Department of Electrical and Computer Engineering
University of Toronto

ECE302 - Probability and Applications

SELECTED PROBLEMS

1. Two numbers are selected at random from the interval [0,1]. Find the probabilty that the second number is greater than 0.75. Justify your answer.

2. A die is tossed and the number of dots facing up is noted. a) Find the probability of the elementary events under the assumption that the face with single dot is twice as likely to be facing up as the rest of the faces b) Find the probability that the outcome of a toss is even under the assumption in part (a) of this problem. Justify your answers.

3. A die is tossed twice and the number of dots facing up noted in the order of occurence. Assuming that all outcomes are equally likely to occur, find the probabilities of the following events: a) A_k: the sum of the two oucomes is k, for $k = 4, 8$. b) B: the outcomes of the two tosses are different.

4. The difference $A - B$ of two events A and B is defined as the event that occurs when A occurs but B does not occur. Show that $P(A - B) = P(A) - P(A \cap B)$.

5. Show that $P[A \cup B \cup C] = P[A] + P[A^c \cap B] + P[A^c \cap B^c \cap C]$.

6. A fair coin is tossed three times. Let A_i be the event "*i-th toss results in heads*". Find the probabilities of the events: A_2, $A_1 \cap A_3$, and $A_1 \cup A_2 \cup A_3$.

7. Eight rooks ("castles") are placed at random on a chess board. Find the probability that no two rooks can take one another (i.e., no two rooks can be in the same row or column). Recall that a chess board has 8 rows and 8 columns and therefore, there are 64 possible positions to place a rook.

8. A coin with probability$P\{heads\} = p = 1-q$ is tossed n times. Show that the probability that the number of "heads" is even equals $0.5[1 + (q - p)^n]$.[Hint: use the formula $\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^k y^{n-k} = (x+y)^n$].

9. If A and B are two events with <u>nonzero</u> probability, state which of the following statements may be true and which must be false:

   (i) $P(A|B) > P(A)$,

   (ii) $P(A|B) < P(A)$,

   (iii) $P(A|B) = P(A)$.

   Justify your answers.

10. Suppose there exists a (fictitious) test for cancer with the following properties. Let

A= event that the test states that tested person has cancer.

B= event that person has cancer.

It is known that $P(A|B) = P(A^c|B^c) = 0.95$ and $P(B) = 0.005$. Is the test a good test? Justify your answer.

11. Suppose that you have two coins, one biased, one fair, but you do not know which coin is which. Coin 1 is biased. It comes up heads with probability 3/4, while coin 2 will flip heads with probability 1/2. Suppose you pick a coin at random and flip it. Let C_i denote the event that coin i is picked. Let H and T denote the possible outcomes of the flip. Given that the outcome of the flip is a head, what is $P[C_1|H]$, the probability that you picked the biased coin? Given that the outcome is a tail, what is the probability $P[C_1|T]$ that you picked the biased coin?

12. Define what is meant (mathematically) by the *independence* of the three events A_1, A_2, A_3. What is meant when we state that these three events are *mutually exclusive* ? If A_1 and A_2 are two mutually exclusive events with respective probabilities p_1 and p_2, find $P(A_1 A_2)$.

13. Let A and B be two events defined in the sample space S. Then, show analytically the following

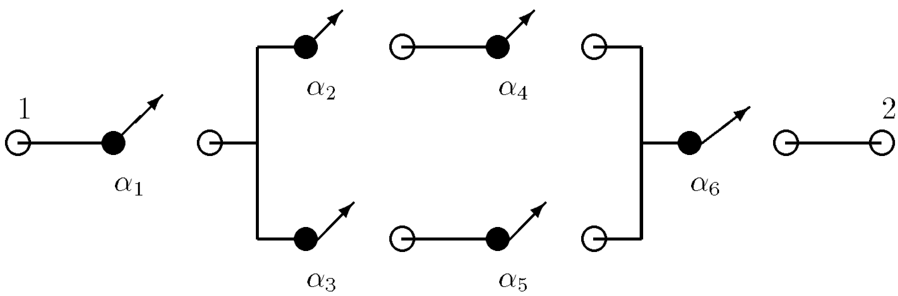
    i) If A and B are statistically independent, then so are A and B^c, B and A^c, and A^c and B^c.

    ii) If both $P(A)$ and $P(B)$ are nonzero, then the events A and B cannot be both mutually exclusive and statistically independent.

14. The circuit shown in the figure below represents a telephone communications link. Switches α_i, $i = 1, \ldots, 6$ are open or closed and operate independently. The probability that a switch is closed is the same for all switches and is equal to p. Let A_i represent the event that switch α_i is closed.

    i) In terms of the A_i's write the event that there exists at least one closed path from 1 to 2.

    ii) Calculate the probability of there being at least one closed path from 1 to 2.



15. A company has three machines B_1, B_2, and B_3 for making 1 $k\Omega$ resistors. it has been observed that 80% of resistors produced by B_1 are within 50Ω of the nominal value.

Machine B_2 produces 90% of resistors within 50Ω of the nominal value. The percentage for machine B_3 is 60%. Each hour, machine B_1 produces 3000 resistors, B_2 produces 4000 resistors, and B_3 produces 3000 resistors. All the resistors are mixed together at random in one bin and packed for shipment.

a) What is the probability that the company ships a resistor that is within 50Ω of the nominal value?

b) What is the probability that an acceptable resistor comes from machine B_3?

16. Consider an experiment that consists of testing two intergrated circuits that come from the same silicon wafer and observing in each case whether a circuit is accepted (a) or rejected (r). We re given that

$$P(ra) = 0.01, \ P(ar) = 0.01, \ P(aa) = 0.97.$$

Find the probability of the events A=*second circuit rejected* and B=*first circuit rejected*. Then find the probability that the second circuit is rejected given that the first circuit is a reject.

17. In an experiment with equiprobable outcomes, the event space is $S = \{1, 2, 3, 4\}$. Are the events $A_1 = \{1, 2, 3\}$, $A_2 = \{2, 3, 4\}$, and $A_3 = \textit{null event}$ independent?

18. Show that the null event is statistically independent from any other event in S.

19. A fair coin is tossed repeatedly, each toss being independent of the others. It is found that the results of the first ten tosses are respectively H,T,T,T,T,T,T,T,T,T. What is the probability that an eleventh toss will produce a "T"(tails)?

20. A process for producing integrated circuit chips is found to produce four defects on the average scattered over $1000\mu m \times 1000\mu m$ region of a wafer.

i) What is the appropriate random variable to model this problem? Explain your answer.

ii) Find the probability that a chip of size $500\mu m \times 500\mu m$ will contain more than two defects.

iii) Find the probability that 6 chips, each of size $100\mu m \times 250\mu m$ will all be free of defect.

21. In most published books, typographical errors are a fact of life. Suppose that, on the average, 320 typographical errors occur in a book of 500 pages. What is the probability that four pages selected at random will be error free?

22. How many raisins should be put into one kilogram lump of dough, in order that the probability is at least 0.99 that a 40 gram bun will contain at least one raisin? Hint: Determine first the probability of a raisin be in 40 grams of dough, then employ Binomial law.

23. One-tenth of 1% of a certain type of RAM chip are defective. A student needs 50 chips for a certain board. How many should she buy in order for there to be a 99% chance or greater of having at least 50 working chips?

24. The number of orders waiting to be processed is given by a Poisson random variable with parameter $\alpha = \lambda/n\mu$ where λ is the average number of orders that arrive in a day, μ is the number of orders that can be processed by an employee per day, and n is the number of employees. Let $\lambda = 3$ and $\mu = 1$. Find the number of employees required so the probability that more than 4 orders are waiting is less than 90%. What is the probability that there are no orders waiting ?

25. The lifetime X of a light bulb is a random variable with

$$P(X > t) = 1/(1+t), \qquad t \geq 0$$

Suppose three new light bulbs are installed at time $t = 0$. At time $t = 1$ all three light bulbs are still working. Find the probability that at least one light bulb is still working at time $t = 9$.

26. Prove the memoryless property of the geometric random variable:

$$P[X \geq k + j | X > j] = P[X \geq k], \qquad j, k > 0$$

27. Prove the memoryless property of the exponential random variable:

$$P[X \geq x + a | X \geq a] = P[X \geq x], \qquad x, a > 0$$

28. Let X be a Laplacian random variable with p.d.f. $f_X(x) = \frac{\alpha}{2} \cdot e^{-\alpha|x|}$, $-\infty < x < \infty$, $\alpha > 0$.

   (a) Calculate the third-order moment of X.

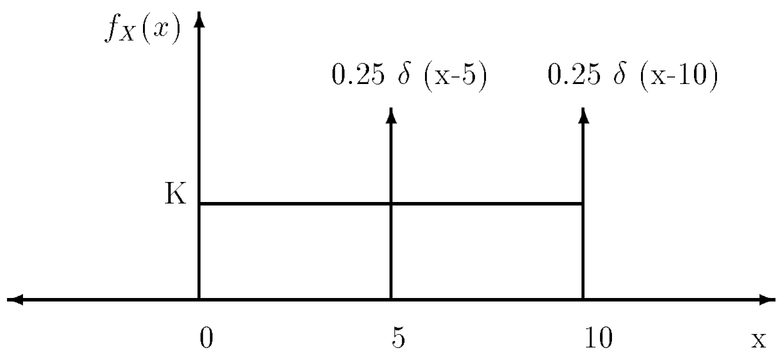
   (b) Let Y be a random variable defined as follows: Y is equal to X with probability p and equal to $-X$ with probability $q = 1 - p$. What is the the p.d.f. $f_Y(y)$?

29. The pdf of a mixed random variable is shown in the figure.

   a. What is the constant K?

   b. Compute $P[X \leq 5]$, $P[5 \leq x < 10]$.

   c. Draw the distribution function $F_X(x)$.



30. The cumulative distribution function (CDF) of a random variable Y is

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{1}{3} + \frac{2y}{9}, & 0 \leq y < 3 \\ 1, & y \geq 3 \end{cases}$$

a. Find and draw the probability density (pdf) function $f_Y(y)$.

b. Find the mean and variance of the random variable Y.

c. What are the probabilities $P[Y < 1]$, $P[0 < Y \leq 1]$?

31. Let X be a random variable with Cumulative Distribution Function (CDF) $F_X(x)$. Let Y be the output of a clipping circuit with the characteristic $Y = g(X)$ where

$$g(X) = \begin{cases} 4, & x \leq 0 \\ 6, & x > 0 \end{cases}$$

Find and draw the CDF $F_Y(y)$ and the pdf $f_Y(y)$.

32. Let X be a Gaussian random variable with mean μ and standard deviation σ. Let $Y = aX + b$ where a and b are two arbitrary constants.

a. Find the probability density (pdf) function $f_Y(y)$

b. Find the mean and variance of the random variable Y.

c. Assuming that $\mu = 1$ and $\sigma^2 = 2$ find the $P[0 < X \leq 2]$. (From tables the value of the standard Gaussian CDF $\Phi(0.7071) = 0.76$).

33. A two level waveform (0,1) is made analog because of the effect of additive Gaussian noise of zero mean and unit variance. A decoder samples the analog waveform $x(t)$ at t_0 and decodes according to the following rule

$$Y = \begin{cases} 1, & if \quad x(t_0) \geq \frac{1}{2} \\ 0, & if \quad x(t_0) < \frac{1}{2} \end{cases}$$

What is the Probability Mass Function, $P_Y(y)$?. Note that from tables of the Q-function we find that $Q(0.5) = 0.31$.

34. Let X be a real random variable uniformly distributed over (-1,1). Find

a) The density function of $Y = X^2$.

b) The density function of $Z = \sqrt{Y}$.

c) The n-th order moment of Y.

d) The characteristic function of X.

35. Let X and Y be two continuous random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-\lambda y}, & 0 \le x < y \\ 0, & otherwise \end{cases}$$

where λ is a positive constant.

a) Find the marginal probability density function $f_Y(y)$.

b) Find the conditional probability density function $f_{X|Y}(x|y)$.

c) Find the ratio $E\{X\}/E\{Y\}$.

36. Let X and Y be two continuous random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} cxy, & 0 \le x, y \le 1 \\ 0, & otherwise \end{cases}$$

where c is a positive constant. Define the event A as $Y > X$.

a) Find c.

b) Find the probability of the event A, $P[A]$.

c) Find the conditional probability density function $f_{Y|A}(y|A)$.

37. Consider the probability space $(S, \mathcal{F}, P)$. Let $S = \{\zeta_1, \ldots, \zeta_5\} = \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$ with $P[\zeta_i] = \frac{1}{5}, i = 1, \ldots, 5$. Define the random variables: $X(\zeta) \triangleq \zeta$ and $Y(\zeta) \triangleq \zeta^3$. Are the variables X and Y independent?

38. Suppose we define a random variable as $X = X_1 + X_2$, where X_1 and X_2 are independent Poisson random variables with parameters α_1 and α_2, respectively. What is the distribution of the random variab le X? (Note that if X_i is Poisson of parameter α_i, then $P(X_i = k) = \frac{\alpha_i^k}{k!} e^{-\alpha_i}$, $k = 0, 1, 2, \ldots$ and $\Phi_{X_i}(\omega) = e^{\alpha_i (e^{j\omega - 1})}$).

39. Let X and Y be independent, identically distributed random variabl es with

$$f_X(x) = \begin{cases} ae^{-ax}, & x \ge 0 \\ 0, & otherwise \end{cases}$$

Compute the probability density function, $f_Z(z)$ of the random variable $Z = X - Y$.

40. Consider a random variable X which is Gaussian with zero mean and un it variance. Define a new random variable Y as follows: Y is equal to X with the probability 0.3 and to $-X$ with the probability 0.7.

i) What is the probability density function $f_Y(y)$?

ii) What is the probability density function of the random variable $Z = X + Y$?

41. Suppose we define a random variable as $X = X_1 + X_2 + \ldots + X_N$, where the $\{X_i\}, \quad i = 1, \ldots, N$ are i.i.d. (independent identically distributed), random variables. N is also a random variable independent of the $\{X_i\}$ and taking the discrete values $\{0, 1, 2, \ldots\}$. Find $E\{X\}$. (*Hint: Use conditional expectation*)

42. Let X and Y be two random variables with joint probability density function (pdf)

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 < y < x < 1 \\ 0, & otherwise \end{cases}$$

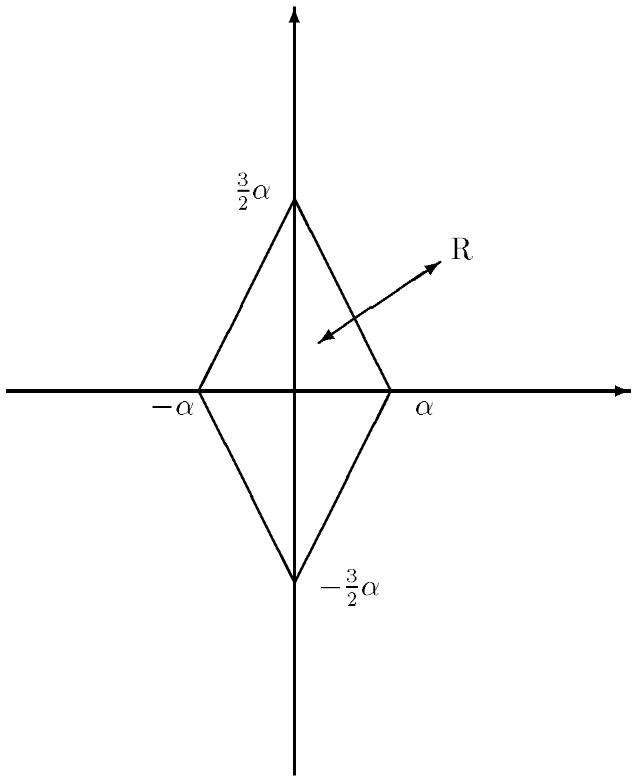
a) Sketch the joint pdf's values in the x-y plane and find the $P(X+Y) > 1$.

b) Find the density function of the random variable $Z = max(X, Y)$.

c) Find and sketch the joint pdf of the random variables

$$Z = X + 2Y, \qquad W = 3X - Y.$$

43. The random variables X and Y have a joint-probability density $f_{X,Y}(x,y)$ which is **constant** over the shaded region R shown in the Figure.



a) What is the value of the joint pdf $f_{X,Y}(x,y)$ over its region of support?

b) Determine the individual probability densities $f_X(x)$ and $f_Y(y)$.

c) Show whether or not X and Y are independent random variables.

d) Suppose that the random variables X and Y are independent and have a joint-probability density $f_{X,Y}(x,y)$ which is constant over a bounded region R of the (x,y) plane. Suppose also that the random variable X can only assume values in the x interval (a,b), while the random variable Y can only assume values in the y interval (c,d). What *must* the shape of region R then be?

44. Let X and Y be two random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} x+y, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & otherwise \end{cases}$$

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a) Find the correlation $E(XY)$ (2 points)

b) Find the covariance $Cov(X,Y)$. (2 points)

45. Let X and Y be two random variables with joint probability mass function

$P_{X,Y}(x,y)$	$y = 40$	$y = 60$
$x = 1$	0.15	0.1
$x = 2$	0.3	0.2
$x = 3$	0.15	0.1

a) Find the correlation $E(XY)$ (1 point)

b) Find the covariance $Cov(X,Y)$. (2 points)

c) Find the correlation coefficient $\rho_{X,Y}$. (1 point)

46. Let $\mathbf{X} = [X_1, X_2]$ be a zero mean random vector with covariance matrix $\mathbf{K}_X$. Also, let $\mathbf{Y} = \mathbf{X} \cdot \mathbf{A}$ where $\mathbf{A}$ is a 2x2 matrix with constant elements. What is the mean and what is the covariance matrix of $\mathbf{Y}$?

47. In the previous problem, let X_1 and X_2 be jointly Gaussian random variables. What can we say about the marginal distribution of Y_1? If X_1 and X_2 are correlated random variables, then is it possible that Y_1 and Y_2 are independent random variables?

48. The random variables X and Y have joint probability density function

$$f_{X,Y}(x,y) = ce^{-(2x^2-4xy+4y^2)}$$

i) What are the expected values $E(X)$ and $E(Y)$?

ii) Find $\rho_{X,Y}$, the correlation coefficient of X and Y.

iii)What are the variances $Var(X)$ and $Var(Y)$?

iv) What is the constant c?

v) Are X and Y independent?