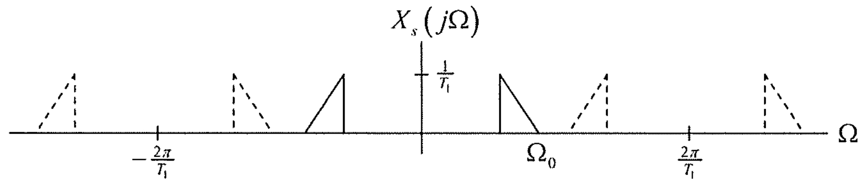
**4.21.** A. The impulse-train signal $x_s(t)$ has spectrum $X_s(j\Omega)$ given by

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left[j\left(\Omega - k\frac{2\pi}{T_1}\right)\right].$$

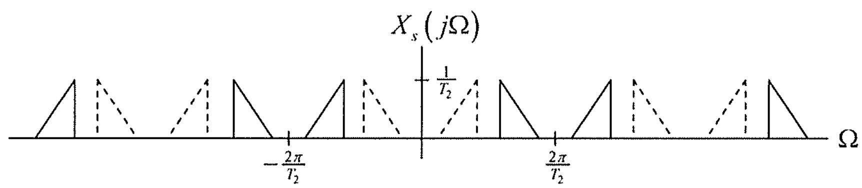
An example is shown below.



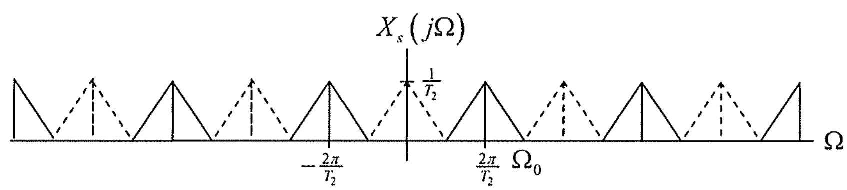

We will have $x_r(t) = x_c(t)$ provided $T_1 \le \frac{\pi}{\Omega_0}$.

B. We will have $x_o(t) = x_c(t)$ under any of the following circumstances:

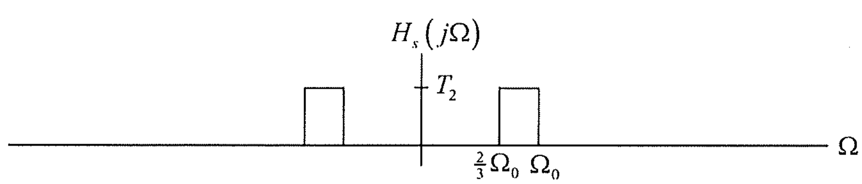
1. As illustrated above, $T_2 \le \frac{\pi}{\Omega_0}$.
2. As illustrated below, $\frac{1.5\pi}{\Omega_0} \le T_2 \le \frac{2\pi}{\Omega_0}$.




3. As illustrated below, $T_2 = \frac{3\pi}{\Omega_0}$.




The frequency response of the filter that is needed to recover $x_c(t)$ is shown below.



**4.23.** Appears in: Fall04 PS1, Fall02 PS1.

### Problem

Note that in OSB and 6.341 $\Omega$ denotes continuous-time frequency and $\omega$ denotes discrete-time frequency.

Figure 1 shows a continuous-time filter that is implemented using an LTI discrete-time filter with frequency response $H(e^{j\omega})$.

(a) If the CTFT of $x_c(t)$, namely $X_c(j\Omega)$, is as shown in Figure 2 and $\omega_c = \frac{\pi}{5}$, sketch and label $X(e^{j\omega})$, $Y(e^{j\omega})$ and $Y_c(j\Omega)$ for each of the following cases:

(i) $1/T_1 = 1/T_2 = 2 \times 10^4$

(ii) $1/T_1 = 4 \times 10^4$, $1/T_2 = 10^4$

(iii) $1/T_1 = 10^4$, $1/T_2 = 3 \times 10^4$

(b) For $1/T_1 = 1/T_2 = 6 \times 10^3$, and for input signals $x_c(t)$ whose spectra are bandlimited to $|\Omega| < 2\pi \times 5 \times 10^3$ (but otherwise unconstrained), what is the maximum choice of the cutoff frequency $\omega_c$ of the filter $H(e^{j\omega})$ for which the overall system is LTI? For this maximum choice of $\omega_c$, specify $H_c(j\Omega)$.

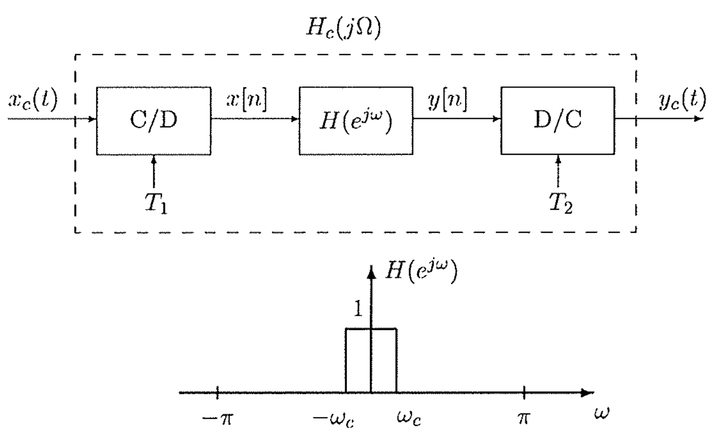



Figure 1: Problem .

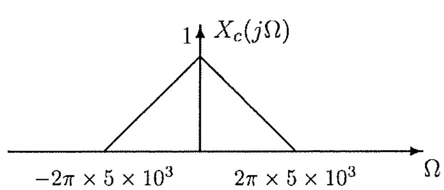
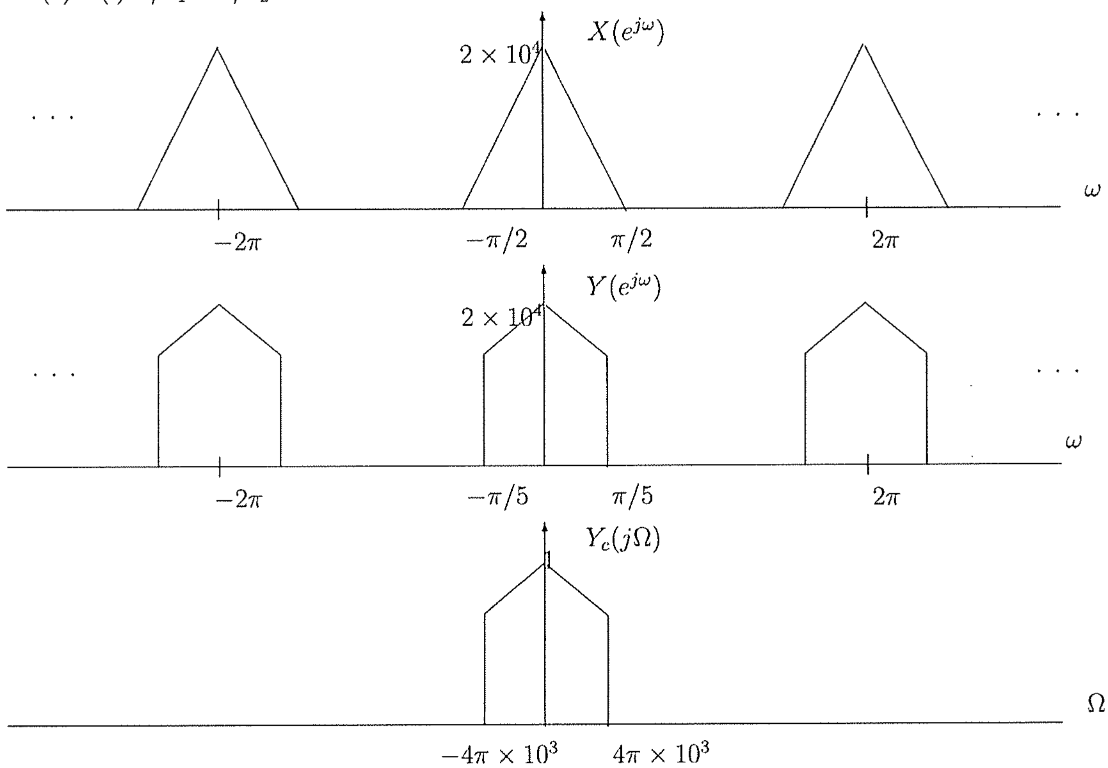


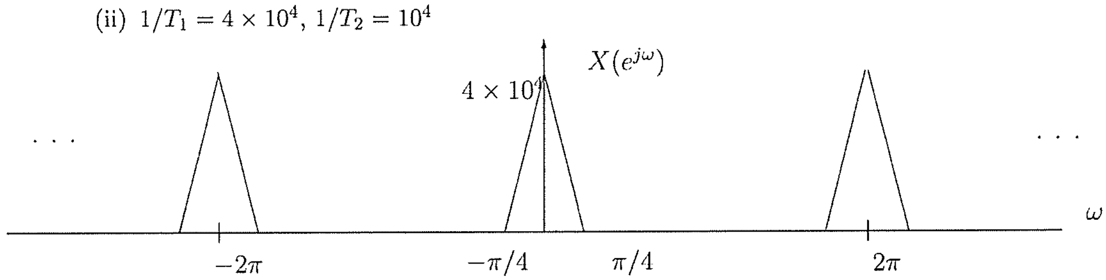

Figure 2: Problem , part (a).

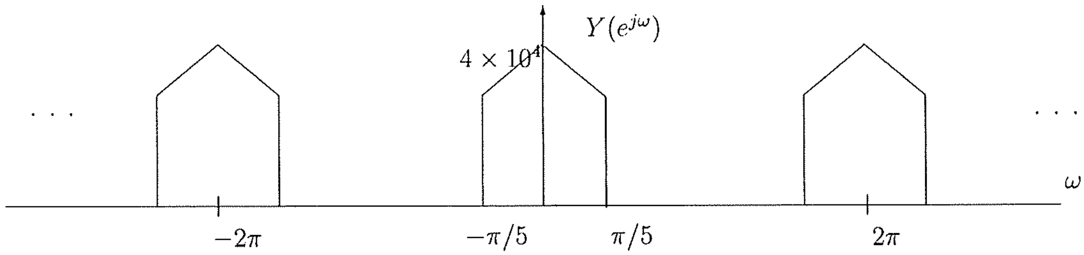
**Solution from Fall04 PS1**

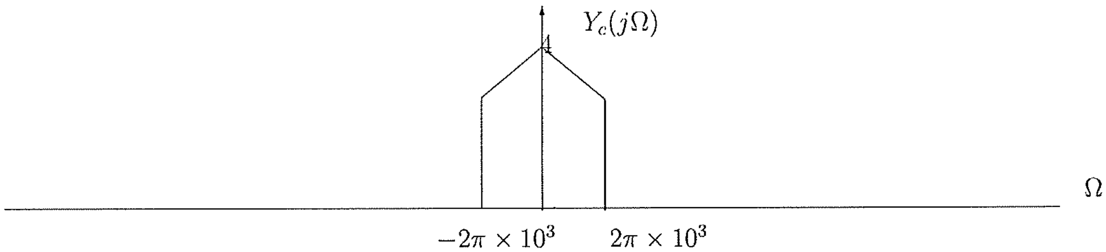
(a) (i) $1/T_1 = 1/T_2 = 2 \times 10^4$



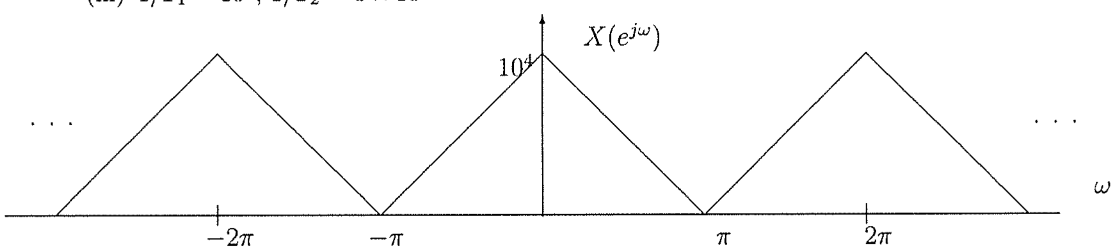

(ii) $1/T_1 = 4 \times 10^4$, $1/T_2 = 10^4$

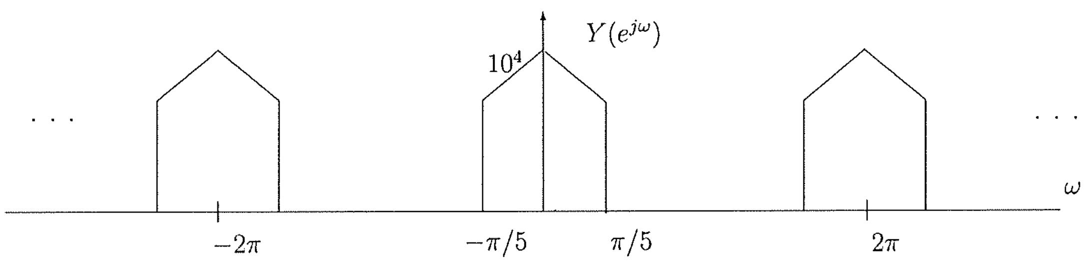


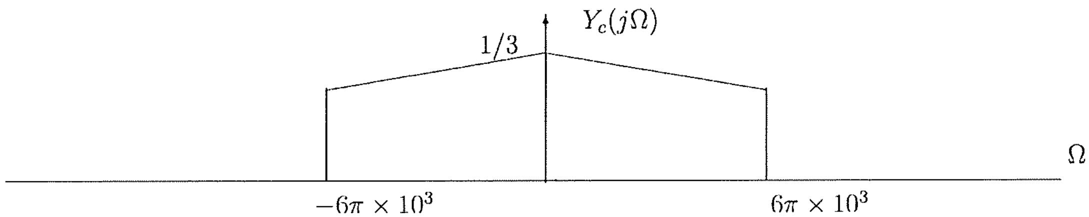



(iii) $1/T_1 = 10^4$, $1/T_2 = 3 \times 10^4$

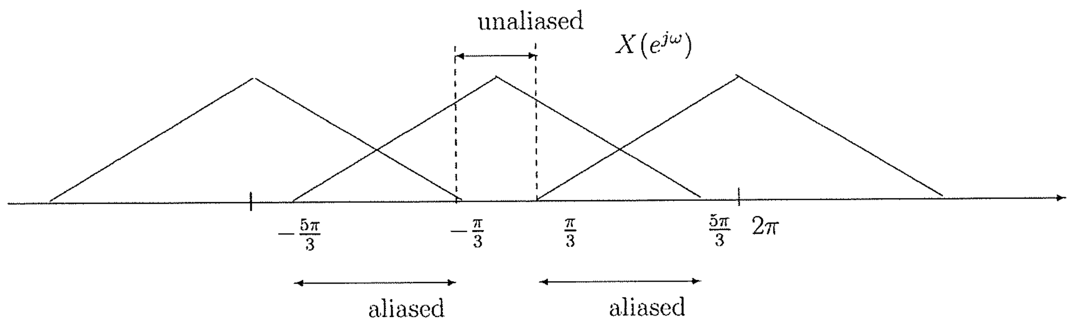


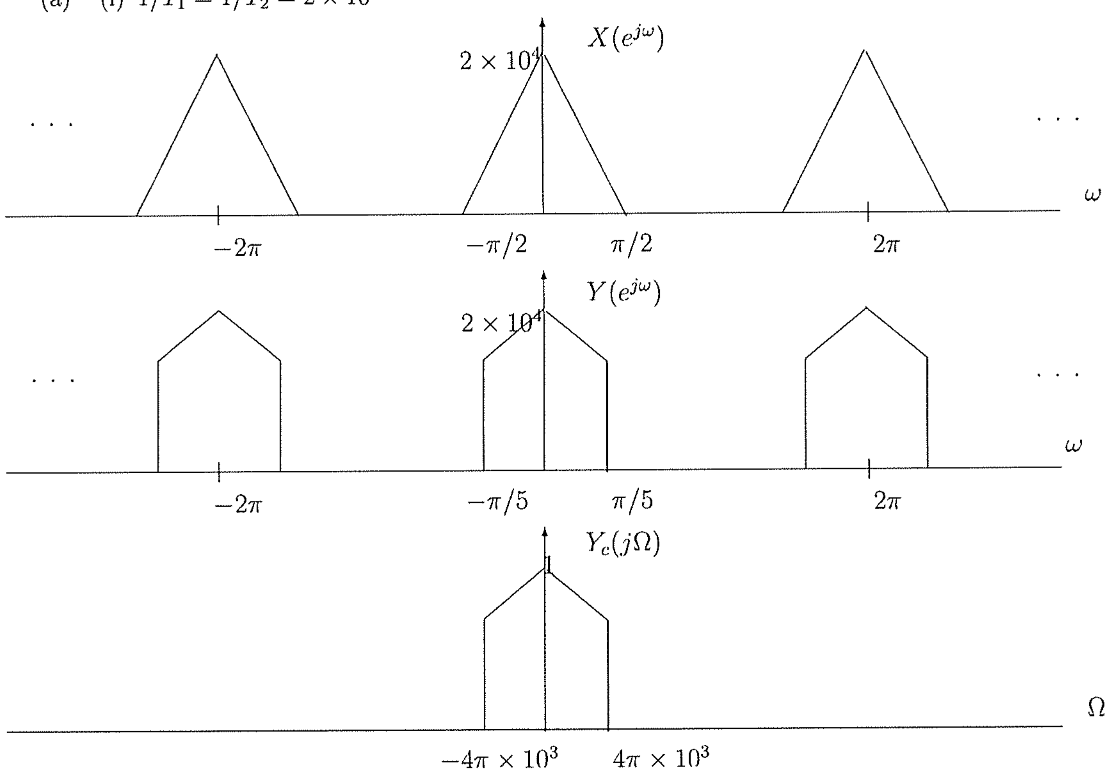
(b) From the figure below, it can be seen that the only portion of the spectrum which remains unaffected by the aliasing is $|\omega| < \pi/3$. So if we choose $\omega_c < \pi/3$, the overall system is LTI with a frequency response of

$$H_c(j\Omega) = \begin{cases} 1 & \text{for } |\Omega| < \omega_c \times 6 \times 10^3 \\ 0 & \text{otherwise.} \end{cases}$$

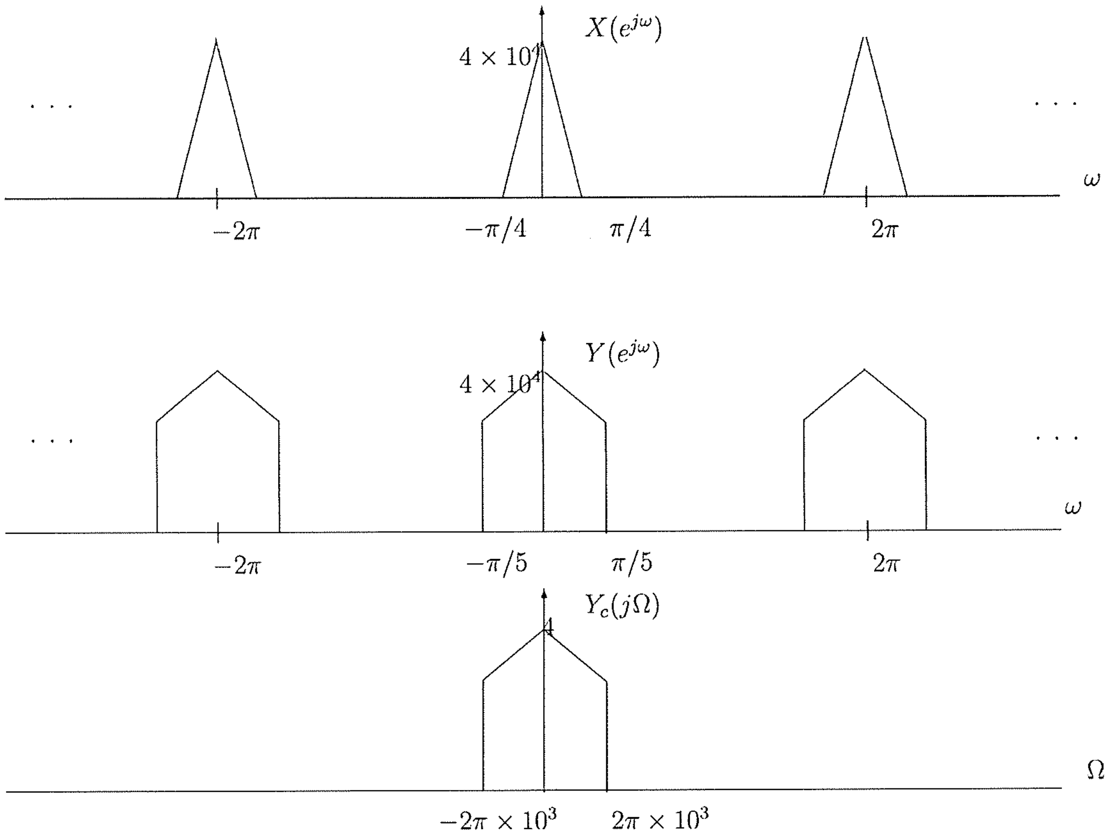



**Solution from Fall02 PS1**

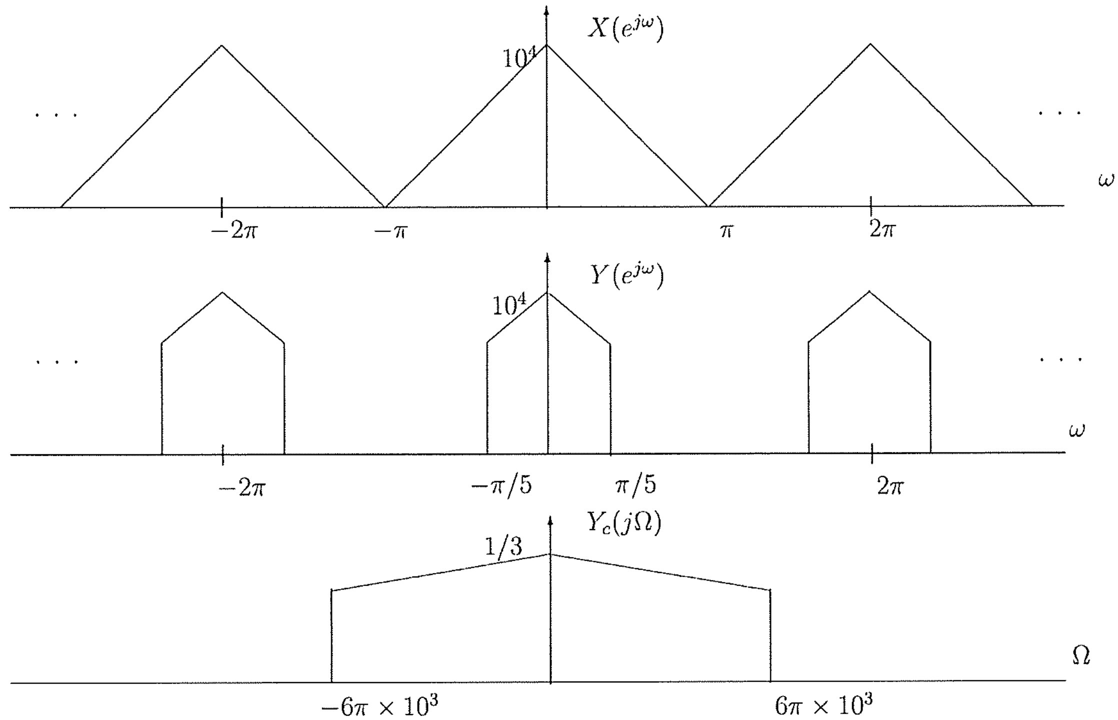
(a) (i) $1/T_1 = 1/T_2 = 2 \times 10^4$



(ii) $1/T_1 = 4 \times 10^4$, $1/T_2 = 10^4$

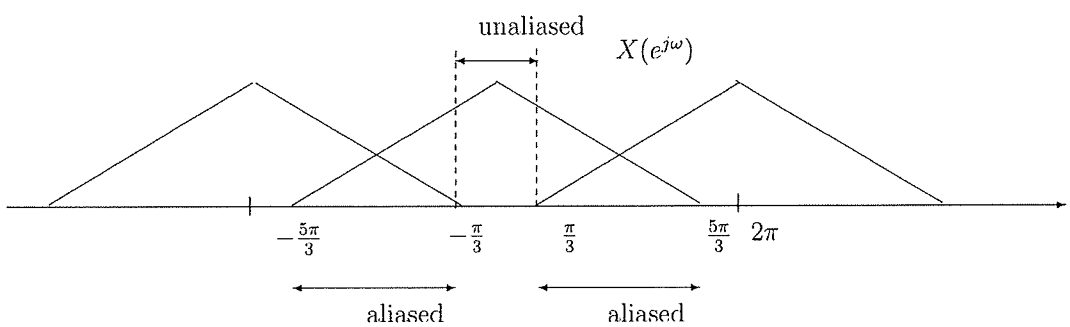


(iii) $1/T_1 = 10^4$, $1/T_2 = 3 \times 10^4$

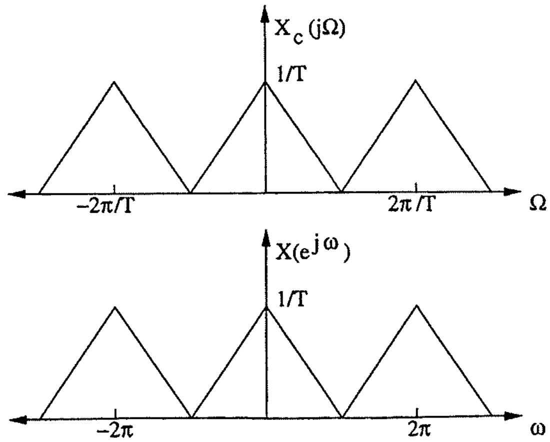



(b) From the figure below, it can be seen that the only portion of the spectrum which remains unaffected by the aliasing is $|\omega| < \pi/3$. So if we choose $\omega_c < \pi/3$, the overall system is LTI with a frequency response of

$$H_c(j\Omega) = \begin{cases} 1 & \text{for } |\Omega| < \omega_c \times 6 \times 10^3 \\ 0 & \text{otherwise.} \end{cases}$$



**4.31.** (a) $x_s(t) = x_c(t)s(t) \Rightarrow X_s(j\Omega) * s(j\Omega)$



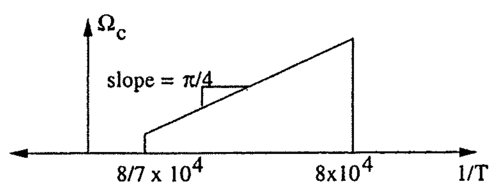

(b) Since $H_d(e^{j\omega})$ is an ideal lowpass filter with $\omega_c = \frac{\pi}{4}$, we don't care about any signal aliasing that occurs in the region $\frac{\pi}{4} \le \omega \le \pi$. We require:

$$\begin{aligned} \frac{2\pi}{T} - 2\pi \cdot 10000 &\ge \frac{\pi}{4T} \\ \frac{1}{T} &\ge \frac{8}{7} \cdot 10000 \\ T &\le \frac{7}{8} \times 10^{-4}\text{sec} \end{aligned}$$

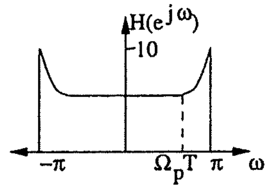
Also, once all of the signal lies in the range $|\omega| \le \frac{\pi}{4}$, the filter will be ineffective, i.e., $\frac{\pi}{4} \le T(2\pi \times 10^4)$. So, $T \ge 12.5\mu$sec.

(c)

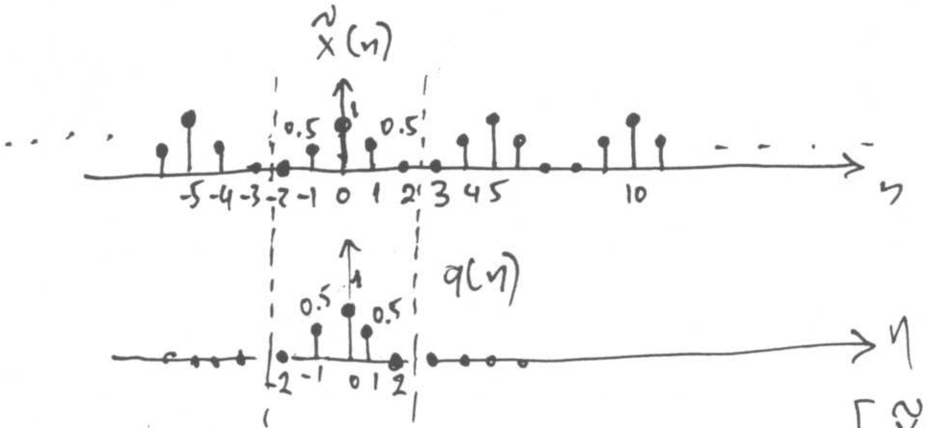
$$\Omega = \frac{\omega}{T} \Rightarrow \Omega_c = \frac{\pi}{4T}$$



**4.62.** Since we want $W(e^{j\omega})$ to equal $X(e^{j\omega})$, then $H(e^{j\omega})$ must compensate for the drop offs in $H_{aa}(j\Omega)$.



(3)




**1st way:** Direct application of DTFS pair

$$\begin{cases} \tilde{X}(k) = \sum_{n=\langle N\rangle} \tilde{x}(n) e^{-j\frac{2\pi k n}{N}} \\ \tilde{x}(n) = \frac{1}{N}\sum_{k=\langle N\rangle} \tilde{X}(k) e^{j\frac{2\pi k n}{N}} \end{cases}$$

$n, k = 0, \ldots, N-1$

Here $N = 5$. Thus

$$\tilde{X}(k) = \sum_{n=-2}^{2} \tilde{x}(n) e^{-j\frac{2\pi k n}{N}} = 0 + 0.5\, e^{+j\frac{2\pi k}{5}} + 1 \cdot e^{j0} + 0.5\, e^{-j\frac{2\pi k}{5}} + 0 =$$

$$= 1 + 0.5\left[ e^{j\frac{2\pi k}{5}} + e^{-j\frac{2\pi k}{5}} \right] \Rightarrow$$

$$\Rightarrow \boxed{\tilde{X}(k) = 1 + \cos\frac{2\pi k}{5} \qquad k = 0, \pm 1, \ldots}$$

DTFS ↓ DTFT

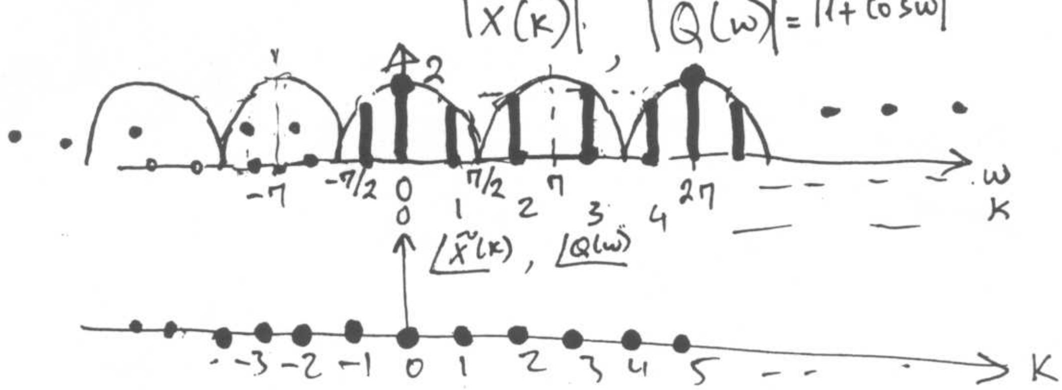
**2nd way:** Let $q(n) = \tilde{x}(n)$, $n = -2, \ldots, 0, \ldots 2$. Then $\tilde{X}(k) = Q\left(\omega = \frac{2\pi k}{5}\right)$

Thus

$$Q(\omega) = \sum_{n=-2}^{2} q(n) e^{-j\omega n} = 0 + 0.5\, e^{j\omega} + 1 \cdot e^{j0} + e^{-j\omega} + 0.$$

$$= 1 + 0.5\left[ e^{j\omega} + e^{-j\omega} \right] = 1 + \cos\omega \geq 0 \text{ for all } \omega$$

Then $\boxed{\tilde{X}(k) = 1 + \cos\left(\frac{2\pi k}{5}\right)}$




→ Period over ω (2π)

→ Period over k (5 samp…