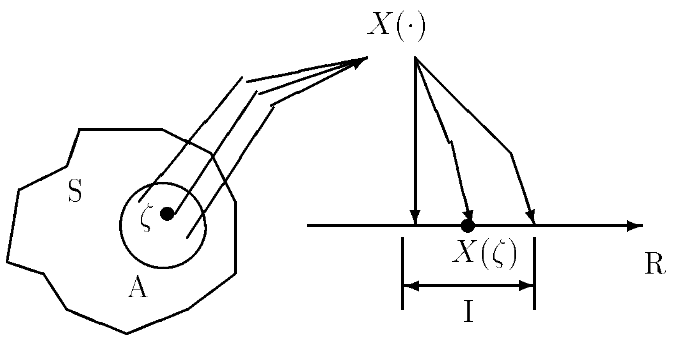
THE CONCEPT OF A RANDOM VARIABLE (R.V.)



$X(\cdot)$: is a function
i.e., to every $\zeta \in S$ corresponds
only one $X(\zeta) \in R$

Event A in S belongs to a Borel Field and might include artribary elements (i.e., voltage or red balls ...)

Event I in R belongs to the Borel Algebra, i.e., the minimal Borel Field generated by the intervals $\{X(\zeta) \leq x\}$, $\forall x \in R$.

DEFINITION: A real Random Variable over a probability space $(S, \Im, P)$ is a process (function) of assigning a real number $X(\zeta)$ to every outcome ζ. In other words, it is a function mapping $X : S \rightarrow R$

$X(\cdot)$ should satisfy the following conditions.

1) For all $x \in R, \{\zeta / X(\zeta) \leq x, \zeta \in S\} \in \Im$
i.e., the event $\{X(\zeta) \leq x\}$ must correspond to an event in $\Im$ and thus a probability measure can be assigned.

2) $P[X(\zeta) = -\infty] = P[X(\zeta) = \infty] = 0$
(However, in general there must be cases where we desire $\pm\infty$ to have non-zero probability, for example, when describing waiting time until the occurence of a random event)

Note: In practice, contition 1 is hard not to satisfy. If the sample space is countable then almost any assignment will do. It is mostly in non-countable sample spaces that possible assinnments might not be valid random variables.

Notation:

1) From now on, when we assign probabilities we mean
$P(X \in I) = P[X(\zeta) \in I] = P[\{\zeta/\zeta \in S \, and \, X(\zeta) \in I\}]$

2) Capital X denotes the r.v., small x the values of the r.v. in R

Example The Indicator random variable

We observe a light incicator in a nuclear plant.

If the light is $\{on\}$, then $X = 1$. If the light is off $\{off\}$, then $X = 0$

Then,

$S = \{on, off\}, \Im = \{\oslash, S, \{on\}, \{off\}\}$

$P(\oslash) = 0, P(S) = 1, P(on) = \frac{1}{4}, P(off) = \frac{3}{4}$

Define the random variable

$$X(l) = \begin{cases} 1, & l \text{ is on} \\ 0, & l \text{ is off} \end{cases} \longrightarrow \begin{cases} P[X = -\infty]=P[\oslash]=0 \\ P[X \leq -1]=P[\oslash]=0 \\ P[X=1]=P[on]=1/4 \\ P[X=0]=P[off]=3/4 \\ P[X \leq 4]=P[S]=1 \\ \dots \\ \dots \\ P[X = \infty]=1-P[X < \infty]=0 \end{cases}$$

It is obvious that every interval in R can be related to an event in $\mathcal{F}$. Hence, X(l) is a valid r.v.

THE DISTRIBUTION FUNCTION

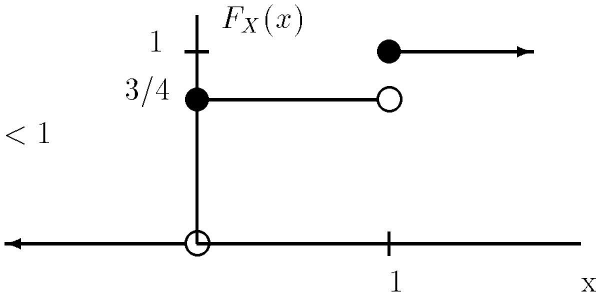
Definition: The PDF [Probability distribution function] or cummulative distribution function, of a r.v. X is defined as :

$$F_X(x) = P\{X \leq x\}$$

(Note that $F_X(x)$ is nothing else but the function that assigns probabilities to the generating events of the Borel Algebra and as such contains all necessary information to calculate the probability of any event in R)

Example: The indicator r.v.

$$F_X(x) = \begin{cases} \mathrm{P}\{\mathrm{S}\}, & 1 \leq x \\ \mathrm{P}\{\mathrm{off}\}, & 0 \leq x < 1 \\ \mathrm{P}\{\emptyset\}, & x < 0 \end{cases}$$



So we can write that a random variable is a funcion $X(\cdot)$,i.e.,

$X : (S, \Im, P) \rightarrow (R, Borel Algebra, F_x)$