ECE 1511, Fall 2019

Problem set 2, Brief solutions

PROBLEM 1

$$S_x(\omega_1,\omega_2) = \sum_{n_1}\sum_{n_2} R_x(n_1,n_2)e^{-j\omega_1 n_1}e^{-j\omega_2 n_2}, \quad \text{Let } R_x(n_1,n_2)=R_x(n_2-n_1) = R_x(\tau)$$

$$= \sum_{n_1}\sum_{n_2} R_x(n_2-n_1)e^{-j\omega_1 n_1}e^{-j\omega_2 n_2} = \sum_{n_1}\sum_{\tau+n_1} R_x(\tau)e^{-j\omega_1 n_1}e^{-j\omega_2 n_1}e^{-j\omega_2 \tau}$$

$$= \sum_{n}\underbrace{\sum_{\tau+n_1} R_x(\tau)e^{-j\omega_1 \tau}}_{S_x(\omega_1)} e^{-j(\omega_1+\omega_2)n_1} =$$

$$= S_x(\omega_1)\cdot\sum_{n} e^{-j(\omega_1+\omega_2)n_1} = \begin{cases} 2\pi S_x(\omega_1), & \omega_1+\omega_2=0 \\ 0, & \text{otherwise} \end{cases}$$

$$= S_x(\omega_1)\cdot \underline{\underline{2\pi\,\delta(\omega_1+\omega_2)}}$$

PROBLEM 2

Computer exercise. No solution is provided

PROBLEM 3

The phases ϕ_a and ϕ_b are independent and uniformly distributed between $-\pi$ and π. Two cases are considered for the phase ϕ_c.

Case 1. ϕ_c is independent of the other two phases and is also uniformly distributed between $-\pi$ and π. Note that $x[n]$ is of the form given in Table 4.6. For example, the component $s_a[n]$ has the form

$$s_a[n] = Ae^{\jmath\omega_a n}$$

where the complex amplitude is given by

$$A = 1 \cdot e^{\jmath\phi_a}$$

(i.e., the magnitude of A is fixed but the phase is random). It can easily be shown that the complex amplitudes satisfy the necessary orthogonality conditions ([redacted]). Therefore, the power density spectrum consists of three lines as shown in Fig. EX4.9a.

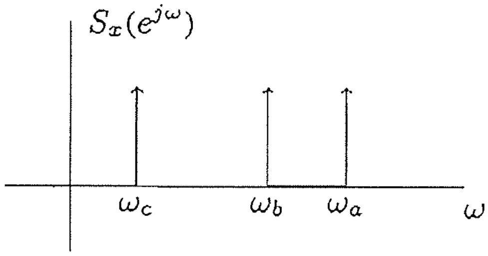

Figure EX4.9a

Now consider the cumulant

$$C_x^{(3)}[l_1, l_2] = \mathcal{E}\{[x[n]^* x[n+l_1] x[n+l_2]\}$$

This involves cross-products of terms such as

$$\mathcal{E}\{s_a^*[n] s_b[n+l_1] s_c[n+l_2]\}$$

Since all of the components are independent and have zero mean, all such cross-terms are zero. The terms involving a single component such as

$$\mathcal{E}\{s_a^*[n] s_a[n+l_1] s_a[n+l_2]\}$$

are also zero. Specifically,

$$\mathcal{E}\left\{e^{-\jmath(\omega_a n+\phi_a)} e^{\jmath(\omega_a(n+l_1)+\phi_a)} e^{\jmath(\omega_a(n+l_2)+\phi_a)}\right\} = e^{\jmath\omega_a(n+l_1+l_2)}\mathcal{E}\left\{e^{\jmath\phi_a}\right\} = 0$$

Thus, since $C_x^{(3)}$ is identically zero, the bispectrum is identically zero.

Case 2. ϕ_a is linearly related to the other two phases; in particular,

$$\phi_a = \phi_b + \phi_c$$

In this case the power density spectrum is exactly the same as before; the coupling between the phase of the components does not destroy the orthogonality between the complex amplitudes.

The cumulant involves expectations of products of the components as before, most of which are zero. However, consider the cross-term

$$\mathcal{E}\{s_a^*[n]s_b[n+l_1]s_c[n+l_2]\} = \mathcal{E}\left\{e^{-\jmath(\omega_a n+\phi_b+\phi_c)}e^{\jmath(\omega_b(n+l_1)+\phi_b)}e^{\jmath(\omega_c(n+l_2)+\phi_c)}\right\}$$

$$= e^{\jmath(\omega_b l_1+\omega_c l_2)}\mathcal{E}\left\{e^{\jmath(\phi_b+\phi_c-\phi_b-\phi_c}\right\} = e^{\jmath(\omega_b l_1+\omega_c l_2)}$$

where we used the fact that $\omega_a = \omega_b + \omega_c$. This term is nonzero and produces an impulse in the bispectrum at $\omega^{(1)} = \omega_b$ and $\omega^{(2)} = \omega_c$, indicating the phase coupling between these frequency components as shown in Fig. EX4.9b.

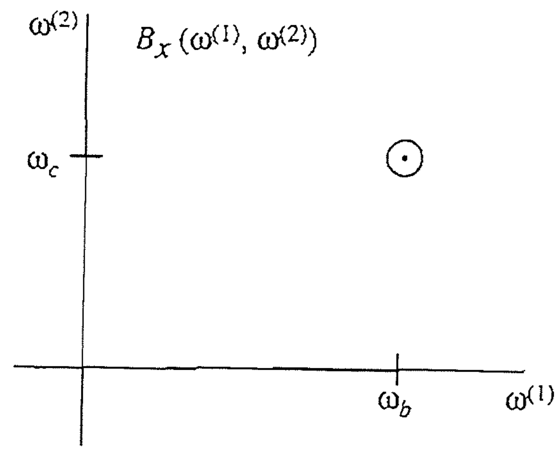

Figure EX4.9b

Further uses of cumulants and higher-order spectra become apparent when transformations of random processes are considered. Linear transformations of random processes produce prescribed effects on all the moments of a random process. If the processes are non-Gaussian, then cumulant analysis can be useful. Nonlinear transformations of even a Gaussian random process produce random processes with higher-order moments different from those of a Gaussian process and require analysis using higher-order moments or cumulants if the effect of the nonlinearity is to be fully accounted for. Thus cumulants and higher-order spectra will probably play an important role in future applications of statistical signal processing.