SINGULAR VALUE DECOMPOSITION (SVD) (1)

Given a non-rectangular matrix **A**: m×n, Rank(**A**)=p≤min(m,n)

1) Form the matrix ($\mathbf{A}^H\mathbf{A}$): n×n. Obtain
- eigenvalues $\{\lambda_i^2\}$
- eigenvectors $\{\mathbf{V}_i\}$

i.e., $(\mathbf{A}^H\mathbf{A}).\mathbf{V}_i = \lambda_i^2 \mathbf{V}_i$, i=1,...p

n×n n×1 1×1 n×1

2) Form the matrix ($\mathbf{A}\mathbf{A}^H$): m×m. Obtain
- eigenvalues $\{\lambda_i^2\}$
- eigenvectors $\{\mathbf{U}_i\}$

i.e., $(\mathbf{A}\mathbf{A}^H).\mathbf{U}_i = \lambda_i^2 \mathbf{U}_i$, i=1,...p

m×m m×1 1×1 m×1

3) **SVD(A)**: $[A = \sum_{i=1}^{p} \lambda_i U_i V_i^H]$, Pseudoinverse: $A^{-1} = \sum_{i=1}^{p} \frac{1}{\lambda_i} V_i U_i^H$

- $\{\mathbf{V}_i\}$, $\{\mathbf{U}_i\}$ are orthonormal sets, i.e., $\sum_i V_i V_i^H = I$ and $\sum_i U_i U_i^H = I$
- $\{\lambda_i\}$, i=1,...,p Real and positive

SINGULAR VALUE DECOMPOSITION (SVD) (2)

- Given a matrix **A**: m×m, Rank(**A**)=p≤m

1) Obtain the eigenvalues $\{\lambda_i\}$ and the eigenvectors $\{\mathbf{Q}_i\}$, i.e.,

$$\underset{m\times m}{\mathbf{A}}\ \underset{m\times 1}{\mathbf{Q}_i} = \underset{1\times 1}{\lambda_i}\underset{m\times 1}{\mathbf{Q}_i} \qquad i=1,\ldots p$$

2) **SVD(A)**: $$[A = \sum_{i=1}^{p} \lambda_i Q_i Q_i^H] \quad , \quad A^{-1} = \sum_{i=1}^{p} \frac{1}{\lambda_i} Q_i Q_i^H$$

(or eigendecomposition)

- If $\mathbf{A}=\mathbf{A}^H$, i.e., Hermitian
  - $\{\lambda_i\}$ are real, positive or zeros
  - $\sum_i Q_i Q_i^H = I$

SINGULAR VALUE DECOMPOSITION (SVD) (3)

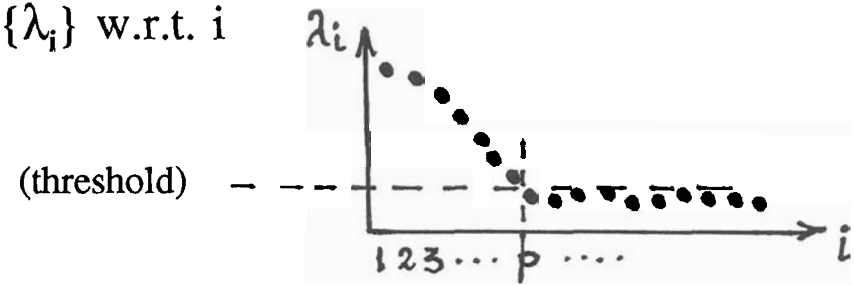
■ LOWER RANK APPROXIMATIONS

Let **A**: m×n, m>n, Rank(**A**)=min(*m*,*n*)=*n*, (**A** is a full rank matrix)

- **SVD(A)**: $A = \sum_{i=1}^{n} \lambda_i U_i V_i^H$

- Order the $\{\lambda_i\}$, i=1,...n in descending order, i.e., $\lambda_1 \geq \lambda \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n$

- Plot $\{\lambda_i\}$ w.r.t. i



- Force λ_i=0 , i>p. Obtain the approximation $\hat{A} = \sum_{i=1}^{p} \lambda_i U_i V_i^H$

- Application: Enhancement of SNR (since the λ_i, i>p due to noise)

EIGENDECOMPOSITION (SVD) OF AUTOCORRELATION MATRIX (1) (for harmonic processes)

Let $$x(n) = \sum_{i=1}^{M} A_i e^{j2\pi f_i(n-1)} + w(n)$$ Let w(n) be independent from exponentials

Then, it is easy to show that the autocorrelation function is

$$R_x(k) = \sum P_i e^{j2\pi f_i k}$$

where, $P_i = A_i^2$, $\sigma_w^2 = E\{|w(n)|^2\}$

- (p-1)^th order autocorrelation matrix

$$\underline{R}_p = \begin{bmatrix} R_x(0) & \cdots & R_x^*(p-1) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ R_x(p-1) & \cdots & R_x(0) \end{bmatrix} \quad (p\times p)$$

Hermitian symmetry

EIGENDECOMPOSITION OF R_p (2) (for harmonic processes)

- Easy to show that: (white noise case)

$$\underline{R}_p = \underbrace{\sum_{i=1}^{M} P_i \underline{s}_i \underline{s}_i^H}_{\underline{S}_p \ (p\times p)} + \underbrace{\sigma_w^2 I}_{\underline{W}_p \ (p\times p)}$$

Signal matrix   Noise matrix

$$\underline{s}_i = [\,1 \;\; e^{j2\pi f_i} \;\; e^{j2\pi f_i 2} \;\; \ldots \;\; e^{j2\pi f_i (p-1)}\,]$$

$\underline{S}_p$: Rank(**M**) (M≤p) [since $\underline{s}_i \underline{s}_i^H$ is of rank(1)], i=1,...M

- $\underline{W}_p$: Rank(p) [$\sigma_w^2 I$ has p nonzero diagonal elements]

$\underline{R}_p$: Rank(p) [since W_p is of rank(p)]

EIGENDECOMPOSITION OF R_p (3) (for harmonic processes)

$$\mathbf{SVD}(\underline{S}_p) : \underline{S}_p = \sum_{k=1}^{p} \lambda_i Q_i Q_i^H$$ (Assuming p>M, then, $\lambda_i=0$, i=M+1,...,p)

- $$\mathbf{SVD}(\underline{W}_p) : \underline{W}_p = \sum_{k=1}^{p} \sigma_w^2 Q_i Q_i^H$$ ({$\mathbf{Q}_i$} is orthonormal set)

  i.e., $\sum_{i=1}^{p} Q_i Q_i^H = I$

- $$\underline{R}_p = \underline{S}_p + \underline{W}_p = \sum_{i=1}^{p} (\lambda_i + \sigma_w^2) Q_i Q_i^H$$

  or

  $$[\underline{R}_p = \underbrace{\sum_{i=1}^{M} (\lambda_i + \sigma_w^2) Q_i Q_i^H}_{\text{Signal subspace}} + \underbrace{\sum_{i=M+1}^{p} \sigma_w^2 Q_i Q_i^H}_{\text{Noise subspace}}]$$

SINUSOIDAL PARAMETER ESTIMATION METHODS (1)

(Also known as "Harmonic Decomposition Methods")

- **ASSUMPTION**

The observed process consists of M (un)damped complex exponentials or sinusoids in noise

→ MODELING

→ PRONY METHOD (Original, Extended)

→ SINGULAR VALUE DECOMPOSITION (SVD)

→ EIGENANALYSIS METHODS

- Signal subspace methods
- Noise subspace methods (Pisarenko, Eigenvector, Music,...)

SINUSOIDAL PARAMETER ESTIMATION METHODS (2)

■ MODELING OF SINUSOIDS (1)

- Trigonometric identity

$$\sin(w_0 n) = 2\cos w_0 . \sin[w_0(n-1)] - \sin[w_0(n-2)], \quad |w_0| \le \pi,\ w_o = 2\pi f_0,\ |f_0| \le \frac{1}{2}$$

Let: $x(n) = \sin(w_0 n)$, then, $x(n) = 2\cos(w_0)\, x(n-1) - x(n-2)$

We can model $x(n)$ as an AR(2) model ($a_{21} = -2\cos w_0$, $a_{22} = 1$) with initial conditions, i.e.,

$$x(n) + a_{21}x(n-1) + a_{22}x(n-2) = \underbrace{d(n)}_{\text{i.c.}} \quad \text{or} \quad X(z)[1 + a_{21}z^{-1} + a_{22}z^{-2}] = \underbrace{D(z)}_{\text{i.c.}}$$

$$A(z) = 0 \rightarrow z_{1,2} = \cos w_0 \pm j\sin w_0 \quad \begin{cases} \nearrow e^{j2\pi f_0} \\ \searrow e^{-j2\pi f_0} \end{cases}$$

Sinusoid: $\sin(w_0 n) \leftrightarrow AR(2)$ ***model with poles at*** $e^{\pm j2\pi f_0}$

SINUSOIDAL PARAMETER ESTIMATION METHODS (3)

■ MODELING OF SINUSOIDS (2)

A) $$x(n) = \sum_{k=1}^{M} \sin \omega_k n \leftrightarrow AR(2M)$$

$$A(z) = \sum_{k=1}^{2M} a_{2M,k} z^{-k} = z^{-2M} [\sum_{k=1}^{2M} a_{2M,k} z^{2M-k}] = z^{-2M} . \prod_{k=1}^{M} (z - \rho_k)(z - \rho_k^*)$$

where, $a_{2M,0} = 1$, $\rho_k = e^{j2\pi f_k}$, $\rho_k^* = e^{-j2\pi f_k}$

B) $$x(n) = \sum_{k=1}^{M} e^{j\omega_k n} \leftrightarrow AR(M)$$

$$A(z) = \sum_{k=1}^{M} a_{M,k} z^{-k} = \ldots = z^{-M} . \prod_{k=1}^{M} (z - \rho_k)$$

where $a_{M,0} = 1$, $\rho_k = e^{j2\pi f_k}$

SINUSOIDAL PARAMETER ESTIMATION METHODS (4)

PRONY METHOD (Baron de Prony, 1975)

- Given N data samples, $x(n)$, $n=1,2,\ldots,N$

  Obtain an estimate: $$\hat{x}(n) = \sum_{k=1}^{M} A_k e^{[(\alpha_k + j2\pi f_k)(n-1) + j\theta_k]} = \sum_{k=1}^{M} h_k z_k^{n-1}$$

  where,

A_k: amplitute	$h_k = A_k e^{j\theta_k}$
α_k: damping factor	$z_k = e^{(\alpha_k + j2\pi f_k)}$
θ_k: initial phase	f_k: frequency

- **Problem:**

  Minimize: $$\sum_{n=1}^{N} | x(n) - \hat{x}(n) |^2 \quad w.r.t. \quad A_k,\ \alpha_k,\ f_k,\ \theta_k,\ k = 1,\ldots,M$$

  - Highly nonlinear system of equations: (high complexity, convergence to local solutions)

SINUSOIDAL PARAMETER ESTIMATION METHODS (5)

■ ORIGINAL PRONY METHOD (1)

- Given N data samples, x(n), n=1,2,...,N

  Assumption: $x(n)$ is exactly equal to the superposition of p complex exponentials

  i.e.: $$x(n) \equiv \sum_{k=1}^{p} h_k z^{n-1}$$

  where, $h_k = A_k e^{j\theta_k}$, $z_k = e^{\alpha_k + j2\pi f_k}$, $(N \geq 2p)$

- Approach: Decouple estimation of $\{h_k\}$ and $\{z_k\}$

  1) First obtain $\{z_k\}$ → $\{\alpha_k\}$, $\{\theta_k\}$

  2) Then obtain → $\{A_k\}$, $\{\theta_k\}$

SINUSOIDAL PARAMETER ESTIMATION METHODS (6)

ORIGINAL PRONY METHOD (2)

1) Consider the polynomial A(z) with zeros $z_k = e^{(\alpha_k + j2\pi f_k)}$, $k = 1,\dots,p$

Then: $A(z) = \sum_{i=0}^{p} a_{p,i} z^{-i} = z^{-p} \sum_{i=0}^{p} a_{p,i} z^{p-i}$, $\quad A(z_k) = 0$, $k = 1,\dots,p$, $a_{p,0} = 1$

- $x(n) = \sum_{k=1}^{p} h_k z_k^{n-1} \Rightarrow x(n-i) = \sum_{k=1}^{p} h_k z_k^{(n-i-1)}$

$$\Rightarrow a_{p,i} x(n-i) = a_{p,i} \sum_{k=1}^{p} h_k z_k^{(n-i-1)}$$

$$\Rightarrow \sum_{i=0}^{p} a_{p,i} x(n-i) = \sum_{i=0}^{p} a_{p,i} . \sum_{k=1}^{p} h_k z_k^{(n-i-1)} = \sum_{k=1}^{p} h_k . \sum_{i=0}^{p} a_{p,i} z_k^{(n-i-1)}$$

$$\Rightarrow \sum_{i=0}^{p} a_{p,i} x(n-i) = \sum_{k=1}^{p} h_k z_k^{(n-1-p)} . \underbrace{\sum_{i=0}^{p} a_{p,i} z_k^{p-i}}_{=0} \equiv 0$$

ORIGINAL PRONY METHOD (3)

Thus, $\sum_{i=0}^{p} a_{p,i}\, x(n-i) = 0 \ , \quad n = p+1,\ldots,N \ ; \quad a_{p,0} = 1$

- For $n=p+1,\ldots,2p$ obtain the linear system of equations

$$\begin{bmatrix} x(p) & x(p-1) & . & . & . & x(1) \\ x(p+1) & x(p) & . & . & . & x(2) \\ . & . & . & . & . & . \\ x(2p+1) & . & & . & . & . & x(p) \end{bmatrix} \begin{bmatrix} a_{p,1} \\ a_{p,2} \\ . \\ a_{p,p} \end{bmatrix} = - \begin{bmatrix} x(p+1) \\ x(p+2) \\ . \\ x(2p) \end{bmatrix}$$

- Solve and obtain the p roots $\{z_k\}$, $k = 1,\ldots,p$ of the polymonial

$$[\sum_{i=0}^{p} a_{p,i}\, z^{p-i} = 0] \ , \quad a_{p,0} = 1$$

- Since $z_k = e^{(\alpha_k + j2\pi f_k)} \ \Rightarrow \ \alpha_k = \ln|z_k| \ , \quad f_k = \frac{1}{2\pi} \tan^{-1}\{\frac{\Im m(z_k)}{\Re e(z_k)}\}$

SINUSOIDAL PARAMETER ESTIMATION METHODS (8)

■ ORIGINAL PRONY METHOD (4)

2) By repeating $x(n) = \sum_{k=1}^{p} h_k z^{n-1}$, $n = 1,\dots,p$ we obtain the linear system of equations

$$\begin{bmatrix} z_1^0 & z_2^0 & \cdot\ \cdot & z_p^0 \\ z_1^1 & z_2^1 & \cdot\ \cdot\ \cdot & z_p^1 \\ \cdot & & \cdot\ \cdot\ \cdot & \\ z_1^{p-1} & z_2^{p-1} & \cdot\ \cdot\ \cdot & z_p^{p-1} \end{bmatrix} \cdot \begin{bmatrix} h_1 \\ h_2 \\ \cdot \\ h_p \end{bmatrix} = \begin{bmatrix} x(1) \\ x(2) \\ \cdot \\ x(p) \end{bmatrix}$$

Solve and obtain $\{h_k\}$, $k = 1,\dots,p$

- Since, $h_k = A_k e^{j\theta_k} \Rightarrow A_k = |h_k|$, $\theta_k = \tan^{-1}\{\frac{\Im m(h_k)}{\Re e(h_k)}\}$, $k = 1,\dots,p$

NOTE: We need $N=2p$ data samples to resolve p complex exponentials

SINUSOIDAL PARAMETER ESTIMATION METHODS (9)

EXTENDED PRONY METHOD (1) (or least squares Prony method)

Given N data samples , $x(n)$, n=1,...,N (N>2p)
- Approximate x(n) with a superposition of exponentials, i.e.,

$$\hat{x}(n) = \sum_{k=1}^{p} h_k z_k^{n-1} \ , \quad h_k = A_k e^{j\theta_k} ; \quad z_k = e^{(\alpha_k + j2\pi f_k)}$$

Approximation error $\mathscr{E}(n) = x(n) - \hat{x}(n)$

Follow similar procedure to that in the original Prony method, i.e., obtain first $\{z_k\}$, then $\{h_k\}$

ALGORITHM:

1) Consider $A(z) = z^{-p} \sum_{i=0}^{p} a_{p,i} z^{p-i}$ with zeros $\{z_k\}$, k=1,...,p

Then, $\sum_{i=0}^{p} a_{p,i} \hat{x}(n-i) = 0 \Rightarrow \sum_{i=0}^{p} a_{p,i} x(n-i) = e(n)$ ~ AR(p)

[where, $e(n) = -\sum_{i=0}^{p} a_{p,i} \mathscr{E}(n-i)$]

SINUSOIDAL PARAMETER ESTIMATION METHODS (10)

■ EXTENDED PRONY METHOD (2)

- Obtain $\{a_{p,k}\}$, $k = 1,\dots,p$ by minimizing the quantity

$$\frac{1}{N-p}\sum_{n=p+1}^{N} |e(n)|^2 = \frac{1}{N-p}\sum_{n=p+1}^{N} \left|\sum_{i=0}^{p} a_{p,i}\,x(n-i)\right|^2$$

(as in the covariance method)

- Obtain the p roots $\{z_k\}$, $k = 1,\dots,p$ of the polynomial $\sum_{i=0}^{p} a_{p,i}\,z^{p-i} = 0,\ a_{p,0} = 1$

- $\alpha_k = \ln|z_k|$, $f_k = \dfrac{1}{2\pi}\tan^{-1}\{\dfrac{\Im m(z_k)}{\Re e(z_k)}\}$, $k = 1,\dots,p$

2) Given $\{z_k\}$, $k = 1,\dots,N$, then,

$$\hat{x}(n) = \sum_{k=1}^{p} h_k z_k^{n-1} = x(n) - \mathscr{E}(n) \text{ is linear w.r.t. } \{h_k\}$$

EXTENDED PRONY METHODS (3)

- Minimize $\sum_{n=1}^{N} |\mathcal{E}(n)|^2$ and obtain the least squares solution:

$$\underline{h} = (\underline{Z}^H \underline{Z})^{-1} \underline{Z}^H \underline{X}$$

where,

$$\underline{Z} = \begin{bmatrix} z_1^0 & z_2^0 & \cdot & z_p^0 \\ \cdot & & \cdot\ \cdot\ \cdot & \\ \cdot & & \cdot\ \cdot\ \cdot & \\ z_1^{N-1} & z_2^{N-1} & \cdot & z_p^{N-1} \end{bmatrix}, \quad \underline{h} = \begin{bmatrix} h_1 \\ \cdot \\ \cdot \\ h_p \end{bmatrix}, \quad \underline{X} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ x_p \end{bmatrix}$$

- Obtain

$$A_k = |h_k|$$

$$\theta_k = \tan^{-1}\left\{\frac{\Im m(h_k)}{\Re e(h_k)}\right\}, \quad k = 1,\ldots,p$$

SINUSOIDAL PARAMETER ESTIMATION METHODS (12)

■ EXTENDED PRONY METHOD (4)

- Poor performance in the presence of additive noise

  (e(n) is not a white sequence)

- p can be chosen by using AR model order selection criteria (or SVD)

- Extended method superior than original with noisy data

PRONY METHOD SPECTRUM

ESD: $\left| DTFT\left(\sum_{k=1}^{p} h_k z_k^{n-1}\right)\right|^2,\ 0 \le f \le 1$ i.e., $\left|\sum_{n}\left(\sum_{k=1}^{p} h_k z_k^{n-1}\right) e^{-j2\pi f_n}\right|^2$

SINUSOIDAL PARAMETER ESTIMATION METHODS (13)

Ex.

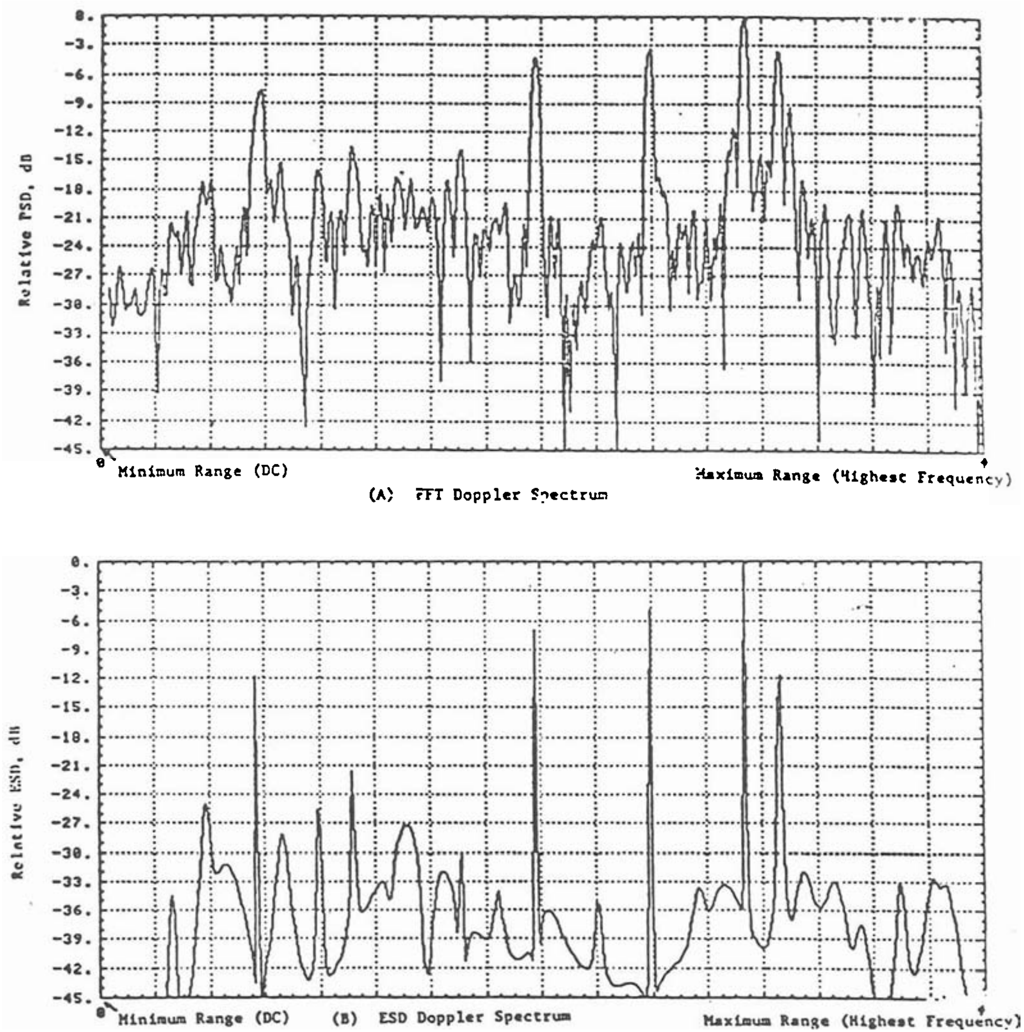


Figure 3. Doppler Radar Example Comparing FFT and ESD Spectra

SINUSOIDAL PARAMETER ESTIMATION METHODS (15)

■ EIGENANALYSIS SIGNAL SUBSPACE METHODS (1)

Given $x(n)$, $n=1,\ldots,N$

1) Obtain the estimates of autocorrelation lags ($\hat{R}_x(-m) = \hat{R}_x^*(m)$)

$$\hat{R}_x(m) = \frac{1}{N}\sum_{n=1}^{N-m} x^*(n)\,x(n+m)\,, \quad m = 0,1,\ldots,p-1$$

2) Form the autocorrelation matrix $\underline{R}_p$

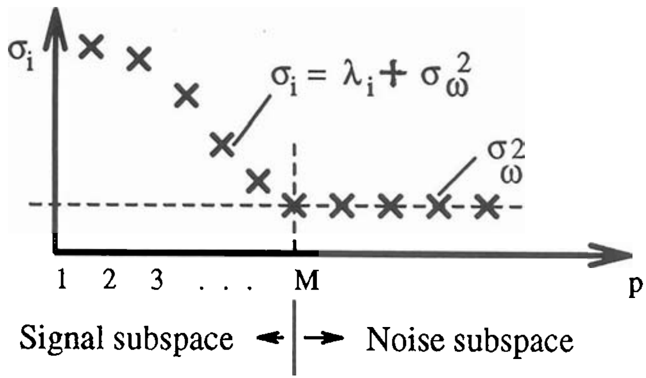
$$\underline{\hphantom{R}}_p = \begin{bmatrix} \hat{R}_x(0) & \cdots & \hat{R}_x^*(p-1) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \hat{R}_x(p-1) & \cdots & \hat{R}_x(0) \end{bmatrix} \quad (p\times p)$$

3) Obtain the eigendecomposition of $\underline{R}_p$: $\quad \hat{\underline{R}}_p = \sum_{i=1}^{p} \sigma_i Q_i Q_i^H$

σ_i: real eigenvalues

■ EIGENANALYSIS SIGNAL SUBSPACE METHODS (2)

4) Order the eigenvalues: $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p$



5) Reconstruct $\hat{\underline{R}}_p$ based only on "principal vectors", i.e., for i=1,...M

$$\hat{\underline{R}}_{p,pc} = \sum_{i=1}^{M} \sigma_i Q_i Q_i^H \qquad \text{and,} \qquad \hat{\underline{R}}_{p,pc}^{-1} = \sum_{i=1}^{M} \frac{1}{\sigma_i} Q_i Q_i^H$$

6) Employ $\hat{\underline{R}}_{p,pc}$ or $\hat{\underline{R}}_{p,pc}^{-1}$ in the spectral estimation methods

SINUSOIDAL PARAMETER ESTIMATION METHODS (17)

EIGENANALYSIS SIGNAL SUBSPACE METHODS (3)

- **MVSE - PC**

$$\hat{P}_{MVSE-PC}(f) = \frac{1}{\underline{E}^H(f)\,\underline{\hat{R}}_{P-PC}^{-1}\,\underline{E}(f)}\ , \quad 0 \le f \le 1$$

- **BT - PC**

$$\hat{P}_{BT-PC}(f) \cong \underline{E}^H(f)\,\underline{\hat{R}}_{P-PC}\,\underline{E}(f)\ , \quad 0 \le f \le 1$$

where, $\underline{E}^T(f) = [\,1\ \ e^{-j2\pi f}\ \ldots\ e^{-j2\pi(p-1)f}\,]$

- **AR - METHODS**

$$\underline{\hat{a}}_{p,PC} = -\,\underline{\hat{R}}_{P-PC}^{-1} \cdot \underline{\hat{r}}_p\ , \quad 0 \le f \le 1$$

{YW, covariance,...)

■ EIGENANALYSIS SIGNAL SUBSPACE METHODS (4)

$$\underline{R}_p = \underbrace{\sum_{i=1}^{M} (\lambda_i + \sigma_\omega^2)\, Q_i Q_i^H}_{\underline{R}_{P-PC}}$$

1) SNR enhancement

2) Less bias in autocorrelation matrix

3) In AR methods with noisy data an increased AR order is necessary. However, a very high order might create "spurious peaks" (because of the high variance in the estimation of the noise eigenvectors). By using $\underline{R}_{P-PC}$ instead of $\underline{R}_p$ we can increase the order of the model without observing the spurious peaks.

EIGENANALYSIS NOISE SUBSPACE METHODS (1)

BASIC IDEA: Signal vectors are orthogonal to vectors in noise subspace

$$\underline{R}_p = \underbrace{\sum_{i=1}^{M} P_i \underline{s}_i \underline{s}_i^H + \sigma_\omega^2 I}_{\sum_{i=1}^{M} (\lambda_i + \sigma_\omega^2) Q_i Q_i^H + \sum_{i=M+1}^{p} \sigma_\omega^2 Q_i Q_i^H} \quad \text{where} \quad s_i^T = [1 \; e^{-j2\pi f_i} \; \ldots \; e^{-j2\pi f_i(p-1)}]$$

- $\{Q_i\}$, $i = 1,\ldots,p$ is an orthogonal set of eigenvectors

- $\{Q_i\}$, $i = 1,\ldots,M$ and $\{s_i\}$, $i = 1,\ldots,p$ span the same signal subspace

Thus, $\{s_i\}$, $i = 1,\ldots,M$ are orthogonal to $\{Q_i\}$, $i = M+1,\ldots,p$

i.e., $$\boxed{s_i^H \cdot \sum_{j=M+1}^{p} c_j Q_j = 0}, \quad 1 = 1,\ldots,M$$

■ EIGENANALYSIS NOISE SUBSPACE METHODS (2)

• PISARENKO METHOD (1) (1972)

(Based on a trigonometric theorem by Caratheodory)

- Given M complex exponentials in white noise

$$R_x(k) = \sum_{i=1}^{M} P_i e^{j2\pi f_i k} + \sigma_\omega^2 \delta(k), \quad k=0,\pm1,\ldots$$

1) Form the (M+1)×(M+1) autocorrelation matrix $\underline{\hat{R}}_{M+1}$ using biased autocorrelation estimates

2) Obtain the eigendecomposition of $\underline{\hat{R}}_{M+1} = \sum_{i=1}^{M+1} \sigma_i Q_i Q_i^H$ or,

$$\underline{\hat{R}}_{M+1} = \underbrace{\sum_{i=1}^{M} (\lambda_i + \sigma_\omega^2) Q_i Q_i^H}_{\sum_{i=1}^{M} P_i \underline{s}_i \underline{s}_i^H} + \underbrace{\sigma_\omega^2}_{\uparrow\ \sigma_{i,\min}} Q_{M+1} Q_{M+1}^H$$

■ PISARENKO METHOD (2)

3) $\boxed{s_i^H \cdot Q_{M+1} = 0}$, $i = 1,2,\ldots,M$ or $\sum_{n=0}^{M} q_{M+1}(n)\, e^{-j2\pi f_i n} = 0$, $i = 1,\ldots,M$

Compute the roots $\{z_k\}$, $k = 1,\ldots,M$ of $\sum_{n=0}^{M} q_{M+1}(n)\, z^{-n}$, $q_{M+1}(0) = 1$

4) Since, $z_k = e^{j2\pi f_k}$, $k = 1,\ldots,M$, then, $f_k = \frac{1}{2\pi} \tan^{-1}\left\{ \frac{\Im m(z_k)}{\Re e(z_k)} \right\}$

5) Obtain power of exponentials by solving the system

$$\begin{bmatrix} e^{j2\pi f_1} & \cdots & e^{j2\pi f_M} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ e^{j2\pi f_1 M} & \cdots & e^{j2\pi f_M M} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ P_M \end{bmatrix} = \begin{bmatrix} \hat{R}_x(1) \\ \hat{R}_x(2) \\ \cdot \\ \hat{R}_x(M) \end{bmatrix}$$

■ PISARENKO METHOD (3)

Comments:

1) If M is overestimated then the method will produce more than the existing frequencies.

   Modification:

   - Form the matrix $\underline{\hat{R}}_p$, $p > M$
   - Obtain and order the eigenvalues: $\sigma_1 > \sigma_2 > \ldots > \sigma_p$
   - Ideally $\sigma_i = \sigma_\omega^2$ *for* $i = M+1, \ldots p$
   - Obtain the eigenvector Q_{M+1} corresponding to $\hat{\sigma}_\omega^2 = \frac{1}{|M-p|} \sum_{i=M+1}^{p} \sigma_i$

   $$\underline{\hat{R}}_p \cdot Q_{M+1} = \hat{\sigma}_\omega^2 \, Q_{M+1}$$

   - Proceed as before

2) Extensions of the method have been proposed [Reddi 1979 Kumaresan 1982]

3) The method looses the phase information.

4) The method is sensitivity to low SNR

PISARENKO METHOD (4) (EXAMPLE)

We are given the autocorrelation lags $\{R(0)=2, R(1)=1, R(2)=0\}$ of a real process consisting of a sinusoid in AWN

Find: 1) A (sinusoid amplitude), 2) f (sinusoid frequency)

3) σ_ω^2 (noise variance)

Solution: $$\underline{R}_3 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

a) Obtain the eigenvalues σ_i, $i = 1,2,3$

$$|\underline{R}_3 - \sigma \underline{I}| = 0 \Rightarrow \begin{bmatrix} 2-\sigma & 1 & 0 \\ 1 & 2-\sigma & 1 \\ 0 & 1 & 2-\sigma \end{bmatrix} = 0 \Rightarrow (2-\sigma)(\sigma^2 - 4\sigma + 3) - (2-\sigma) = 0$$

$$\Rightarrow (2-\sigma)[\sigma^2 - 4\sigma + 2] = 0$$

where, $\sigma_1 = 2 + \sqrt{2}$, $\sigma_2 = 2$, $\sigma_3 = 2 - \sqrt{2}$

i.e., $\sigma_\omega^2 = \sigma_{MIN} = 2 - \sqrt{2}$

■ PISARENKO METHOD (5) (EXAMPLE)

b) Compute the eigenvectors corresponding to $\sigma_{MIN} = 2 - \sqrt{2}$

$$\underline{R}_3 \cdot Q_3 = (2-\sqrt{2})\, Q_3 \Rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} 1 \\ q_1 \\ q_2 \end{matrix} = (2-\sqrt{2}) \begin{matrix} 1 \\ q_1 \\ q_2 \end{matrix} \Rightarrow \begin{matrix} q_1 = -\sqrt{2} \\ q_2 = 1 \end{matrix}$$

c) Find the roots of $z^2 + q_1 z + q_2 = 0 \Rightarrow z_{1,2} = \dfrac{\sqrt{2} \pm j\sqrt{2}}{2}$

Note: $|z_{1,2}| = 1, \quad f = \dfrac{1}{2\pi} \tan^{-1}(1) = \dfrac{1}{2\pi} \cdot \dfrac{\pi}{4} \Rightarrow \boxed{f = \dfrac{1}{8}}$

d) From step 5 with $P_1 = P_2, \; f_1 = f, \; f_2 = -f, \; A = \sqrt{2\sqrt{2}}$

EIGENANALYSIS NOISE SUBSPACE METHODS

• FREQUENCY ESTIMATORS (1)

Assuming:
- $s_i^T = [1 \; e^{j2\pi f_i} \; \dots \; e^{j2\pi f_i (p-1)}]$
- $\{Q_k\}$, $k = M+1,\dots,p$ noise subspace eigenvectors

Then, $s_i^H Q_k = 0$, $\quad i = 1,\dots M$; $\quad k = M+1,\dots p$

- Define the vector $\underline{E}^T(f) = [1 \; e^{j2\pi f} \; \dots \; e^{j2\pi f(p-1)}]$

Then, $|\underline{E}^H(f)\, Q_k|^2 = \underline{E}^H(f)\,[Q_k Q_k^H]\,\underline{E}(f) = 0$, $f = f_1, \dots, f_M$

- The function: $\dfrac{1}{\sum_{k=M+1}^{p} c_k \left|\underline{E}^H(f)\, Q_k\right|^2}$ exhibits peaks at $f = f_1, f_2, \dots, f_M$

and thus can be used as frequency estimator

SINUSOIDAL PARAMETER ESTIMATION METHODS

■ EIGENANALYSIS NOISE SUBSPACE METHODS

FREQUENCY ESTIMATORS (2)

- **MUSIC METHOD** (Schmidt, 1981, 1986)
  (MULTIPLE SIGNAL CLASSIFICATION)

$$\hat{P}_{MUSIC}(f) = \frac{1}{\sum_{k=M+1}^{p} |\underline{E}^H(f)\, Q_k|^2}, \quad 0 \le f \le 1 \quad (i.e., \;\; c_k = 1)$$

- **EIGENVECTOR METHOD** (Johnson, 1982)

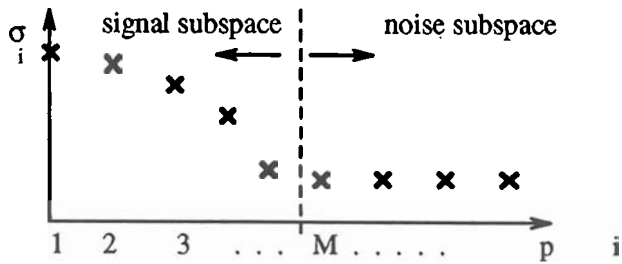
$$\hat{P}_{EV} = \frac{1}{\sum_{k=M+1}^{p} \frac{1}{\sigma_i} |\underline{E}^H(f)\, Q_k|^2}, \quad 0 \le f \le 1$$

- High resolution methods
- EV exhibits less spurious peaks than MUSIC
- Not PSE estimators but only frequency estimators

SINUSOIDAL PARAMETER ESTIMATION METHODS

EIGENANALYSIS METHODS

- **MODEL ORDER SELECTION** (# of complex exponentials)
  1) Form the autocorrelation matrix $\underline{R}_p : p \times p$
  2) Obtain the eigenvalues $\sigma_1 > \sigma_2 > \ldots > \sigma_p$
  3) Choose M as follows:



In practice this approach might not work well

4) Form: $$AIC(i) = (p-1).\ln\left[\frac{\frac{1}{p-i}\sum_{k=i+1}^{p}\sigma_k}{\prod_{k=i+1}^{p}\sigma_k^{-(p-i)}}\right] + i\,(2p-i)$$

Pick $i=M$ so that $AIC_{min}(i)=AIC(M)$