RECURSIVE FILTERING (1)

Objective:

Given $\left[x(k) = s(k) + n(k) \quad , k = 0, 1, \ldots, n \right]$

where: $s(n)$ is a random signal with correlation function $R_s(\ell)$

$n(n)$ is a white noise process with variance σ_n^2, zero mean

$s(n)$ and $n(n)$ are uncorrelated.

Find the best linear-mean-square estimate of $s(n)$, $\hat{s}(n)$

that is find $c_0, \ldots, c_n$ so that

$$\hat{s}(n) = \sum_{i=0}^{n} c_i x(i) \quad \text{by minimizing} \quad E\{|s(n) - \hat{s}(n)|^2\}$$

Obtain $\hat{s}(n)$ recursively:

$$\left[\hat{s}(n) = B_n \hat{s}(n-1) + K_n x(n) \right]$$

Determine conditions for recursive estimation

(Determine values for B_n and K_n)

RECURSIVE FILTERING (2)

- Let $e(n) = s(n) - \hat{s}(n)$, $\sigma^2_{e(n)} = Var[e(n)]$

  From problem assumptions we have :

$$\underline{R_{xs}(\ell)} = E\{s(n)x^*(n-\ell)\} = E\{s(n)\cdot(s^*(n-\ell)+n^*(n-\ell)\} = E\{s(n)s^*(n-\ell)\} = \underline{R_s(\ell)}$$

$$\underline{R_x(\ell)} = E\{x(n)x^*(n-\ell)\} = E\{(s(n)+n(n))(s^*(n-\ell)+n^*(n-\ell))\} = E\{s(n)s^*(n-\ell)\} + E\{n(n)n^*(n-\ell)\} = \underline{R_s(\ell) + \underset{\sigma_n^2\delta(\ell)}{\underset{\parallel}{R_n(\ell)}}}$$

$$\underline{R_{nx}(0)} = E\{x(n)n^*(n)\} = E\{(s(n)+n(n))\cdot n^*(n)\} = E\{n(n)n^*(n)\} = \underline{\sigma_n^2 = R_n(0)}$$

- Apply orthogonality condition: $E\{e(n)x^*(n-\ell)\}=0$, $\ell=0,1,\dots,n$

$\ell=0$: $E\{e(n)[s^*(n)+n^*(n)]\} = \underbrace{E\{e(n)s^*(n)\}}_{\sigma^2_{e(n)}} + E\{e(n)n^*(n)\} = \sigma^2_{e(n)} + E\{(s(n)-\hat{s}(n))n^*(n)\}$

$$= \sigma^2_{e(n)} + E\{[s(n) - B_n\hat{s}(n-1) - K_n x(n)]\cdot n^*(n)\} = 0 \;\dots\Rightarrow \sigma^2_{e(n)} - K_n E\{x(n)n^*(n)\} = 0$$

$$\Rightarrow \sigma^2_{e(n)} - K_n R_n(0) = 0 \Rightarrow \boxed{K_n = \frac{\sigma^2_{e(n)}}{\sigma_n^2}}$$ real and positive.

RECURSNE FILTERING (3)

$\ell > 0$:

$$E\{e(n)x^*(n-\ell)\} = E\{s(n)x^*(n-\ell)\} - B_n E\{\hat{s}(n-1)\, x^*(n-\ell)\} - K_n E\{x(n)x^*(n-\ell)\} = 0$$

$$\Rightarrow R_s(\ell) - B_n E\{[s(n-1) - e(n-1)] \cdot x^*(n-\ell)\} - K_n \big(R_s(\ell) + \underbrace{\cancel{R_n(\ell)}}_{0\ (\ell>0)}\big) = 0 \Rightarrow$$

$$\Rightarrow (1-K_n)\cdot R_s(\ell) - B_n R_s(\ell-1) - 0 = 0 \Rightarrow$$

$$\Rightarrow \left[R_s(\ell) - \frac{B_n}{1-K_n} R_s(\ell-1) = 0 \right], \quad \ell > 0$$

[NOTE: since this is a recursive filter the condition $E\{e(n-1)\,x^*(n-\ell)\} = 0$ has been "achieved" in the previous iteration]

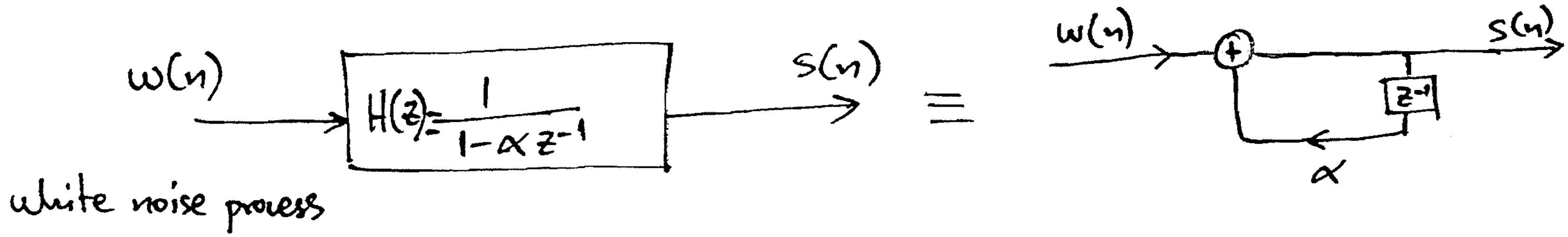
Solution of the above difference equation provides

$$\left[R_s(\ell) = R_s(0)\,\alpha^\ell, \quad \ell > 0 \right] \quad \text{where} \quad \left[\alpha = \frac{B_n}{1-K_n} \right]$$

exponential form of correlation function.

RECURSIVE FILTERING (4)

Consider the signal $s(n) = \alpha s(n-1) + w(n)$



i.e. first order autoregressive model. Then for this signal it is easy to show that

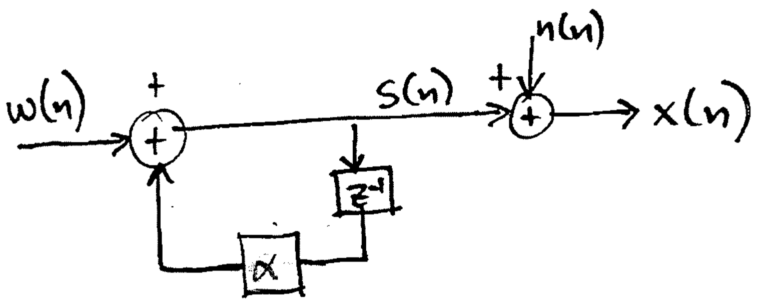
$$R_s(\ell) = \sigma_w^2 \cdot \frac{\alpha^\ell}{1-|\alpha|^2} = R_s(0) \cdot \alpha^\ell$$

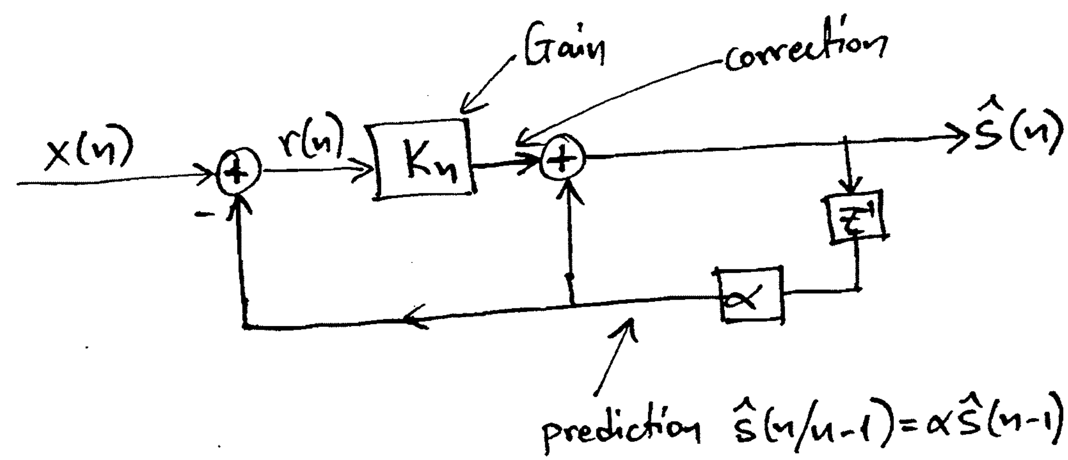
Therefore, for a signal $s(n)$ generated as above a recursive solution to the optimal filtering as defined previously is possible.

RECURSIVE FILTERING (5)

So, $\hat{s}(n) = B_n \hat{s}(n-1) + K_n x(n) = \alpha(1-K_n)\hat{s}(n-1) + K_n x(n)$

or $\left[\hat{s}(n) = \underbrace{\alpha \hat{s}(n-1)}_{\text{forward predictor}} + \underbrace{K_n \cdot [x(n) - \alpha \hat{s}(n-1)]}_{\text{Correction term}} \right]$, $K_n = \dfrac{\sigma^2_{e(n)}}{\sigma^2_n}$





$r(n)$: residual error $\begin{cases} \text{white non-stationary, represents the new information in observations} \\ \text{white stationary at steady state } (n \to \infty) \text{; innovations process of } x(n) \end{cases}$

RECURSIVE FILTERING (6)

To define K_n better let us derive an expression for $\sigma^2_{e(n)}$:

From orthogonality principle: $\sigma^2_{e(n)} = E\{s(n)\cdot e^*(n)\} = \cdots = (1-K_n)[\sigma_s^2 - |\alpha|^2 E\{s(n-1)\hat{s}^*(n-1)\}]$

for $n=0$ (assuming $\hat{s}(-1)=0$) : $\sigma^2_{e(0)} = \dfrac{\sigma_s^2}{1+\sigma_s^2/\sigma_n^2} = \dfrac{\sigma_w^2}{1+\sigma_w^2/\sigma_n^2}$

for $\underline{n>0}$ it can be shown that $\left[\sigma^2_{e(n)} = \dfrac{\sigma_w^2 + |\alpha|^2\sigma^2_{e(n-1)}}{\sigma_n^2+\sigma_w^2+|\alpha|^2\sigma^2_{e(n-1)}}\cdot\sigma_n^2\right]$: Recursive expression.

$\underline{\text{So}}$ $\left[K_n = \dfrac{\sigma_w^2 + |\alpha|^2\sigma^2_{e(n-1)}}{\sigma_n^2+\sigma_w^2+|\alpha|^2\sigma^2_{e(n-1)}},\quad \underline{n>0}\right]$, If $\sigma_w^2 \gg \sigma_n^2 \Rightarrow K_n \rightarrow 1 \Rightarrow \underline{\hat{s}(n)=x(n)}$

K_n, $\sigma^2_{e(n)}$ are computed recursively but can be computed a-priori since they do not depend on $x(n)$

$K_0 = \dfrac{\sigma_w^2}{\sigma_n^2+\sigma_w^2}$

RECURSIVE FILTERING (7)

Example: Constant signal in noise

het $x(k) = s + n(k) \qquad k = 0, 1, \ldots, n$

where, s is a constant $(w(n) = 0 \; , \alpha = 1)$

Then, $s(n) = s(n-1) = s$ and.

$$\hat{s}(n) = \hat{s}(n-1) + K_n \left(x(n) - \hat{s}(n-1) \right)$$

Since $\sigma_w^2 = 0$, $\alpha = 1$,

$$\sigma_{e(n)}^2 = \frac{\sigma_{e(n-1)}^2}{\sigma_n^2 + \sigma_{e(n-1)}^2} \cdot \sigma_n^2 = \frac{\sigma_{e(n-1)}^2}{1 + \frac{\sigma_{e(n-1)}^2}{\sigma_n^2}} < \sigma_{e(n-1)}^2 < \sigma_{e(n-2)}^2$$

Thus, as $n \to \infty \quad \sigma_{e(n)}^2 \to 0$ and $\hat{s}(n) \to s$

RECURSIVE PREDICTION (1)

Given $\left[x(k) = s(k) + n(k) \quad , \quad k = 0, 1, \ldots, n \right]$

estimate, (predict) $s(n+1)$ in the minimum mean square sense. using a linear estimator

where we assume that $s(n) = \alpha s(n-1) + w(n)$,

$s(n), n(n)$ are uncorrelated etc. . . . . .

Define the estimate $\boxed{\hat{s}(n+1/n) = \alpha \hat{s}(n)}$ this is the optimal estimator.

and the error $e(n+1/n) = s(n+1) - \hat{s}(n+1/n)$

Then it can be shown that $E\{x(n-\ell)\, e^*(n+1/n)\} = 0 \quad \ell = 0, 1, 2, \ldots, n.$

and $\sigma^2_{e(n+1/n)} = E\{s(n+1)\, e^*(n+1/n)\} = |\alpha|^2 \sigma^2_{e(n)} + \sigma_w^2$

RECURSIVE PREDICTION (2)

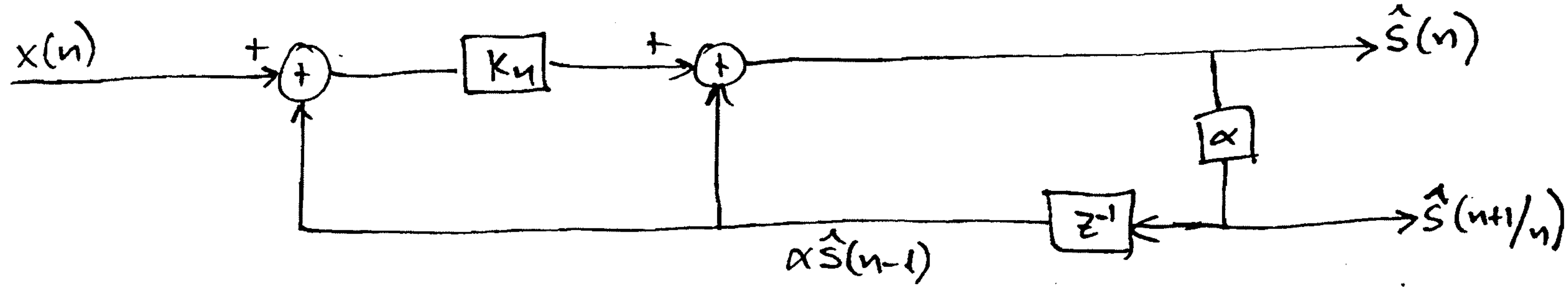
From previous results $\left[\hat{s}(n) = \alpha\hat{s}(n-1) + K_n\left[x(n) - \alpha\hat{s}(n-1)\right]\right]$, $\hat{s}(n+1/n) = \alpha\hat{s}(n)$ **1**

So $$\hat{s}(n+1/n) = \alpha\left(\underbrace{\cdot\alpha\hat{s}(n-1)} + K_n\left[x(n) - \alpha\hat{s}(n-1)\right]\right) \Rightarrow$$

$$\Rightarrow \left[\hat{s}(n+1/n) = \alpha\cdot\hat{s}(n/n-1) + \alpha K_n\left[x(n) - \hat{s}(n/n-1)\right]\right],$$ $$K_n = \frac{\sigma^2_{e(n+1/n)} - \sigma^2_w}{|a|^2\sigma^2_n} = \frac{\sigma^2_{e(n)}}{\sigma^2_n}$$ **2**

i.e., same as before.

Realization **1**

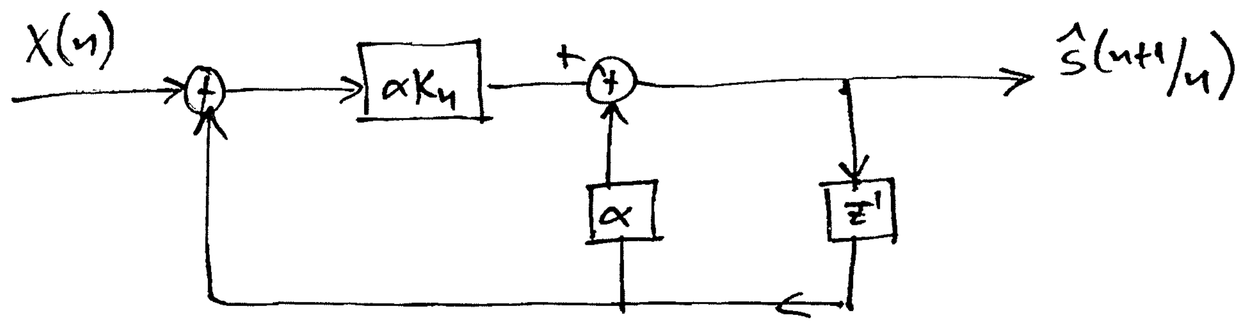


System to perform simultaneous filtering and prediction.

RECURSIVE PREDICTION (3)

- Direct realization of predictor. (recursive realization)

Realization 2



$$\alpha K_n = \frac{\sigma^2_{e(n+1/n)} - \sigma^2_w}{\alpha^* \sigma^2_n} \quad , \quad \sigma^2_{e(n+1/n)} = \sigma^2_w + \frac{|\alpha|^2 \sigma^2_n \sigma^2_{e(n/n-1)}}{\sigma^2_n + \sigma^2_{e(n/n-1)}}$$

* Note: for $\sigma^2_n \to 0$ it can be shown that $K_n \to 1$ and $\hat{s}(n+1/n) = \alpha x(n)$

RECURSIVE FILTERING

- The generalization of the presented recursive procedure to vector valued random processes is Known as the Kalman filter

- The recursive filter is a special case of the Wiener filter that is causal but not time invariant. It begins by processing a single observation and continues to use more samples of the observation process to estimate the desired signal as those samples become available.

- The filter parameters are also computed recursively but are independent of the observed data so they can be computed "off line."

- Assymptotically (or after a large number of recursions) the filter approaches steady state and becomes equivalent to the Infinite length Wiener filter....

Generalization of Kalman filter.

Let $$s(n) = \sum_{k=1}^{p} a(k)\, s(n-k) + w(n)$$

$$y(n) = s(n) + n(n)$$

Let $$\underline{s}(n) = \begin{bmatrix} s(n) \\ s(n-1) \\ \vdots \\ s(n-p+1) \end{bmatrix} \quad p \times 1$$

Then
$$\underline{s}(n) = \begin{bmatrix} a(1) & a(2) & \cdots & a(p) \\ 1 & 0 & & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \vdots & 1 & 0 \end{bmatrix} \underline{s}(n-1) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} w(n)$$

$$y(n) = [1 \ 0 \ 0 \ 0]\, \underline{s}[n) + v(n)$$

state transition matrix

or state eqn $\boxed{\underline{s}(n) = \underline{A}\, \underline{s}(n-1) + \underline{w}(n)}$ —) linear

measurement (observation) $\boxed{y(n) = \underline{C}\, \underline{s}(n) + v(n)}$ ← measurement eqn) lal

$E\{\underline{w}(n) w^H(k)\} = Q_w(n)\, \delta(n-k)$ $\quad E\{v(n) v^H(k)\} = Q_v(n)\, \delta(n-$

Optimum estimate given measurements up to time n

$$\boxed{\hat{\underline{s}}(n) = A\, \hat{\underline{s}}(n-1) + \underline{K}\left[y(n) - \underline{C}\,\underline{A}\,\hat{s}(n-1\right.}$$

Nonstation case : $\hat{s}(n) = \sum_{k=1}^{P} a(k)$

$$\underline{s}(n) = \underline{A}(n-1)\, \underline{s}(n-1) + \underline{w}(n)$$

$$\underline{y}(n) = \underline{C}(n)\, \underline{s}(n) + \underline{v}(n)$$

$$\hat{s}(n) = A(n-1)\hat{s}(n-1) + K(n)[\,y(n) - C(n)\,A(n-1)\,\hat{s}(n-1)\,]$$

DISCRETE KALMAN FILTER

- $x(k) = s(k) + \eta(k)$ — $x(k), s(k)$ jointly stationary; $s(k), \eta(k)$ independent or at least uncorrelated

  Known statistics

- Optimum FIR Wiener Filter → Normal equations (Wiener-Hopf equations)
- Optimum IIR Wiener Filter → Non-causal solution non-realizable but easy to design → performance bounds
- Causal IIR Wiener filtering → Spectral factorization → realizable but more difficult to design.
- Recursive filtering → discrete Kalman Filter Applicable to non-stationary situations as well

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AR(1) modeling of $s(n)$: $s(n) = as(n-1) + w(n)$

- Recursive solution - stationary case

$$\hat{s}(n) = a\,\hat{s}(n-1) + K_n[x(n) - a\,\hat{s}(n-1)]$$

Recursive solution - non-stationary case

$$\hat{s}(n) = a_{n-1}\,\hat{s}(n-1) + K_n[x(n) - a_{n-1}\,\hat{s}(n-1)]$$

Define

$$e(n/n) = s(n) - \hat{s}(n/n) \qquad \text{filtering error}$$
$$e(n/n-1) = s(n) - \hat{s}(n/n-1) \qquad \text{prediction error}$$

$\underline{P}(n/n)$ filtering error covariance matrix

$\underline{P}(n/n-1)$ prediction error covariance matrix

The Discrete Kalman Filter

State equation $\quad \underline{s}(n) = \underline{A}(n-1)\,\underline{s}(n-1) + \underline{w}(n)$

Observation equation $\quad \underline{y}(n) = \underline{C}(n)\,\underline{s}(n) + \underline{v}(n)$

Initialization: $\quad \hat{\underline{s}}(0/0) = E\{\underline{s}(0)\}$

$$\underline{P}(0/0) = E\{\underline{s}(0)\,\underline{s}^H(0)\}$$

Computation: for $n = 1, 2, \ldots$

$$\hat{s}(n/n-1) = \underline{A}(n-1)\,\hat{s}(n-1/n-1)$$

$$\underline{P}(n/n-1) = \underline{A}(n-1)\,\underline{P}(n-1/n-1)\,\underline{A}^H(n-1) + \underline{Q}_w(n)$$

$$\underline{K}(n) = \underline{P}(n/n-1)\,\underline{C}^H(n)\left[\underline{C}(n)\,\underline{P}(n/n-1)\,\underline{C}^H(n) + \underline{Q}_v(n)\right]^{-1}$$

$$\hat{\underline{s}}(n/n) = \hat{\underline{s}}(n/n-1) + \underline{K}(n)\left[\underline{y}(n) - \underline{C}(n)\,\hat{\underline{s}}(n/n-1)\right]$$

$$\underline{P}(n/n) = \left[\underline{I} - \underline{K}(n)\,\underline{C}(n)\right]\underline{P}(n/n-1)$$

EXAMPLE

We wish to estimate a signal $s(n)$ from the noisy observations

$$x(n) = s(n) + \eta(n), \quad n = 0, 1, \dots, N$$

where, $s(n)$ is generated as $s(n) = 0.8\, s(n-1) + \omega(n)$, $\omega(n)$ is white noise with $\sigma_\omega^2 = 0.36$ and $\eta(n)$ is also white with $\sigma_\eta^2 = 1$ and independent from $s(n)$. (That is $\eta(n)$ is observation corrupting noise, $\omega(n)$ is signal generating noise)

– from the equation for $s(n)$ we conclude that $R_s(k) = R_s(0) \cdot 0.8^{|k|}$

$k = 0, \pm 1, \pm 2, \dots$

– The causal wiener filter for the estimation of $s(n)$

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow \hat{d}(n) \quad \big(d(n) = s(n)\big)$$

is given by

$$H(z) = \frac{1}{K H_{ca}(z)} \left[\frac{S_{dx}(z)}{H_{ca}^*\left(\frac{1}{z^*}\right)} \right]_+$$

$$\text{Now} \quad S_{dx}(z) = S_{sx}(z) = S_s(z) = \underset{\text{aver}}{F}\big(R_s(k)\big) = F\left[0.36 \cdot 0.8^{|k|}\right] = \frac{0.36}{(1-0.8z^{-1})(1-0.8z)}$$

$$\text{and} \quad S_x(z) = K H_{ca}(z) H_{ca}^*\left(\frac{1}{z^*}\right) = 1 + S_s(z) = \dots$$

$$= \underbrace{1.6}_{K} \; \underbrace{\frac{(1-0.5z^{-1})}{(1-0.8z^{-1})}}_{H_{ca}(z)} \; \underbrace{\frac{(1-0.5z)}{(1-0.8z)}}_{H_{ca}^*(1/z^*)}$$

__Note:__ Since $x(n)$ is real: $H_{ca}^{*}\left(\frac{1}{z^{*}}\right) = H_{ca}(z^{-1})$

__Therefore:__ $H(z) = \frac{1}{k\, H_{ca}(z)} \cdot \left[\frac{S_{dx}(z)}{H_{ca}(z^{-1})}\right]_{+} =$

$$= \frac{1\,(1-0.8z^{-1})}{1.6\,(1-0.5z^{-1})} \cdot \left[\frac{0.36 \cdot \cancel{(1-0.8z)}}{(1-0.8z^{-1})\,\cancel{(1-0.8z)}\,(1-0.5z)}\right]_{+}$$

$$= \frac{1\,(1-0.8z^{-1})}{1.6\,(1-0.5z^{-1})} \cdot \left[\frac{0.36}{(1-0.8z^{-1})(1-0.5z)}\right]_{+} =$$

$$= \underbrace{\frac{0.6}{1-0.8z^{-1}}}_{\text{causal}} + \underbrace{\frac{0.3}{z^{-1}-0.5}}_{\text{un-causal}}$$

$$\Rightarrow H(z) = \boxed{\frac{1}{1.6}\,\frac{\cancel{(1-0.8z^{-1})}}{(1-0.5z^{-1})}\,\frac{0.6}{\cancel{(1-0.8z^{-1})}}} =$$

$$H(z) = \frac{0.375}{1-0.5z^{-1}} \Rightarrow \boxed{h\{n\} = 0.375\left(\frac{1}{2}\right)^{n} u(n)}$$

Since $H(z) = \frac{D(z)}{X(z)} \Rightarrow \boxed{d(n) = 0.5\,d(n-1) + 0.375\,x(n)}$

and $MMSE = R_d(0) - \sum_{l=0}^{\infty} h(l)\, R_{dx}(l) = 1 - \frac{3}{8}\sum_{l=0}^{\infty}\left(\frac{1}{2}\right)^{l}(0.8)^{l} = \underline{0.375}$

Now consider a recursive Kalman filter solution:

$$s(n) = 0.8\, s(n-1) + w(n) \quad \Rightarrow \quad A = 0.8$$

$$x(n) = s(n) + w(n) \qquad\qquad C = 1$$

and we can write the equations

$$\hat{s}(0) = E\{s(0)\} = 0$$

$$P(0/0) = E\{s(0)\, s^{+}(0)\} = 1$$

Then,

$$\hat{s}(n) = 0.8\, \hat{s}(n-1) + k(n)\left(x(n) - 0.8\, \hat{s}(n-1)\right)$$

$$P(n/n-1) = 0.8^2\, P(n-1/n-1) + 0.36$$

$$K(n) = P(n/n-1)\left[P(n/n-1) + 1\right]^{-1}$$

$$P(n/n) = \left[1 - K(n)\right] P(n/n-1)$$

We can find that $\lim\limits_{n \to \infty} k(n) \rightarrow \underline{0.375}$ as expected