Hw #2 brief solutions

(1)

$$\underline{C_y}(\tau) = E\left\{\begin{bmatrix} x(0) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \end{bmatrix} \begin{bmatrix} x(0) & x(2) & x(3) & x(4) & x(5) \end{bmatrix}\right\} =$$

zero-mean process

$C_y(k) = C_y(-k)$ real process

$$= \begin{bmatrix} E\{x^2(0)\} & E\{x(0)x(2)\} & \cdots \\ E\{x(2)x(0)\} & E\{x^2(2)\} & \cdots \\ & & \end{bmatrix} = \begin{bmatrix} C_y(0) & C_y(2) & \cdots & C_y(5) \\ C_y(2) & C_y(0) & & C_y(3) \\ C_y(3) & C_y(1) & & \\ & & & C_y(0) \end{bmatrix}$$

(a) — Not Toeplitz

(b) — Symmetric

(c) — Yes since $C_y(1)$ is already present in the first matrix.

(2) $x(n) = A\cos w_0 n + w(n)$

A: $G \sim (0, \sigma_A^2)$; $w(n)$: white (σ_w^2); independent

a) $R_x(n,k) = E\{x(n)x(n+k)\} = E\{A^2\cos(w_0 n)\cos(w_0(n+k))\} +$
$E\{w(n)\,\cancel{A\cos w_0(n+k)}\}^{0} + E\{\cancel{A\cos w_0 n}\cdot w(n+k)\}^{0}$
$+ E\{w(n)\,w(n+k)\} =$

$$= \begin{cases} E\{A^2\}\{\cos^2(w_0 n)\} + E\{w^2(n)\} & k=0 \\ E\{A^2\}\cos w_0 n \cos(w_0(n+k)) + \cancel{E\{w(n)w(n+k)\}}^{0} & k\neq 0 \end{cases}$$

$$= \begin{cases} \sigma_A^2\cos^2 w_0 n + \sigma_w^2 & k=0 \\ \sigma_A^2\cos(w_0 n)\cos(w_0(n+k)) & k\neq 0 \end{cases}$$

b) This process is not stationary so the PSD is not defined

c) $R_x(n,k) = E\{A^2\cos(\omega_0 n+\varphi)\cos(\omega_0(n+k)+\varphi)\} +$

$+ E\{w(n)\,w(n+k)\} = ---$

$--- = \frac{\sigma_A^2}{2}\cos(\omega_0 k) + \sigma_w^2\,\delta(k) = R(k)$

This process is stationary $\longrightarrow S_x(\omega) = F\{R(k)\}$

$\Rightarrow S_x(\omega) = \frac{\sigma_A^2}{2}\delta(\omega+\omega_0) + \frac{\sigma_A^2}{2}\delta(\omega-\omega_0) + \sigma_w^2$

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$T = 1ms \longrightarrow f_s = \frac{1}{10^{-3}} = 1000\,Hz$

a) $f_{MAX} = \frac{f_s}{2} = 500\,Hz$

b) $f \leq \frac{f_s}{2} = 500\,Hz$

c) $S_x(\Omega) = \int_{-\infty}^{\infty} R_{x_c}(\tau)\,e^{-j\omega\tau}\,d\tau =$

$= ------ = \frac{2\sigma_x^2\,\tau_0}{1+\Omega^2\tau_0^2}$

We want $10\log_{10}\frac{S_x(f=0)}{S_x(f=500)} > 40 \rightarrow$

$-----$

$\Rightarrow \tau_0 > 31.8\ ms$

BONUS question solution

$$P(\omega) = \lim_{N\to\infty} \frac{1}{2N+1} E\left\{\left|\sum_{n=-N}^{N} x(n) e^{-j\omega n}\right|^2\right\} =$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} E\left\{ \left(\sum_{n=-N}^{N} x(n) e^{-j\omega n}\right)\left(\sum_{l=-N}^{N} x(l)e^{-j\omega l}\right)^* \right\} =$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} E\left\{ \sum\sum_{n,l=-N}^{N} x(n)x^*(l) e^{-j\omega(n-l)}\right\} =$$

$$= \lim_{N\to\infty} \frac{1}{2N+1}\sum\sum_{n,l=-N}^{N} \underbrace{E\{x(n)x^*(l)\}}_{R(n-l)} e^{-j\omega(n-l)} =$$

by placing $n-l=\tau$

$$= \lim_{N\to\infty} \frac{1}{2N+1}\sum_{\tau=-2N}^{2N} (2N-|\tau|-1)\, R(\tau)e^{-j\omega\tau} =$$

$$= \lim_{N\to\infty} \frac{1}{2N+1}(2N-1)\sum_{\tau=-2N}^{2N} R(\tau)e^{-j\omega\tau} - \lim_{N\to\infty}\frac{1}{2N+1}\sum_{\tau=-2N}^{2N}|\tau|R(\tau)e^{-j\omega\tau}$$

tends to $\to 1$

$$= \sum_{\tau=-\infty}^{\infty} R(\tau)e^{-j\omega\tau} - \cancel{\lim_{N\to\infty}\frac{1}{2N+1}\sum_{\tau=-2N}^{2N}|\tau|R(\tau)e^{-j\omega\tau}} \to 0$$

$$= \sum_{\tau=-\infty}^{\infty} R(\tau)e^{-j\omega\tau} = P(\omega)$$

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provided that $|\tau|R(\tau)$ decays fast enough as $\tau\to\infty$

Condition for equivalence of PS definitions