0.1 JOINTLY GAUSSIAN RANDOM VARIABLES

Definition: Two real random variables X and Y are said to be jointly Gaussian if their joint density function can be written in the form:

$$f_{XY}(x,y) = Ae^{-\frac{1}{2}[x-x_0,y-y_0]\wedge^{-1}[x-x_0,y-y_0]^T} = Ae^{-\frac{1}{2}\mathbf{x}^T\wedge^{-1}\mathbf{x}} \tag{1}$$

Where, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and $\wedge^{-1}$ is some positive semidefinite matrix (recall that M is said to be positive semidefinite if $\mathbf{x}^T M\mathbf{x} \geq 0$ for all $\mathbf{x} \neq 0$) and A is some constant so that the total volume under $f_{XY}(x,y)$ is equal to one.

Statement: If

$$\wedge^{-1} = \frac{1}{(1-p^2)}\begin{bmatrix} \frac{1}{\sigma_x^2} & \frac{-p}{\sigma_x\sigma_y} \\ \frac{-p}{\sigma_x\sigma_y} & \frac{1}{\sigma_y^2} \end{bmatrix} \qquad (|p| \leq 1) \tag{2}$$

then it can be shown that the marginal densities of X and Y are

$$\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi}\sigma_x}e^{-(x-x_0)^2/2\sigma_x^2} \\ f_Y(y) &= \frac{1}{\sqrt{2\pi}\sigma_y}e^{-(y-y_0)^2/2\sigma_y^2} \end{aligned} \tag{3}$$

and that $A = \frac{1}{[2\pi\sigma_x\sigma_y\sqrt{1-p^2}]}$

Proof:

To prove the above statement, let us first show that $\wedge^{-1}$ must have the form indicated by (2). We first recall for any matrix M,

$$\mathbf{x}^T M\mathbf{x} = \mathbf{x}^T\{\frac{1}{2}(M+M^T)\}\mathbf{x}$$

so that we may without loss of generality assume that $\wedge^{-1}$ is a symmetric matrix. Given a symmetric matrix $M = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$ then this will be a positive semidefinite matrix if and only if $ab - c^2 \geq 0$ and $a \geq 0$, $b \geq 0$. Excluding the case where a and b are zero and writing $a = \frac{1}{\sigma_x^2}, b = \frac{1}{\sigma_y^2}, for, \sigma_x, \sigma_y \geq 0$, then since $c^2 \leq ab = \frac{1}{\sigma_x^2\sigma_y^2}$ it follows that $c = \frac{-p}{\sigma_x\sigma_y}$ for some

$|p| \leq 1$. Scaling the parameters σ_x, σ_y by replacing them with $\sigma_x/\sqrt{1-p^2}$ and $\sigma_y/\sqrt{1-p^2}$ allows us to write any positive semidefinite matrix with nonzero entries as:

$$\frac{1}{(1-p^2)}\begin{bmatrix} \frac{1}{\sigma_x^2} & \frac{-p}{\sigma_x\sigma_y} \\ \frac{-p}{\sigma_x\sigma_y} & \frac{1}{\sigma_y^2} \end{bmatrix}$$

for some $\sigma_x > 0$, $\sigma_y > 0$ and $|p| \leq 1$. Thus $\wedge^{-1}$ can be assumed to have this form.
(Note that the case with α or b equal to zero can be shown to impossible since it would not be possible to find $\wedge$ with the required property in that sense)

To find the marginal density function of X we recall that:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy \tag{4}$$

Then by calculating.

$$\begin{aligned} \tfrac{1}{2}\mathbf{x}^T \wedge^{-1} \mathbf{x} &= \tfrac{1}{2(1-p^2)}\left[\tfrac{(x-x_0)^2}{\sigma_x^2} + \tfrac{(y-y_0)^2}{\sigma_y^2} - \tfrac{2p(x-x_0)(y-y_0)}{\sigma_x\sigma_y}\right] \\ &= \ldots = \tfrac{(x-x_0)^2}{2\sigma_x^2} + \tfrac{1}{2(1-p^2)}\left[\tfrac{p(x-x_0)}{\sigma_x} - \tfrac{(y-y_0)}{\sigma_y}\right]^2 \end{aligned}$$

and replacing into (1) and (4) we obtain $f_X(x)$ as given by (3) Due to the symmetry conditions by reversing the roles of X and Y we obtain $f_Y(y)$ as given by (3).
From the relations (1) to (4) we make the observation that in order X and Y to be independent, i.e., $f_{XY}(x,y) = f_X(x)f_Y(y)$ it is sufficient and necessary that $p = 0$

A final observation is that, while we have shown that two jointly Gaussian r.v.s will have Gaussian marginal density functions the inverse is not necessarily true. That is, Gaussian marginal distributions do not imply that r.v.s are jointly Gaussian (unless they are independent or <u>every</u> linear combination of the two r.v.s is Gaussian) Also note that $\wedge$ is nothing else but the covariance matrix of Y, X
i.e.,

$$\wedge = \begin{bmatrix} \sigma_x^2 & p\sigma_x\sigma_y \\ p\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$$