DIGITAL 1-D

POWER SPECTRUM

ESTIMATION

OBJECTIVE:

- Given N observations {x(0), x(1), ..., x(N-1)} of a single realization of a wide-sense stationary random process, estimate power spectral density in the range

$$0 \leq f \leq \frac{1}{T},$$

(T: sampling period)

- Power spectral analysis provides an "estimate", rather than an exact representation of the power spectral density ("empirical")

CLASSICAL METHODS

1. PERIODOGRAM (SHUSTER)

(Direct, deterministic definition)

- Mean periodogram (Welch)

2. CORRELOGRAM (BLACKMAN - TUKEY)

(Indirect, stochastic definition)

- Smooth correlogram (Bartlett)

● **TRADE-OFF BETWEEN BIAS - VARIANCE OF ESTIMATOR**

Small bias - Acceptable variance <u>(YES)</u>

PERIODOGRAM (1) (Schuster, 1898)

- **PSD DEFINITION FOR STOCHATIC SIGNALS**

$$P(f) = \lim_{M\to\infty} E\left\{\frac{1}{2M+1}\left|\sum_{n=-M}^{M} x(n)e^{-j2\pi fn}\right|^2\right\}, \qquad 0 \le f \le 1$$

- **BY NEGLECTING E{.} AND ASSUMING WE ARE GIVEN {x(0),...,x(N-1)}**

$$\hat{P}_{PER}(f) = \frac{1}{N}\left|X(f)\right|^2 = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\,e^{-j2\pi fn}\right|^2, \qquad 0 \le f \le 1$$

- **BY USING DFT (implemented with FFT)**

$$\hat{P}_{PER}(k) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)e^{-j2\pi \frac{k}{N} n}\right|^2, \qquad k=0,1,\dots,N-1$$

PERIODOGRAM (2)

MEAN OF ESTIMATOR:

$$E\{\hat{P}_{PER}(f)\} = E\{\frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\omega n} \cdot \sum_{k\ 0}^{N-1} x^*(k)\, e^{j\omega k}\} =$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} E\{x(n)\, x^*(k)\}\, e^{-j\omega(n-k)} =$$

$$R(n\text{-}k)$$

$$= \frac{1}{N}\sum_{m=-(N-1)}^{N-1} (N-|m|)\, R(m)\, e^{-j\omega m} = \sum_{m=-(N-1)}^{N-1} \frac{N-|m|}{N} R(m)\, e^{-j\omega m}$$

(m=n-k, $\omega=2\pi f$)

Let $N \to \infty$, then,

$$E\{\hat{P}_{PER}(f)\} \to \sum_{m=-(N-1)}^{N-1} R(m)\, e^{-j\omega m} = P(f)$$

PERIODOGRAM (3)

- For N finite $\hat{P}_{PER}(f)$ is biased estimator

$$E\{\hat{P}_{per}(f)\} = F\{w(m) . R(m)\} = W(f) * P(f)$$

where, $w(m) = \dfrac{N - |m|}{N}$ is the Bartlett window,

F{..} denotes Fourrier transform

- $\hat{P}_{PER}(f)$ is assymptotically unbiased estimator

- White noise case ($R(m)=\sigma_x^2\,\delta(m)$) $\hat{P}_{PER}(f)$ is unbiased even for finite N

PERIODOGRAM (4)

VARIANCE OF ESTIMATOR: (x(n): real)

$$E\{\hat{P}_{PER}(f_1).\hat{P}_{PER}(f_2)\} = (\frac{1}{N})^2 \sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} E\{x(k)x(l)x(m)x(n)\}.e^{-j[\omega_1(k-l)+\omega_2(m-n)]}$$

- Difficult to analyze
- Easier calculations assuming x(n) is white and Gaussian

In general [Kay, (1988), Oppenheim & Schafer (1991)]

$$E\{\hat{P}_{PER}(f_1)\,\hat{P}_{PER}(f_2)\} = P(f_1)\,P(f_2)\,.\,[\,(\frac{\sin N\pi\,(f_1+f_2)}{N\sin\pi\,(f_1+f_2)})^2 + (\frac{\sin N\pi\,(f_1-f_2)}{N\sin\pi\,(f_1-f_2)})^2\,]$$

$$VAR\{\hat{P}_{PER}(f)\} = COV\{\hat{P}_{PER}(f)\hat{P}_{PER}(f)\} = P^2(f)\,[\,1+(\frac{\sin N2\pi f}{N\sin 2\pi f})^2\,]$$

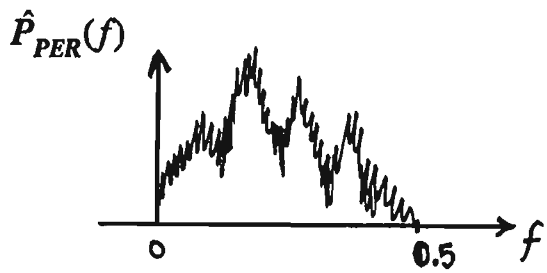
- Let N $\rightarrow \infty$, then, $VAR\{\hat{P}_{PER}(f)\} \approx P^2(f)$:not consistent estimator

- Let $f_1 = \frac{m}{N}$, $f_2 = \frac{n}{N}$, then, $COV[\hat{P}_{PER}(f_1)\,.\,\hat{P}_{PER}(f_2)] = 0$

PERIODOGRAM (5)

$$\hat{P}_{PER}(k) = \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi \frac{k}{N} n} \right|^2 , \qquad k=0,\ldots,N-1$$

1. Assymtotically unbiased estimator $(E\{\hat{P}_{PER}(k)\} \xrightarrow{N \to \infty} P(k))$
2. Not consistent estimator $(VAR\{\hat{P}_{PER}(k)\} \xrightarrow{N \to \infty} P^2(k))$
3. Extension by periodicity (underlaying assumption for FFT operations)
4. Rapidly fluctuating as N increases
5. Easy to implement (FFT)
6. Resolution: $\sim \frac{1}{N}$



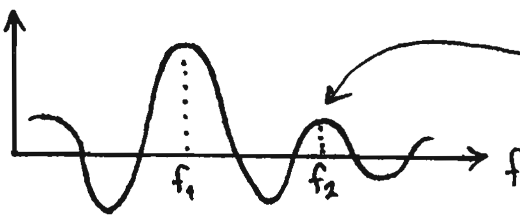
PERIODOGRAM (6)

- **Use of windows:**

$$\hat{P}_{PER}(k) = \frac{1}{N} \left| \sum_{n=0}^{N-1} w(n)\, x(n)\, e^{-j2\pi \frac{k}{N} n} \right|^2, \qquad k=0,\dots,N-1$$

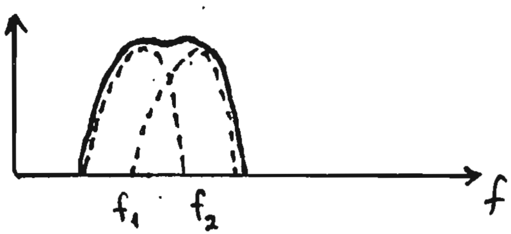
- **w(n)x(n) → W(f)*X(f)**

- **Spectral leakage caused by the the window sidelobes**



Signal at $f=f_1$ "perceived" as a signal at $f=f_2$ (bias)

- **Resolution bandwidth depends on the main lobe width**



Two equal strength main lobes separated in frequency by less than the 3-dB bandwidths of the main lobe will exhibit a single spectral peak.
(6-dB bandwidth separation required)

WINDOWS {w(n)} (1)

- **Spectral leakage vs. Resolution**

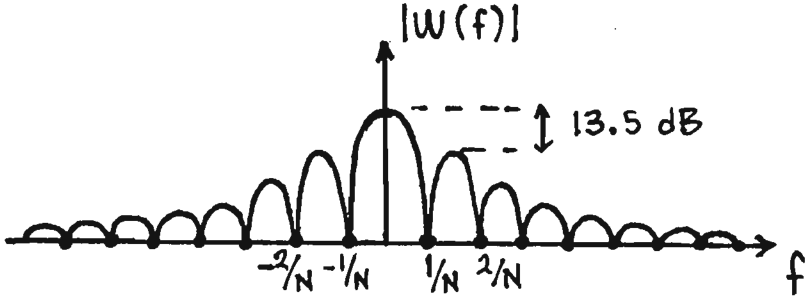
  ↑ ↑

  Low sidelobes vs. Narrow main lobe

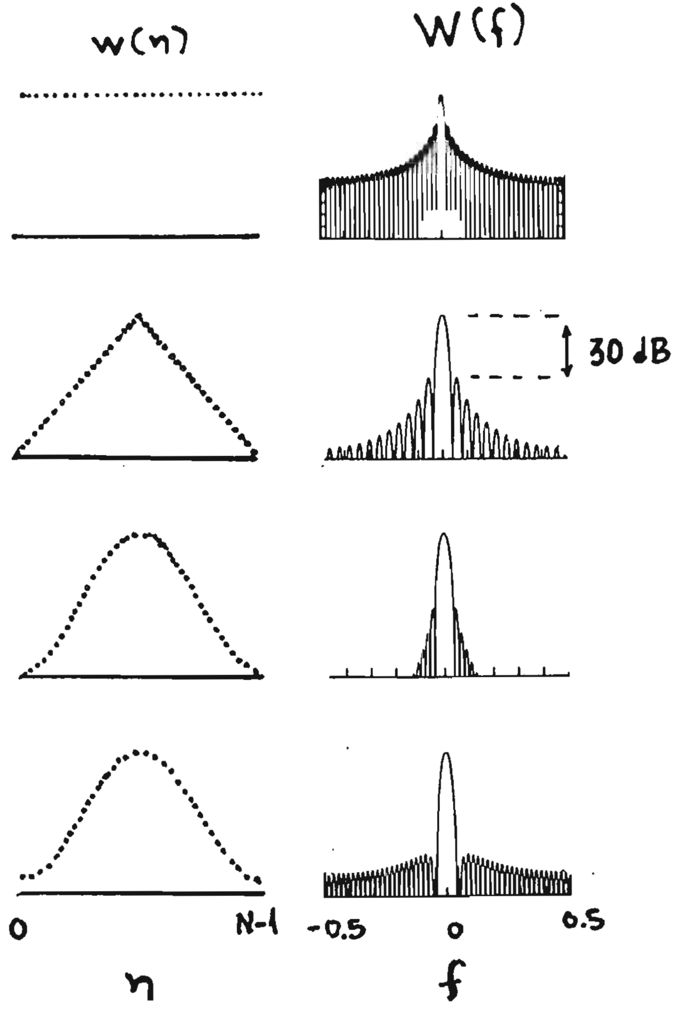
- **The rectangular window is the "default" window**

  - Characterized by high sidelobes and narrow mainlobe

$$w(n)=\begin{cases} 1, & n=0,\dots,N-1 \\ 0, & \text{otherwise} \end{cases}$$



WINDOWS (2)



Rectangular

$w(n)=1$

Triangular (Bartlett)

$w(n)=1-|F(n)|$

Hanning

$w(n)= 0.5 + 0.5\cos[2\pi F(n)]$

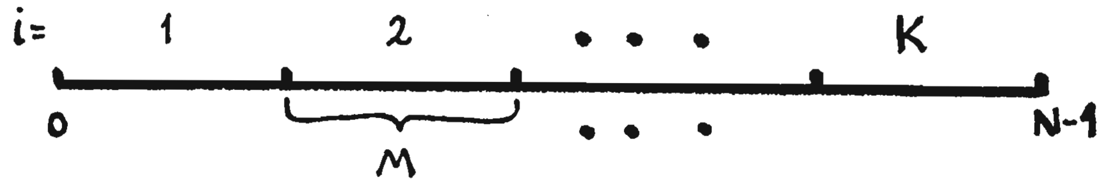
Hamming

$w(n)= 0.54 + 0.46\cos[2\pi F(n)]$

$$F(n) = \frac{n-\frac{(N-1)}{2}}{\frac{N-1}{2}}, \qquad n=0,\ldots,N-1$$

AVERAGED PERIODOGRAM (Welch, 1967)

- Given x(n), n=0,...,N-1 , K=N/M



- Obtain periodogram of each segment of length M

$$\hat{P}^{i}_{PER}(f) = \frac{1}{M} \left| \sum_{n=(i-1)M}^{iM-1} x(n)\, e^{-j2\pi f n} \right|^2 , \qquad i=1,2,\ldots,K$$

- Average over periodograms of all segments

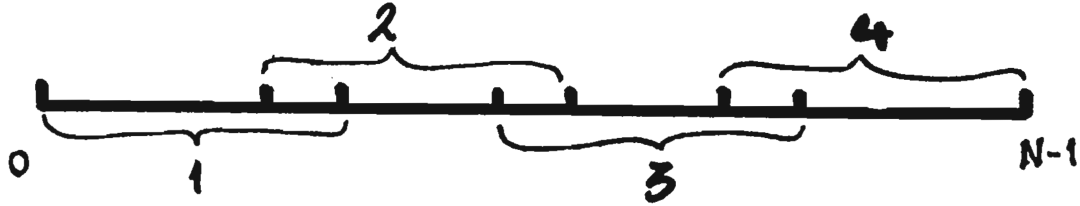
$$\hat{P}^{AV}_{PER}(f) = \frac{1}{K} \sum_{i=1}^{K} \hat{P}^{i}_{PER}(f)$$

AVERAGED PERIODOGRAM (2)

- Less resolution than periodogram
- Assymptotically unbiased estimator
- Less variance

$$VAR[\hat{P}_{PER}^{AV}(f)] = \frac{1}{K} VAR[\hat{P}_{PER}^{i}(f)]$$

- FFT based calculation
- Windowing can be applied to each segment
- Overlapping segments for further reduction of variance



BLACKMAN - TUKEY METHOD (1958) (1)

- **Given x(n), n=0,...,N-1**
- **Form biased autocorrelation estimates** $\left(M \simeq \frac{N}{10} \text{ to } \frac{N}{5} \right)$

$$\hat{R}_x(m) = \frac{1}{N} \sum_{n=0}^{N-m-1} x^*(n)\, x(n+m)\,, \qquad m=0,1,\ldots,M$$

$$\left(\hat{R}_x(-m) = \hat{R}_x^*(m) \right)$$

- Obtain PSD estimate

$$\hat{P}_{BT}(f) = \sum_{m=-M}^{M} \underset{\substack{\uparrow \\ \text{window}}}{w(m)}\, \hat{R}_x(m)\, e^{-j2\pi fm}$$

- **Equivalent to periodogram if w(m)=1 and M=N-1**

BLACKMAN - TUKEY METHOD (2)

- **Unbiased autocorrelation estimate $R_x(m) = \frac{N}{N-m}\hat{R}_x(m)$ is not a positive definite sequence and does not guarantee non-negative power spectral density $\hat{P}_{BT}(f)$**

- **Assymptotically unbiased $\hat{P}_{BT}(f) \xrightarrow{M \to \infty} \hat{P}_{BT}(f)$**

- **Variance: $VAR[\hat{P}_{BT}(f)] \approx \frac{P^2(f)}{N}\sum_{k=-M}^{M} |w(k)|^2$**

- **Smooth spectra - Sidelobe activity** ($w(m).R_x(m) \to W(f)*P_x(f)$)

- **Resolution ~ 1/M**

- **Zero padding extension of observed data** (underlaying assumption)

- **Efficient calculation with FFT**

BLACKMAN - TUKEY METHOD (3)

- **Zero padding is necessary to produce more "dense" spectral estimates**

  (*i.e.*, more DFT points)

$$\begin{cases} \hat{R}_x(m), & m=0,1,\ldots,M \quad \text{(biased estimates)} \\ \hat{R}_x(m) = 0, & m=M+1,\ldots,L \\ \hat{R}_x(-m) = \hat{R}_x^*(m) & \end{cases}$$

  Then,

$$\hat{P}_{BT}(k) = \sum_{m=-L}^{L} w(m)\,\hat{R}_x(m)\,e^{-j2\pi \frac{k}{(2L+1)} m}, \qquad k=0,\ldots,2L$$

- **Zero padding increases only spectral "visibility" but not resolution**

BLACKMAN - TUKEY METHOD (4)

- **Implementation using FFT**
  (Most FFT's are designed for positive time instants)

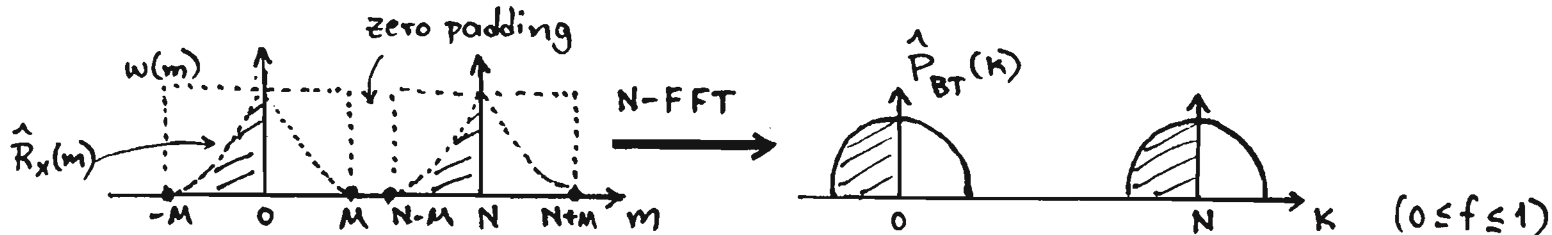
- **Given** $\hat{R}_x(m)$; m=-M,...,0,...M

- **Generate:**

$$\hat{R}_x^f(m) = \begin{cases} \hat{R}_x(m), & m=0,\dots,M \\ 0, & m=M+1,\dots,N-M-1 \\ \hat{R}_x(m-N), & m=N-M,\dots,N-1 \end{cases}$$

- **Obtain N-point FFT**

$$\hat{P}_{BT}(k) = \sum_{m=0}^{N-1} w^f(m)\,\hat{R}_x^f(m)\,e^{-j2\pi\frac{k}{N}m}, \qquad k=0,\dots,N-1 \qquad (0 \le f \le 1)$$



MINIMUM VARIANCE SPECTRAL ESTIMATOR (MVSE) (1)

(Capon, 1969) , (Lacoss, 1971)

- **Also known as: Maximum Likelihood Method (MLM)**
  - ML estimation of sinusoid amplitude $x(n) = Ae^{j\phi} . e^{j2\pi f_0 n} + z(n)$;

  f_0 is known, $n=0,1,...,M-1$

  Thus: $\underline{x} = Ae^{j\phi}\,\underline{E}(f_0) + \underline{z}$

  - $\underline{E}(f_0) = [\,1, e^{-j2\pi f_0}, \ldots, e^{-j2\pi (M-1) f_0}\,]$

  - *z(n)*: Complex, zero-mean Gaussian
    known autocorrelation matrix $\underline{R}_z = E\{\underline{z}\,\underline{z}^H\}$

  - **MLE of $A_\phi = Ae^{j\phi}$ (unbiased estimator):** $\hat{A}_\phi^{MLE} = \dfrac{\underline{E}^H(f_0)\,\underline{R}_z^{-1}\,\underline{x}}{\underline{E}^H(f_0)\,\underline{R}_z^{-1}\,\underline{E}(f_0)}$

[Maximize $P[x \mid A_\phi]$ $\rightarrow$ minimize $(\underline{x} - A_\phi E(f_o))^H R_z^{-1} (\underline{x} - A_\phi E(f_0))$]

Note: it means probability of $\underline{x}$ given A_ϕ

MINIMUM VARIANCE SPECTRAL ESTIMATOR (MVSE) (2)

■ Linear minimum variance unbiased estimate (LMVU) of $A_\phi = Ae^{j\phi}$

$$\underline{a} = [a_0, \ldots, a_{M-1}]^T, \quad \underline{x} = [x(0), \ldots, x(M-1)]^T$$

Problem definition

Minimize $VAR\{\hat{A}_\phi^{LMVU}\} = \underset{1\times M}{\underline{a}^H} \; \underset{M\times M}{\underline{R}_z} \; \underset{M\times 1}{\underline{a}}$, s.t.c. $\underline{a}^H \underline{E}(f_0) = 1$

(condition for unbiased estimator)

- Sinusoid (f_0) passes through filter undistorted
- Solution: $\hat{\underline{a}}_{OPT} = \dfrac{\underline{R}_z^{-1} \underline{E}(f_0)}{\underline{E}^H(f_0)\underline{R}_z^{-1}\underline{E}(f_0)}$, and thus, $\hat{A}_\phi^{LMVU} = \hat{A}_\phi^{MLE}$
- Minimum variance: $VAR\{\hat{A}_\phi^{LMVU}\}_{MIN} = \dfrac{1}{\underline{E}^H(f_0)\underline{R}_z^{-1}\underline{E}(f_0)}$

MINIMUM VARIANCE SPECTRAL ESTIMATOR (MVSE) (3)

■ FILTERING INTETPRETATION OF LMVU ESTIMATOR (1)

Let z(n) be white noise ($\underline{R}_z=\sigma^2\underline{I}$). Then,

$$\hat{A}_{\phi}^{LMVU} = \frac{\underline{E}^H(f_0)\,\underline{x}}{\underline{E}^H(f_0)\,\underline{E}(f_0)} = \frac{1}{M}\sum_{n=0}^{M-1} x(n)\,e^{-j2\pi f_0 n} = h(n) * x(n)\,\big|_{n=0}$$

$$h(n) = \begin{cases} \frac{1}{M}e^{j2\pi f_0 n}, & n=-(M-1),\dots,-1,0 \\ 0, & \text{otherwise} \end{cases}$$

$$H(f) = \frac{\sin[M\pi(f-f_0)]}{M\sin[\pi(f-f_0)]}\,e^{j(M-1)\pi(f-f_0)}$$

Narrowband bandpass

Filter centered at $f=f_0$
($H(f_0)=1$ according to constraint

MINIMUM VARIANCE SPECTRAL ESTIMATOR (MVSE) (4)

■ FILTERING INTETPRETATION OF LMVU ESTIMATOR (2)

- **For white noise**: $VAR\{\hat{A}_{\phi}^{LMVU}\}_{MIN} = \frac{\sigma^2}{M}$, for all $0 \leq f_0 \leq 1$

- **For colored noise**: The effect of $\underline{R}_z$ is to produce a frequency response with high attenuation in bands where the noise power is high and low attenuation in bands where the noise power is low,

  i.e., $VAR\{\hat{A}_{\phi}^{LMVU}\}_{MIN} = Q(f_0)$

■ **PERIODOGRAM ESTIMATOR**: The corresponding narrowband and bandpass filter H(f) centered at $f=f_0$ is independent of f_0 *in all cases* and identical to that of LMVU for white noise.

MINIMUM VARIANCE SPECTRAL ESTIMATOR (MVSE) (5)

- **LMVU** estimator: A filter that allows frequency components around f_0 while rejects other frequency components in an optimal way

Let x(n) be a WSS process

$$\min E\{|y(n)|^2\} \equiv \min[\underline{a}^H \underline{R}_x \underline{a}], \qquad s.t.c. \qquad \underline{a}^H \underline{E}(f) = 1$$

$\underline{E}(f)=[1, e^{-j2\pi f}, \ldots, e^{-j2\pi f(M-1)}]^T$

Minimum variance $$\hat{P}_{MVSE\ or\ MLM}(f) = \frac{1}{E^H(f)\,\underline{R}_x^{-1}\,E(f)} \qquad 0 \le f \le 1$$

[*i.e.*, for $f=f_0$, $\hat{P}_{MVSE}(f_0)$ is the power at the output of the optimal filter with coefficients $\underline{\hat{a}}_{OPT}$]

MINIMUM VARIANCE SPECTRAL ESTIMATOR (MVSE) (6)

1. Higher resolution than classical methods (Periodogram, BT,...)

2. Let $R_x(k) = P e^{j2\pi f_0 k} + \sigma_z^2 \delta(k)$. Then,

$$\hat{P}_{MVSE}(f_0) = \frac{\sigma_z^2}{M}\left(1 + \frac{MP}{\sigma_z^2}\right) \simeq P, \qquad \text{(if SNR}= \frac{P}{\sigma_z^2} \text{ is high)}$$

3. Area under peak of estimator $\sim \sqrt{P}$ (i.e., $\hat{P}_{MVSE}(f)$ is not the true PSD)

4. In practice biased estimate of the autocorrelation matrix is utilized

5. Pseudolinearity property near peaks

MAXIMUM ENTROPY SPECTRAL ESTIMATOR (1)

- Given the autocorrelation lags $R_x(m)$, m=0,...,M

Problem definition

Maximize $$\int_0^1 \ln P_x(f)\, df$$ Entropy/sample for a zero-mean Gaussian process

s.t.c $$\int_0^1 P_x(f)\, e^{j2\pi fm}\, df = R_x(m)\,, \qquad m=0,1,\ldots,M$$

Interpretation

Maximum randomness outside the known range of autocorrelations

(extrapolation of $R_x(m)$, m=M+1,...)

- Flattest spectrum among all spectra with the given $R_x(m)$, m=0,1,...,M

MAXIMUM ENTROPY SPECTRAL ESTIMATOR (2)

- **Solution:**

$$\hat{P}_x^{MESE}(f) = \frac{\sigma_M^2}{\left|1 + \sum_{n=1}^{M-1} a_{M,n} e^{-j2\pi fn}\right|^2}$$

- $\{a_{M,n}\}$, n=1,...,M and σ_M^2 are obtained by solving the "Normal Equation"

$$\begin{bmatrix} R_x(0) & R(-1) & R(-2) & \dots & R(-M) \\ R_x(1) & R(0) & R(-1) & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ R_x(M) & R(M-1) & . & \dots & R(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_{M,1} \\ . \\ . \\ . \\ a_{M,M} \end{bmatrix} = \begin{bmatrix} \sigma_M^2 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

MAXIMUM ENTROPY SPECTRAL ESTIMATOR (3)

- Autocorrelation matrix is Hermitian Toeplitz ($R(-m) = R^*(m)$)

  Solve the Normal equations with Levinson Recursion

- Equivalent to AR spectral estimators for Gaussian processes and

  known autocorrelations

- Higher resolution than MVSE , PER , BT methods

RESOLUTION

Let: $x(n)$, $n=0,1,\ldots,M-1$

$R_x(m)$, $m=0,\pm1,\ldots,\pm M$

- **CLASSICAL:** $$\Delta f \geq \frac{1}{M}$$

- **MVSE :** $$\Delta f \geq \frac{1/\sqrt{SNR}}{(2M+1)}$$

- **MESE :** $$\Delta f \geq \frac{1/SNR}{(2M+1)}$$

EXAMPLE (1)

We are given the time series data

$$\sqrt{20}\cos(2\pi 0.15n) + \sqrt{20}\cos(2\pi 0.25n) + \sqrt{10}\cos(2\pi 0.3n + \frac{\pi}{4}) + w(n), \quad n = 1,\ldots,N$$

- Calculate and draw the following spectrum estimators (in log scale in the range $0 \le f \le \frac{1}{2}$)

  - Periodogram } rectangular window
  - Blackman - Tukey } Bartlett window
  - MVSE

  Consider the following 2 cases:

  i) N=32, 128, 512; w(n): white, Gaussian, $\sigma_w^2=1$, zero-mean

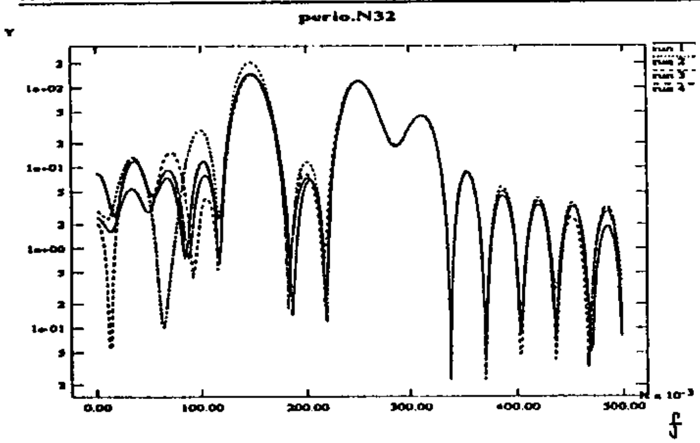
  ii) N=32, 128, 512; w(n): colored, Gaussian, $w(n) = \sum_k q(k)\,z(n-k)$

  where:

  $$q(n) = 0.227\delta(n) + 0.460\delta(n-1) + 0.688\delta(n-2) + 0.460\delta(n-3) + 0.227\delta(n-4)$$

  z(n): white, Gaussian, $\sigma_w^2=1$, zero-mean

PERFORMANCE COMPARISONS (1)



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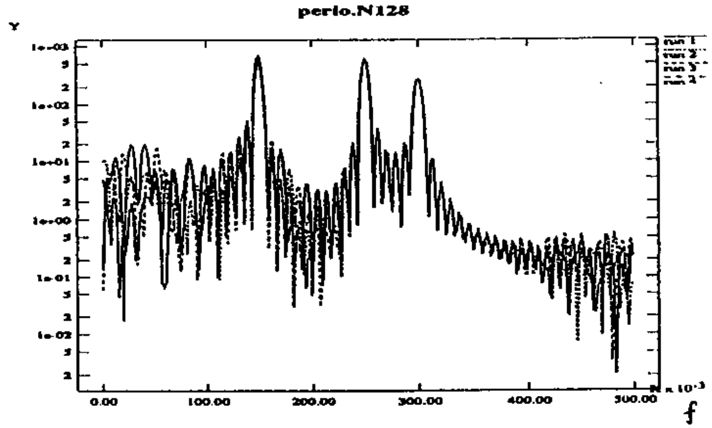
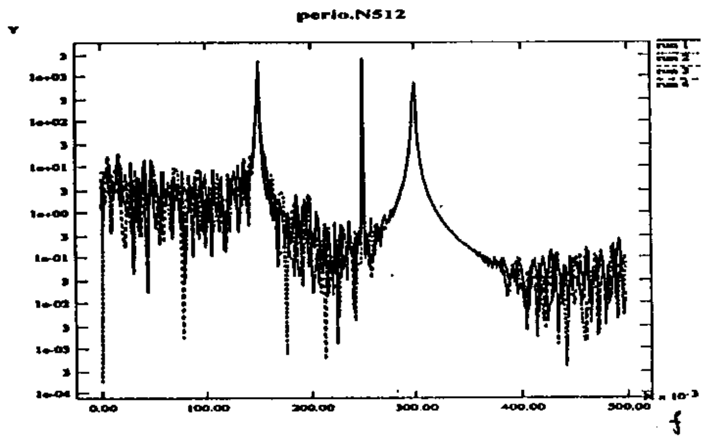


(b)



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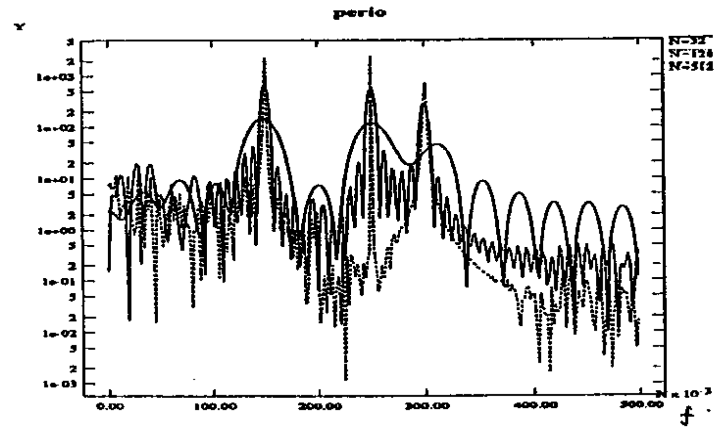
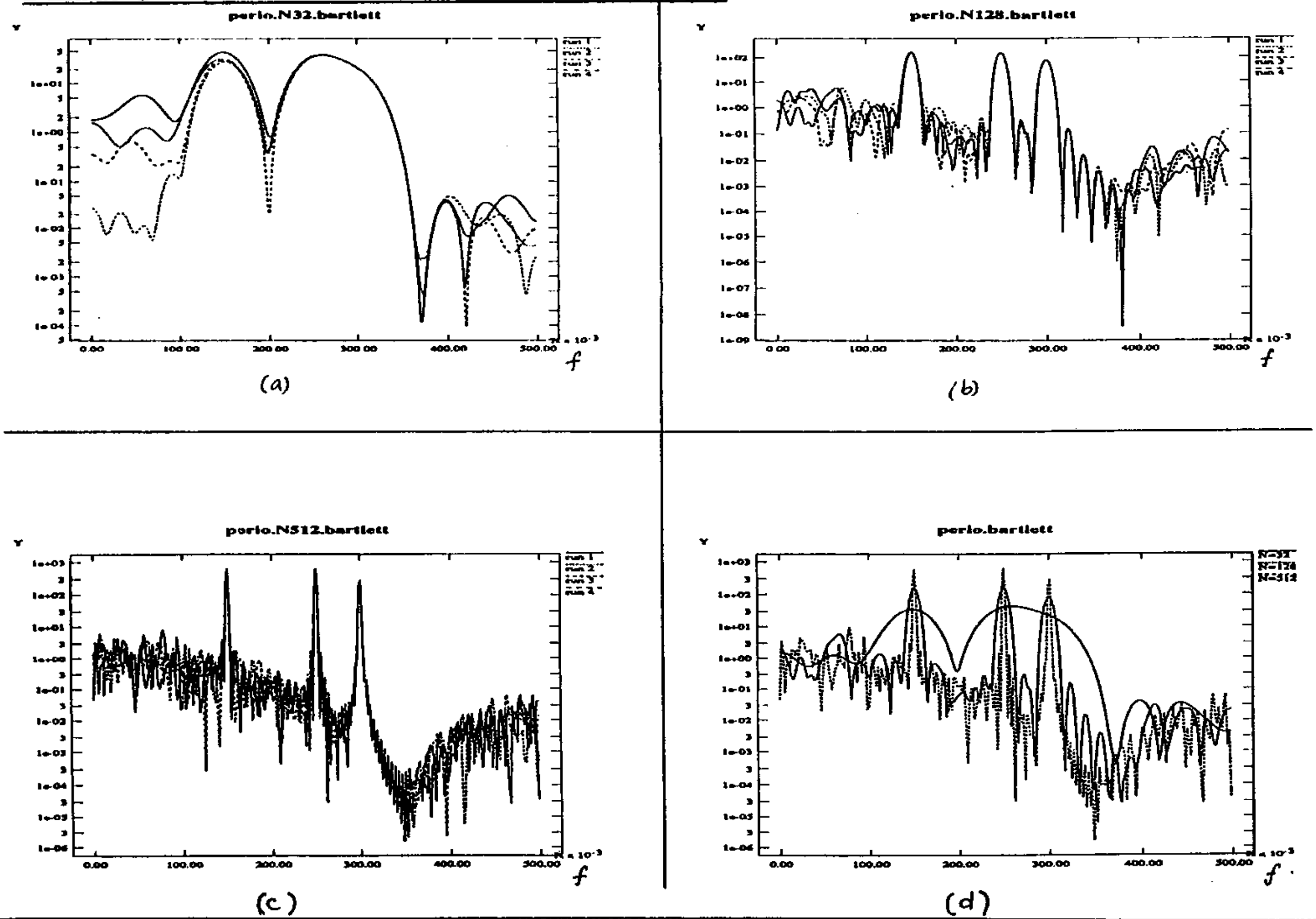
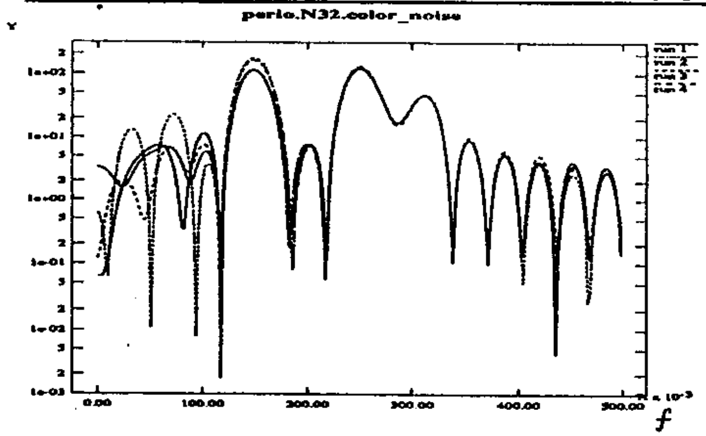


(d)

PERFORMANCE COMPARISONS (2)



PERFORMANCE COMPARISONS (3)



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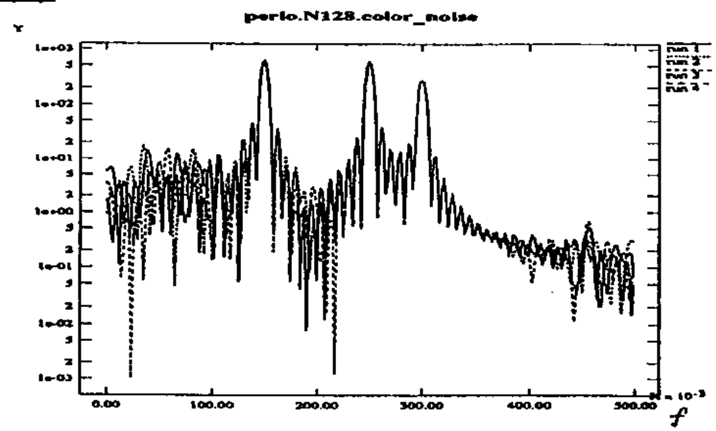
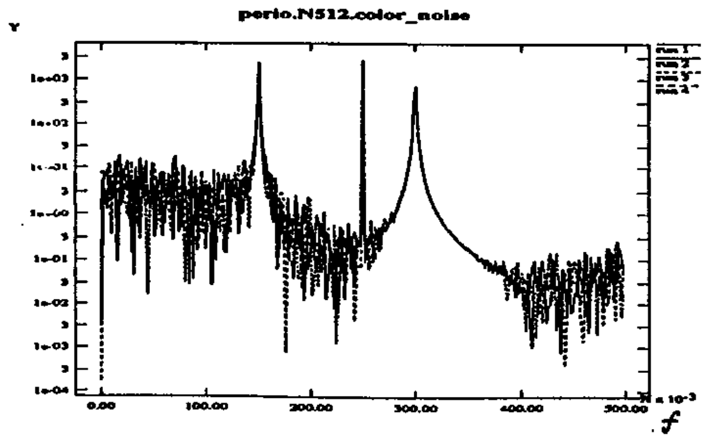


(b)



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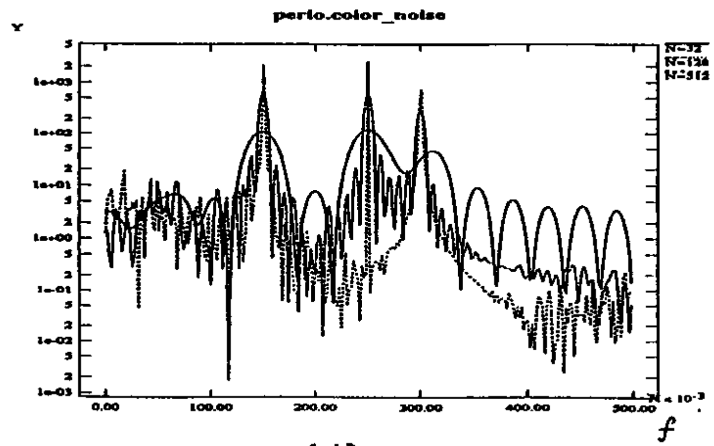
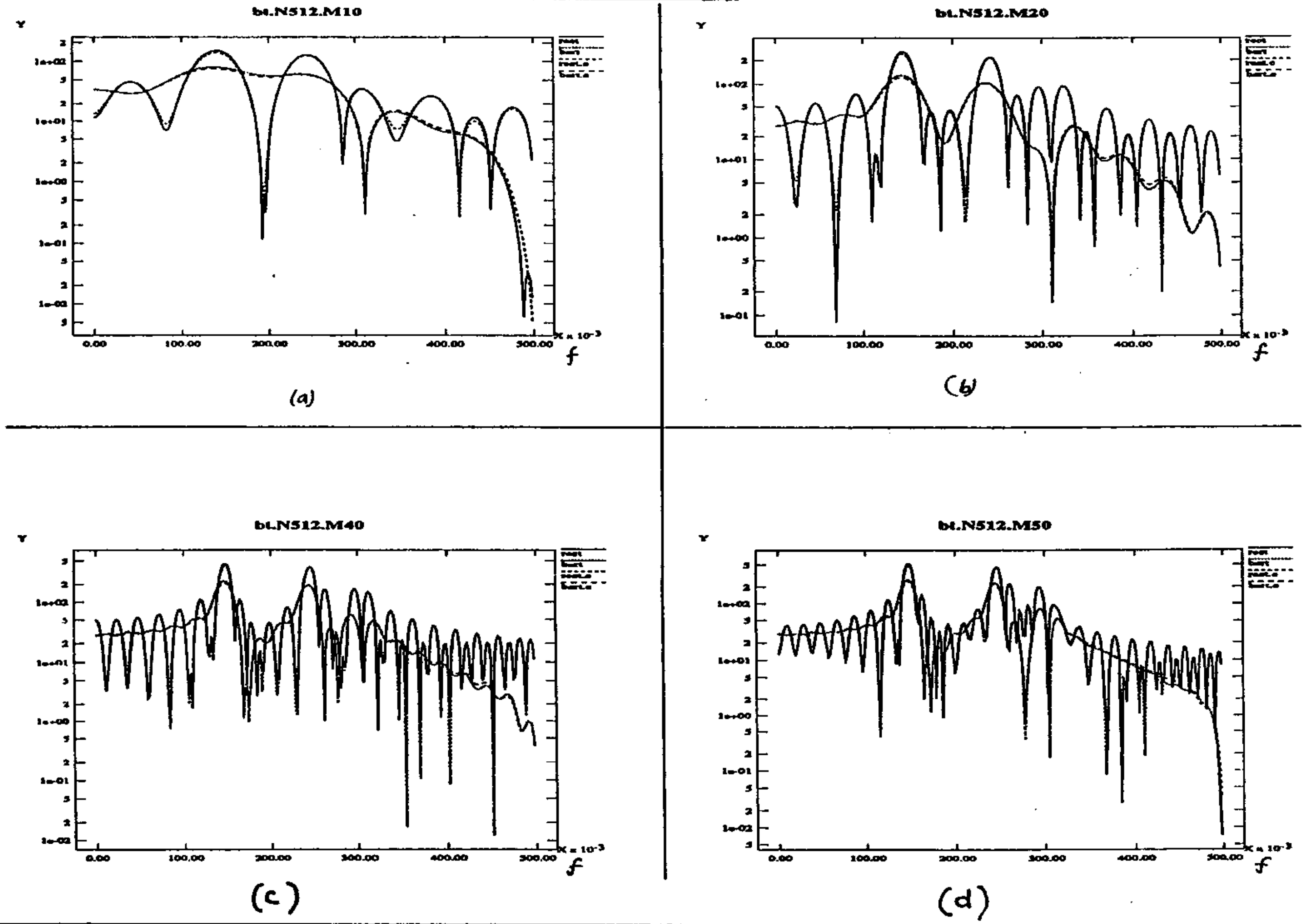
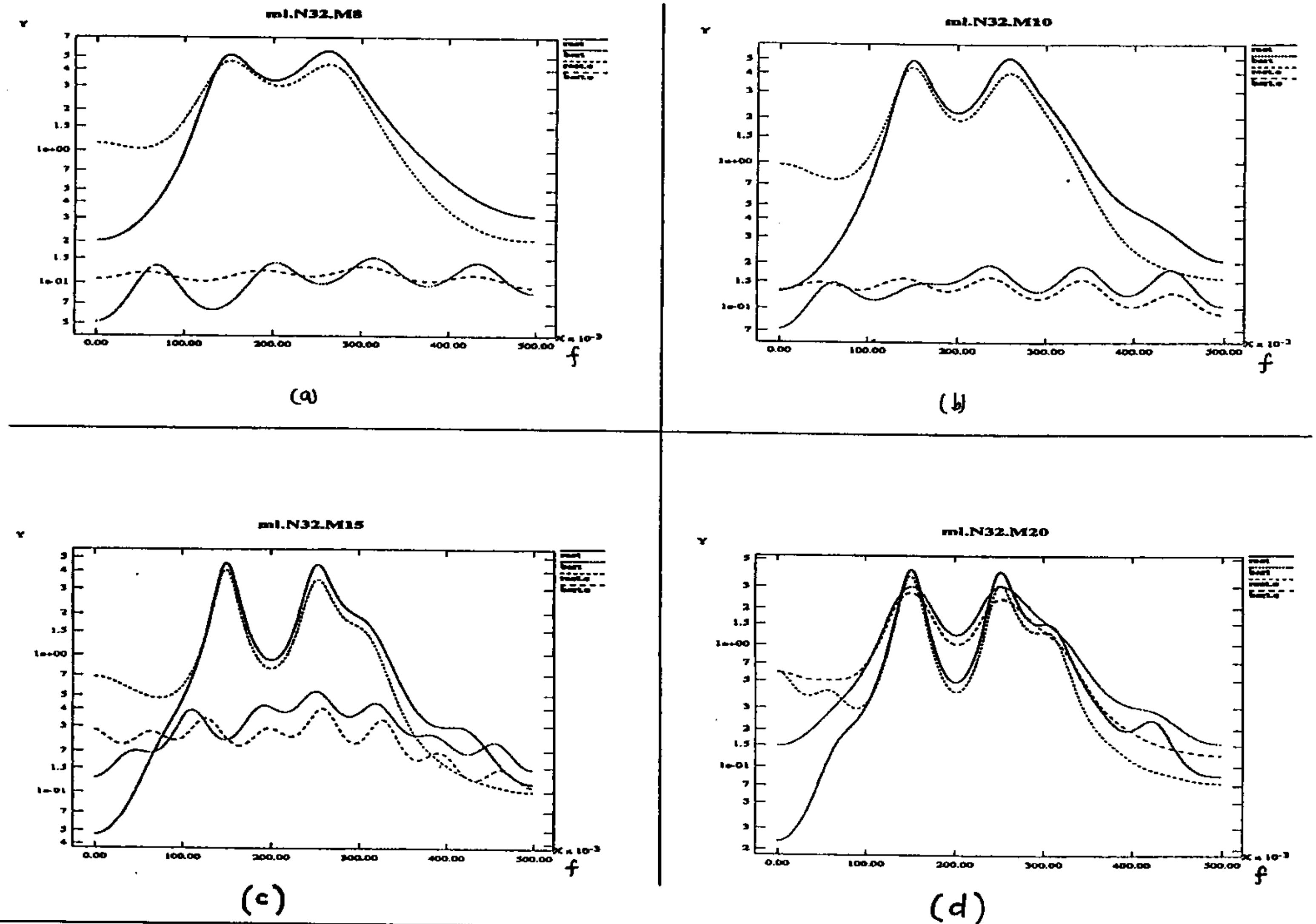


(d)

PERFORMANCE COMPARISONS (4)



PERFORMANCE COMPARISONS (5)



PERFORMANCE COMPARISONS (6)