 0 \rightarrow \boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)|H(\omega)|^2 d\omega \geq 0}$

Suppose we choose $H(\omega)$ as follows $H(\omega) = \begin{cases} 1, & \omega_1 \leq \omega \leq \omega_2 \\ 0, & \text{otherwise} \end{cases}$

then $\int_{\omega_1}^{\omega_2} S_x(\omega)\,d\omega \geq 0$ and by taking $\omega_1 - \omega_2 \rightarrow 0$ and ω_1 to span the region $-\pi \leq \omega_1 \leq \pi$ we easily conclude that



$S_x(\omega) \geq 0$ (Proof of positivity property of PSD.)

OTHER INPUT-OUTPUT SPECTRAL RELATIONS

* Let $y(n) = h(n) * x(n)$ and $x(n)$ be cyclostationary of period N.

Then $R_y^{\alpha}(\ell) = R_h^{\alpha}(\ell) * R_x^{\alpha}(\ell)$ where $R_h^{\alpha}(\ell) = \sum_k h^*(k)h(k+\ell)e^{-j2\pi\alpha k}$.

It is easy to show that

$$S_y^{\alpha}(\omega) = \underbrace{\left[H(\omega)H^*(\omega - 2\pi\alpha)\right]}_{\text{No Hermitian symmetry.}} \cdot S_x^{\alpha}(\omega) \quad , \quad \alpha = \frac{i}{N}, \ i = 0, \pm 1, \ldots$$

for $\alpha = 0$: $S_y^0(\omega) = \underbrace{|H(\omega)|^2}_{\text{Hermitian symmetry}} \cdot S_x^0(\omega)$: Equivalent relation for stationary process.

* Similar relations hold in the z-transform domain that is.

$$S_y^{\alpha}(z) = H(z)H^*(z \cdot e^{-j2\pi\alpha}) S_x^{\alpha}(z)$$

$$S_y^0(z) = |H(z)|^2 \cdot S_x^0(z)$$

SPECTRAL RELATIONS FOR HIGHER-ORDER CORRELATIONS

Let $y(n) = h(n) * x(n)$ where $x(n)$ a stationary process of third-order.

Then, $L_y(l_1, l_2) \triangleq E\{y(n)y(n+l_1)y(n+l_2)\} = \left[\sum_k h(k)\, h(k+l_1) h(k+l_2)\right] * L_x(l_1, l_2)$

By taking two dimensional Fourier transform of $L_y(l_1, l_2)$ we can show that

$$\underset{\substack{\uparrow \\ \text{Bispectrum of } y(n) \\ [\text{F.T.}[L_y(l_1,l_2)]]}}{B_y(\omega_1, \omega_2)} = \underbrace{H(\omega_1) \cdot H(\omega_2) H(-\omega_1 - \omega_2)}_{\substack{\text{2-D F.T. of} \\ \sum_k h(k)h(k+l_1)h(k+l_2)}} \cdot \underset{\substack{\uparrow \\ \text{Bispectrum of } x(n) \\ [\text{F.T.}[L_x(l_1,l_2)]]}}{B_x(\omega_1, \omega_2)}$$

* If $h(n)$ is real then $H(-\omega_1 - \omega_2) = H^*(\omega_1 + \omega_2)$

* $-\pi \le \omega_1 \le \pi$, $-\pi \le \omega_2 \le \pi$, $-\pi \le \omega_1 + \omega_2 \le \pi$.