SOLUTIONS

ELE 431S

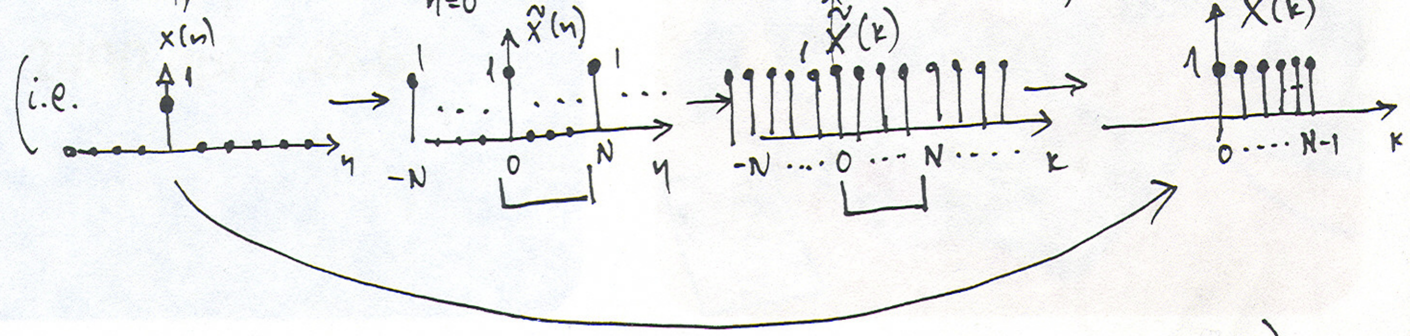
PROBLEM ASSIGNMENT ON DFT 3

1)

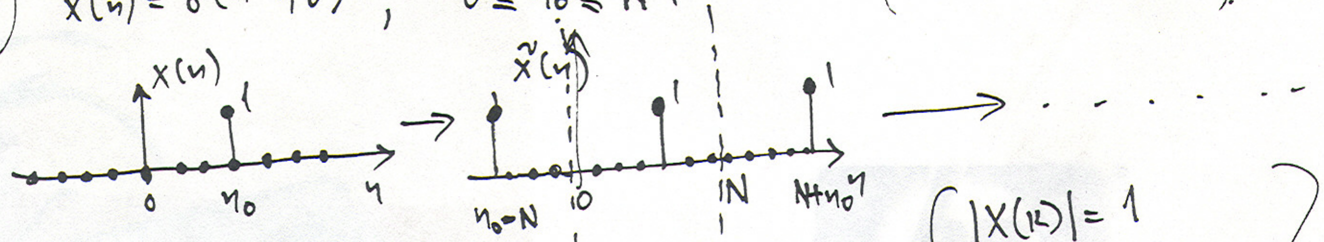
a) $x(n)=\delta(n)$ and let $N \geq 1$ (no time aliasing)

Then, $X(k)=\sum_{n=0}^{N-1}\delta(n)e^{-j\frac{2\pi kn}{N}} = e^{-j\frac{2\pi k 0}{N}} = 1$, $k=0,\dots\dots,N-1$

(i.e.



(b) $x(n)=\delta(n-n_0)$, $0\leq n_0 \leq N-1$ (no time aliasing)



$$X(k)=\sum_{n=0}^{N-1}\delta(n-n_0)e^{-j\frac{2\pi kn}{N}} = e^{-j\frac{2\pi k n_0}{N}} \Rightarrow \begin{cases} |X(k)| = 1 \\ \angle X(k) = -\frac{2\pi k n_0}{N} \\ k=0,1,2,\dots,N-1. \end{cases}$$

(c) $x(n)=a^n$, $0\leq n\leq N-1$

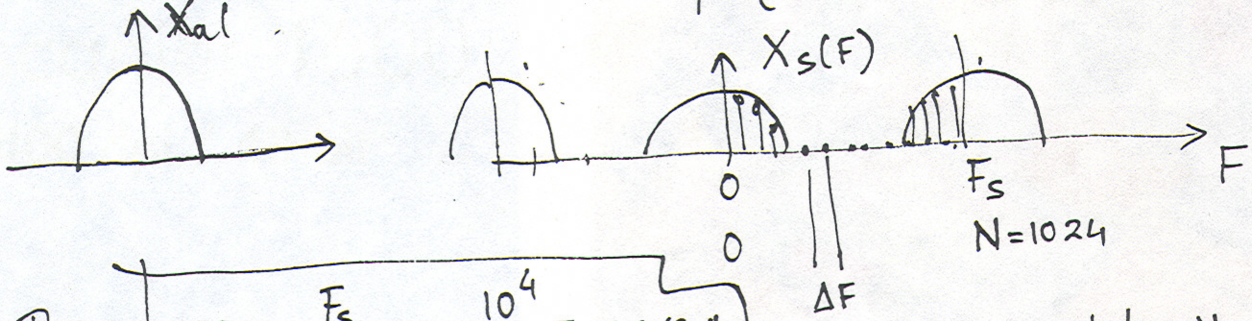
N-DFT : no time aliasing.

$$X(k)=\sum_{n=0}^{N-1}a^n e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{N-1}\left(a\cdot e^{-j\frac{2\pi k}{N}}\right)^n = \frac{1-\left(ae^{-j\frac{2\pi k}{N}}\right)^N}{1-ae^{-j\frac{2\pi k}{N}}} = \frac{1-a^N}{1-ae^{-j\frac{2\pi k}{N}}}.$$

$$\Rightarrow \boxed{X(k) = \frac{1-a^N}{1-ae^{-j\frac{2\pi k}{N}}} \;,\; k=0,\dots,N-1}$$

② $\left.\begin{array}{l} x_a(t) \\ F_s = 10\text{ kHz} \end{array}\right\} \Rightarrow \quad X_s(\Omega) = \frac{1}{T}\sum_{\ell} X_a(\Omega + \ell\, 2\pi F_s)$

or $X_s(F) = \frac{1}{T}\sum_{\ell} X_a(F + \ell F_s)$



Thus, $\boxed{\Delta F = \frac{F_s}{1024} = \frac{10^4}{1024} = 0.976\text{ Hz}}$

[N-DFT : take N samples in the interval $0 \le F \le F_s$ or $0 \le \Omega \le \Omega_s$ or $0 \le f \le 1$ or $0 \le \omega \le 2\pi$]

③ $x(n) \xrightarrow{N} \tilde{x}(n) = \sum_{\ell} x(n+\ell N) \ldots$

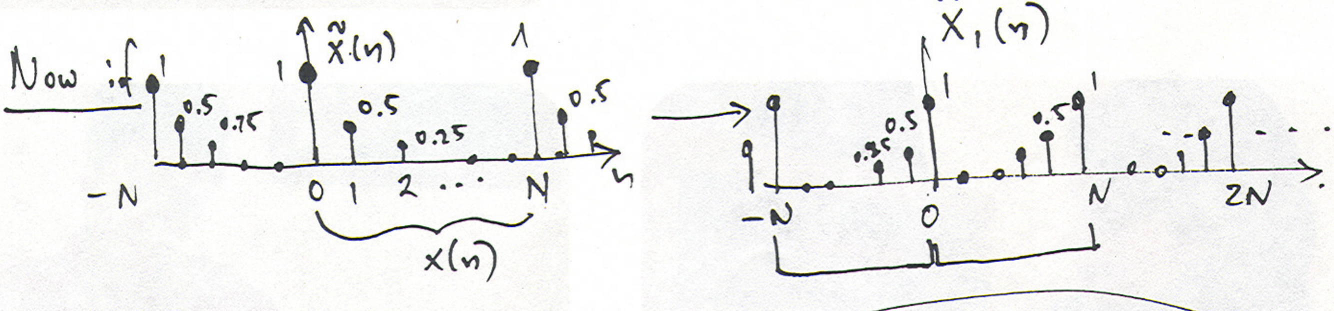
$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j\frac{2\pi k n}{N}} \quad , k = 0, \ldots, N-1$$

Now let

$$\tilde{x}_1(n) = \sum_{k=0}^{N-1} \tilde{X}(k)\, e^{-j\frac{2\pi k n}{N}} = N \underbrace{\left[\frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}(k)\, e^{j\frac{2\pi k(-n)}{N}}\right]}_{x(-n)}$$

$\Rightarrow \boxed{\tilde{x}_1(n) = N\, \tilde{x}(-n)}$ $\quad n = 0, \ldots, N-1$ or any other period of N samples.

Now if



Thus we can also write that $x_1(n) = N\, \tilde{x}(N-n)$

$n = 0, \ldots, N-1$

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$$x(n) = \left(\frac{1}{2}\right)^n u(n) \xleftrightarrow{\text{DTFT}} X(\omega)$$

$$y(n) = \begin{cases} \neq 0 & , n = 0, 1, \ldots, 9 \\ 0 & , \text{otherwise} \end{cases} \xleftrightarrow{\text{10-DFT}} Y(k) \qquad k = 0, 1, \ldots, 9$$

Also. $Y(k) = X\left(\omega = \frac{2\pi k}{10}\right)$, $k = 0, 1, \ldots, 9$.

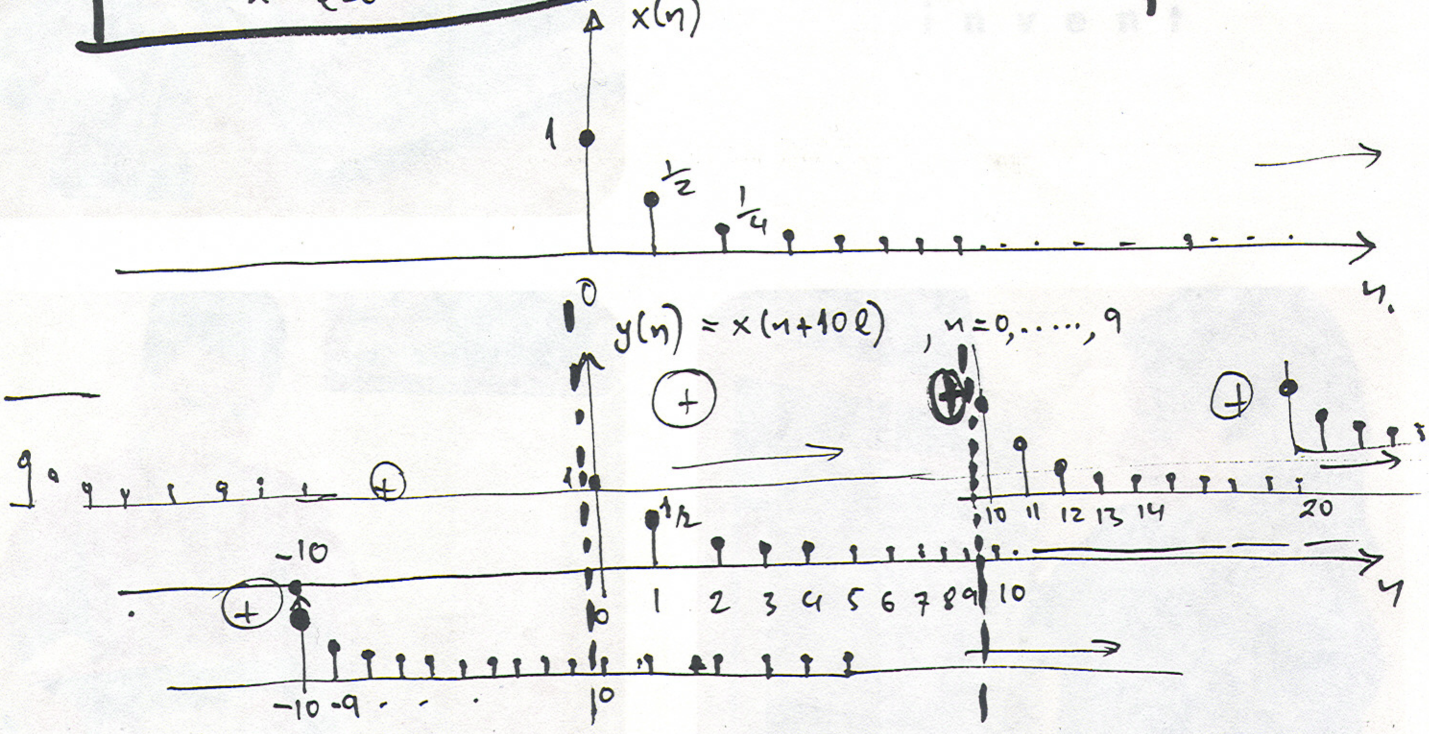
Then we know that in time domain

$$y(n) = \left[\sum_{\ell=-\infty}^{\infty} x(n+10\ell)\right] R_{10}(n) = \left[\sum_{\ell=-\infty}^{\infty} \left(\frac{1}{2}\right)^{(n+10\ell)} u(n+10\ell)\right] R_{10}(n)$$

$$\Rightarrow y(n) = \frac{1}{2^n} \cdot \sum_{\ell=-\infty}^{\infty} \left(\frac{1}{2}\right)^{10\ell} u(n+10\ell) \quad , \quad n = 0, 1, \ldots, 9.$$

From graph we note that only the replicas for $\ell \geq 0$ contribute in the area from $n = 0, 1, \ldots, 9$. Thus

$$\boxed{y(n) = \frac{1}{2^n} \sum_{\ell=0}^{\infty} \left(\frac{1}{2}\right)^{10\ell} u(n+10\ell) \quad , n = 0, 1, \ldots, 9}$$



Note that for $n=0,1,\dots,9$ and $\ell=0,1,\dots\infty$

$$u(n+10\ell) = 1 \quad \underline{\text{always}}$$

Thus, $$y(n) = \frac{1}{2^n}\sum_{\ell=0}^{\infty}\left(\frac{1}{2}\right)^{10\ell} = \frac{1}{2^n}\sum_{\ell=0}^{\infty}\left(\frac{1}{2^{10}}\right)^{\ell} = \frac{1}{2^n}\frac{1}{1-\frac{1}{2^{10}}} = \frac{2^{10}}{2^n}\frac{1}{2^{10}-1}.$$

Therefore,
$$y(0) = \frac{2^{10}}{2^{10}-1}$$
$$y(1) = \frac{2^{9}}{2^{10}-1}$$
$$\vdots$$
$$y(9) = \frac{2}{2^{10}-1}$$

5. $$x(n) = \begin{cases} \neq 0, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

We want to calculate $X(k) = X\left(\omega = \frac{2\pi k}{M}\right)$, $k=0,\dots,M-1$
i.e., take M equally spaced samples in $0 \le \omega \le 2\pi$.

Also, $\underline{M<N}$

How can we obtain the M points using only <u>one</u> M-DFT?

We know that: $$\left[\sum_{\ell} x(n+\ell M)\right] R_M(n) \xleftrightarrow{\text{M-DFT}} X\left(\omega = \frac{2\pi k}{M}\right) \cdot R_M(k)$$

Thus: ① Form $\sum_{\ell} x(n+\ell M)\, R_M(n)$
② Take M-point DFT.

6 $x(n)$, $y(n)$

$x(n): \begin{cases} \neq 0 & , 0 \le n \le 7 \\ 0 & , \text{other.} \end{cases}$, $y(n): \begin{cases} \neq 0, & 0 \le n \le 19 \\ 0, & \text{otherwise.} \end{cases}$

Now zero padd $x(n)$ with 12 zeros and calculate:

$$\left[\sum_{l=-\infty}^{\infty} x(n+20l)\right] R_{20}(n) \xleftrightarrow{\text{20-DFT}} X(k)$$

$k=0,\dots,19$

Also $$\underbrace{\left[\sum_{l=-\infty}^{\infty} y(n+20l)\right] R_{20}(n)}_{y(n)} \xleftrightarrow{\text{20-DFT}} Y(k)$$

let $r(n) = \text{IDFT}[X(k)Y(k)] = x(n) \underset{20}{\circledast} y(n)$

Let also $r_L(n) = x(n) * y(n)$ <u>linear convolution</u>.

Note: $r(n): \begin{cases} \neq 0 & , 0 \le n \le 19 \\ 0 & , \text{otherwise} \end{cases}$

$r_L(n): \begin{cases} \neq 0 & , 0 \le n \le 26 \\ 0 & , \text{otherwise} \end{cases}$

Thus, $r(n) = r_L(n)$ for $n = 7, \dots, 19$

$$r(n) = \left[\sum_{l=-\infty}^{\infty} r_L(n+l20)\right] R_{20}(n)$$