1 PROBABILITY SPACE

• Probability space is a triple:

- Sample Space

- Event Space

- Probability Measure

* **SAMPLE SPACE** (S): The set of all possible outcomes of an experiment.

* **EVENT SPACE** $\Im$: It must be a Field (or Algebra),i.e.,

$$[if A \in \Im \; then \; A^c \in \Im] \tag{1}$$

$$[if A, B \in \Im \; then \; A + B \in \Im] \tag{2}$$

then, the properties $AB \in \Im$, $\{\emptyset\} \in \Im$, $S \in \Im$, $A - B \in \Im$ follow easily.

* The set of all subsets of S in an algebra (the largest)

* Sometimes we need an algebra smaller than the largest (for example in those cases where S has infinite number of subsets)

Example: Let $S = \{1, 2, 3\}$. Largest Algebra has $2^3 = 8$ elements. Assuming that we are interested only on the subset $\{1\}$ we must choose as event space the algebra $\{\{\emptyset\}, \{1\}, \{2, 3\}, S\}$ which is the smallest algebra containing $\{1\}$.

THEOREM: Given a collection $\Im_1$ of subsets of S then there exists a smallest algebra containing $\Im_1$ termed the minimal algebra.

Lemma Let $\Im_1$ and $\Im_2$ be two algebras. Then the intersection $\Im_1\Im_2$ is also an algebra.

Proof: Assuming $a, b \in \Im_1 \, and \, \Im_2$ show that :

$$a^c, b^c \in \Im_1\Im_2$$

$$a + b \in \Im_1\Im_2$$

Conclusion: The minimal algebra containing $\Im_1$ is the intersection of all algebras containing $\Im_1$.

NOTES:

1. For the finite sample space case there is no difficulty in choosing the event space to be the largest algebra.

2. Although finite intersections and unions of elements in an algebra are also in the algebra, infinite unions or intersections need not be.

3. Borel Field (or σ -Algebra) is a field which is closed under any infinite, however, countable set of unions, intersections and other combinations. Thus, if $A_1, \ldots, A_n, \ldots$ belong to $\Im$ then,

   $\bigcup_{i=1}^{\infty} A_i \in \Im$ and $\bigcap_{i=1}^{\infty} A_i \in \Im$

   Consequently, in more general terms the event space must be a Borel Field.

$\star$ **PROBABILITY MEASURE** $(P(\cdot))$

We assign probabilities to those subsets of S that form a Borel Field (i.e., only to events)

The probability measure is a finite measure that satisfies AXIOMS 1, 2 and 3, i.e.,

$[0 \leq P(A) \leq 1, P(S) = 1, \, if \; AB = \{\oslash\} \; then \; P(A+B) = P(A) + P(B)]$

Properties resulting from the axioms:

$P(A^c) = 1 - P(A)$, $P(\oslash) = 0$, $P(B - A) = P(B) - P(AB)$

$P(A+B) = P(A) + P(B) - P(AB)$ (proof: $A + B = A + (B - A) = \ldots$)

$P(A+B) \leq P(A) + P(B)$ (subadditivity property)

* The events A and B are called "equal events" or "events equal with probability 1" if they have same elements, i.e.,

$P(A) = P(B) = P(AB)$

* The events A and B are "equal in probability" if

$P(A) = P(B)$

However, A and B might not have same elements.

* Let $\{A_i\}_{i=1,2,\ldots,\infty}$ be a countable collection of pairwise disjoint sets $(A_iA_j = \{\oslash\})$ $(i \neq j)$. Then

$P[\bigcup_{i=1}^{\infty} A_i] = \sum_{i=1}^{\infty} P(A_i)$: (σ -additivity of the probability measure.)

THE BOREL ALGEBRA

If S consists of uncountably infinite elements then is is impossible to define a probability measure for this space. Such an example is the set of real numbers $\Re$ ("the real line"). It can be shown that if we consider all subsets of the real line as events we cannot define a probability measure.

To overcome this difficulty, we consider as the event space the minimal σ -algebra generated by the intervals.

$$\{x/x \leq x_1\} \forall x_1 \in \Re \qquad \text{(BOREL ALGEBRA)}$$

This algebra will contain as well all intervals $x_1 \leq x \leq x_2, x_1 < x < x_2$, all points, all countable unions of intervals, etc. In fact, it contains almost any set one could think of and thus, it is broad enough for any application.

A probability measure is defined for the events $\{x/x \leq x_1\}$.

* Example: let $a(x)$ be such that $\int_{-\infty}^{\infty} a(x)dx = 1, \qquad a(x) \geq 0$

Then:

$$P\{x \leq x_1\} = \int_{-\infty}^{x_1} a(x)dx$$

satisfies all probability axioms. $\rightarrow$ (could you check ?)