Student Name: Student Number:

University of Toronto
Faculty of Applied Science and Engineering

MIDTERM EXAMINATION
ECE431, Digital Signal Processing
October 22 2012, 6:15-7:45 pm, HA316
Examiner: D. Hatzinakos

Exam type A
Calculators are allowed

In a frequency coding scheme, symbols (letters, numbers, etc.) are represented by a combination of two frequencies F_1 and F_2 where $F_1 \neq F_2$ and each taking one out of 10 possible values, e.g., 250 Hz, 750 Hz, 1250 Hz, 1750 Hz, 2250 Hz, 2750 Hz, 3250 Hz, 3750 Hz, 4250 Hz and 4750 Hz. In the transmitter continuous time signals xa(t)= $[\ \sin(2\pi F_1 t) + \sin(2\pi F_2 t)\ ]$, t=0,...,T sec are formed and transmitted consequently every T secs. In the receiver detection of the transmitted symbols is based on discrete signal processing techniques. In other words assuming perfect synchronization is possible, T sec long signal segments are extracted from the incoming signal, each segment is ideally and uniformly sampled with a period of Ts secs and then a N-point DFT is computed and plotted to detect the underlying two frequencies and from those the transmitted symbol.

a) Let T/Ts=L. Write an expression for the discrete signal x(n)=xa(n Ts), n=0,...,L-1. Choose an appropriate value for Ts and then calculate the normalized frequencies f_i, $i=1,2,\ldots,10$. corresponding to the real frequencies F_i, $i=1,2,\ldots,10$. (2 points).

$x[n] = x_a[nT_s] \Rightarrow x(n) = \sin(2\pi F_1 n T_s) + \sin(2\pi F_2 n T_s)$ $n=0,1,\ldots,L-1$

$= \sin(2\pi f_1 n) + \sin(2\pi f_2 n)$

where $f_1 = F_1 T_s$

$f_2 = F_2 T_s$

We choose T_s so that $F_s = \frac{1}{T_s} \geq 2$ max frequency content of $x_a(t)$

Let us choose $F_s = 10\,000$ samples/sec. Then

$f_i = F_i T_s = F_i 10^{-4} = 0.025, 0.075, 0.125, 0.175, 0.225, 0.275,$

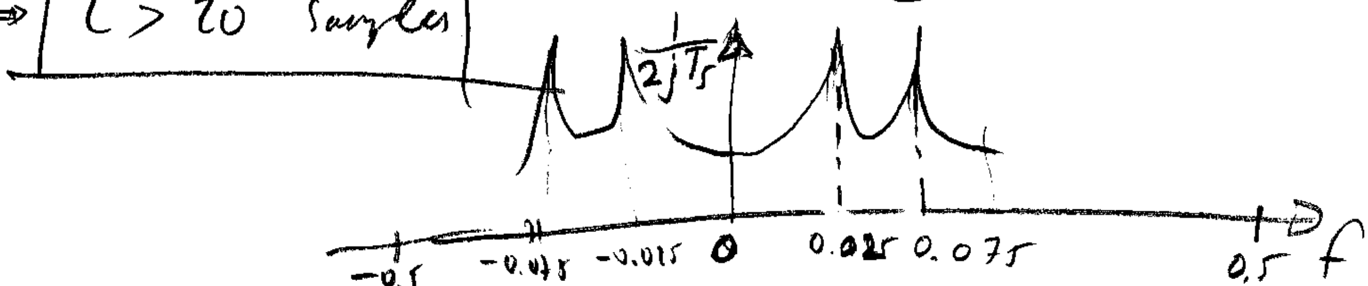
0.325, 0.375, 0.425, 0.475

Student Name: Student Number:

b) What should be the minimum value of L (or the corresponding length T) so that our frequency detector has sufficient resolution to detect any two of the frequencies that form the signal? Assuming that L and Ts have been properly chosen, draw approximately the DTFT of a signal segment containing the first two frequencies.(2 points)

The minimum distance between two components is $\Delta f = 500$ Hz or equivalently $\Delta f = 0.05$ cycles/sample. So required frequency resolution should be $\Delta f < 0.05 \Rightarrow \frac{1}{L} < 0.05 \Rightarrow L > \frac{1}{0.05} \Rightarrow$

$\Rightarrow \boxed{L > 20 \text{ samples}}$



c) What is the relation between the DTFT $X(f)$ and the N-DFT $X(k)$, k=0,1, ….,N-1of x(n), n=0,1,…,L-1? In your opinion, what should be the value of N so that we achieve best detection of the frequencies that form the signal? What are the corresponding values of k of the N-DFT that correspond to the frequencies to be detected? (2 points)

$\rightarrow X(k) = X(f)\Big|_{f=\frac{k}{N}}$, $k = 0, 1, \ldots, N-1$

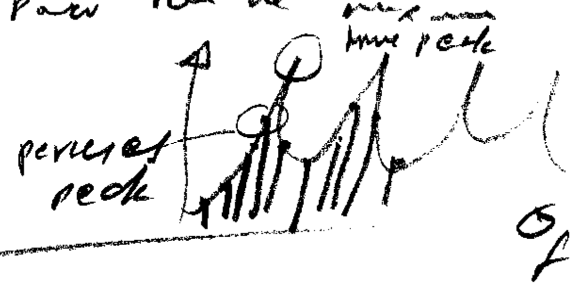
N should be such that we hit the values of 0.025, 0.075, ...

So: $\frac{1}{N} = 0.025 \Rightarrow N = \frac{1}{0.025} = \frac{1000}{25} = 40$

The values of k corresponding to the desired freqs will be $k = 1, 3, 5, 7, \ldots, 19$

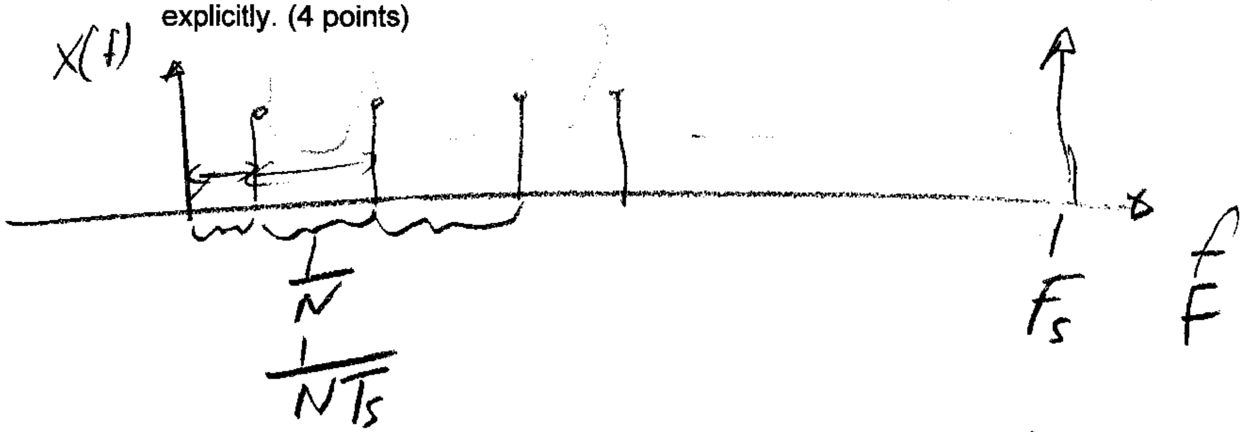
d) As N in part (c) increases, one observes that the maximum values of the N-DFT do not necessarily correspond to those values of k that represent the true locations of the sinusoidal frequencies . Furthermore the estimated sinusoids seem to be of different power. Why does this happen? (2 points)

As N increases the N-DFT provides a more dense approximation of the DTFT. However, the corresponding samples of the DTFT do not necessarily correspond to the location of the desired peaks. So we may get samples that show less power than the max true peak but also at the "wrong" location. e.g.



Student Name: Student Number:

e) Given x(n), n=0,1,…,L-1, describe a DSP procedure to calculate only those values of $X(f = 250Ts + k500Ts)$, k=0,1,…,9 corresponding to the 10 possible frequencies in the signal. Do you think that this is a more efficient way to estimate and detect the true frequencies in the observed signal? Please explain explicitly. (4 points)



If we use a 10-DFT then we want the spacing between samples to be $\frac{1}{NT_s} = 500$. So the

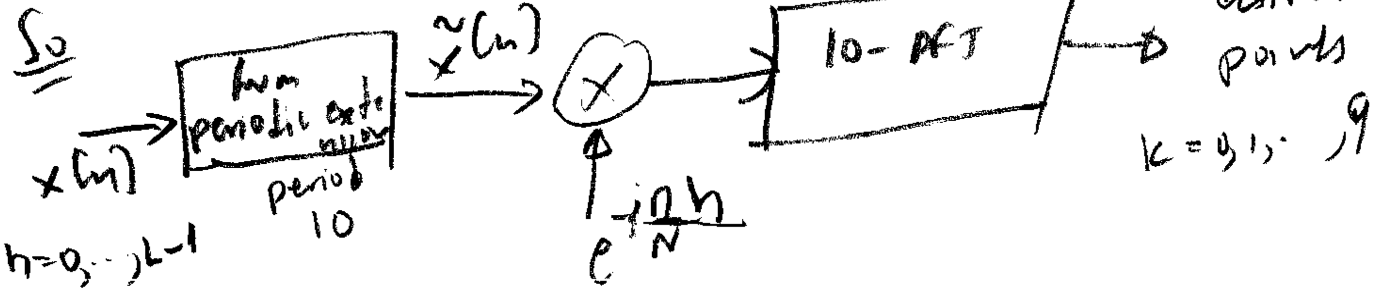
$$X(f = 250T_s + k500T_s) = X\left(f = \frac{1}{2N} + \frac{k}{N}\right), \quad k=0,1,\dots,9$$

Also, since $N < L$ we must use the time aliased signal $\tilde{x}(n) = \sum_{l} x(n+10l)$ in the calculation

Then, $$X(k) = \sum_{n=0}^{9} \tilde{x}[n] e^{-j2\pi\left(\frac{1}{2N}+\frac{k}{N}\right)n} = \sum_{n=0}^{9} \tilde{x}[n] e^{-j\frac{\pi n}{N}} e^{-j\frac{2\pi kn}{N}}$$

$k=0,\dots,9$

So



Student Name: Student Number:

f) Given the 10 values of the DFT of part (e) describe a procedure to interpolate 10 more values in the DFT domain equally spaced in between the 10 original values (4 points)

zero padding in time domain (N zeros) $\xleftrightarrow{N\text{-DFT}}$ Interpolation in DFT domain (N new values

So Given the 10-DFT

$$X(k) \;\; k=0,\dots,9 \longrightarrow \boxed{10\text{-IDFT}} \xrightarrow{\tilde{x}(n)} \;\; n=0,1,\dots,9 \longrightarrow \boxed{\begin{array}{c}\text{zero pad}\\ \text{10 zeros}\end{array}} \longrightarrow \boxed{20\text{-DFT}} \xrightarrow{\hat{X}(k)} \;\; k=0,1,\dots,19$$

Notice that the new values will not correspond to true values of the DTFT $X(w)$ since we have used $\tilde{x}(n)$ but there is nothing better we can do

Student Name: Student Number:

g) Suppose that in an effort to improve detection (i.e. decide the presence or not of a frequency component), you try an averaging technique as follows:

$$X_{avr}(k) = [\alpha X(k-1) + X(k) + \alpha X(k+1)]/(1+2\alpha)$$

Where, k=1,2,…,N-2 and α is a constant of amplitude less than 1. Averaging in this manner is equivalent to multiplying the signal x(n) by a new window w(n) before computing the DFT. Derive a simple formula for w(n). (4 points)

$$X(k) \xleftrightarrow{N\text{-}IDFT} x[n]$$

$$X(k-1) \xleftrightarrow{N\text{-}IDFT} x[n]e^{j\frac{2\pi n}{N}}$$

$$X(k+1) \xleftrightarrow{N\text{-}IDFT} x[n]e^{-j\frac{2\pi n}{N}}$$

$$X_{avr}(k) \xleftrightarrow{IDFT} \frac{x[n]\left[1 + ae^{j\frac{2\pi n}{N}} + ae^{-j\frac{2\pi n}{N}}\right]}{1+2a} =$$

$$= \frac{x[n]}{1+2a} \cdot \left(1 + a\left(e^{j\frac{2\pi n}{N}} + e^{-j\frac{2\pi n}{N}}\right)\right) =$$

$$= \frac{x[n]}{1+2a} \cdot \left(1 + 2a\cos\frac{2\pi n}{N}\right) =$$

$$= x[n] \cdot \underbrace{\frac{1+2a\cos\left(\frac{2\pi n}{N}\right)}{1+2a}}_{w[n]}, \quad n = 0, 1, \ldots, N-1$$