

THE CONCEPT OF FREQUENCY

A) SINUSOIDS

- Continuous time ($-\infty < t < \infty$)
- $x(t) = A \cos[\Omega t + \theta] = A \cos[2\pi F t + \theta]$
- F : cycles/sec
- Periodic with period $T = \frac{1}{F}$, i.e.,
 $x(t) = x(t+T) = x(t+eT)$, T : fundamental period
- F takes any value: $-\infty < F < \infty$
- Distinct sinusoids are obtained from any distinct frequencies in $-\infty < F < \infty$
- The highest rate of oscillation is achieved when $\Omega \rightarrow \infty$, or $F \rightarrow \infty$

- Discrete time ($n=0, \pm 1, \pm 2, \dots$)

$$x(n) = A \cos[\omega n + \theta] = A \cos[2\pi f n + \theta]$$

$$f: \text{cycles/sample}$$

- Periodic with period N samples, i.e.

$$x(n) = x(n+N) = x(n+eN)$$

N : fundamental period

$$A \cos(\omega n + \theta) = A \cos(\omega(n+N) + \theta) \Rightarrow$$

$$\Rightarrow \omega n = \omega(n+N) \pm 2\pi k \Rightarrow \boxed{f = \pm \frac{k}{N}}$$

i.e., f must be rational

$$\text{— Let } \omega_1 = \omega_2 + 2\pi k \text{ : Then } \cos[\omega_1 n + \theta] = \cos[\omega_2 n + \theta]$$

Thus distinct discrete sinusoids can be obtained only in intervals of 2π , i.e. $\{-\pi \leq \omega \leq \pi\}$ or $0 \leq \omega \leq 2\pi$.

(principal range) i.e. $\left\{-\frac{1}{2} \leq f \leq \frac{1}{2}\right\}$ or $0 \leq f \leq 1$.

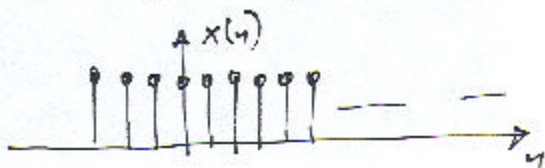
- The highest rate of oscillation for a discrete sinusoid is achieved when $\omega = \pi$ or $f = \frac{1}{2}$.

Ex. Let $x(n) = \cos \omega_0 n$

$$\omega_0 = 0, \pi/8, \pi/4, \pi/2$$

$$f = 0, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}$$

a) $\omega_0 = 0$, $f = 0$, period $N = \infty$



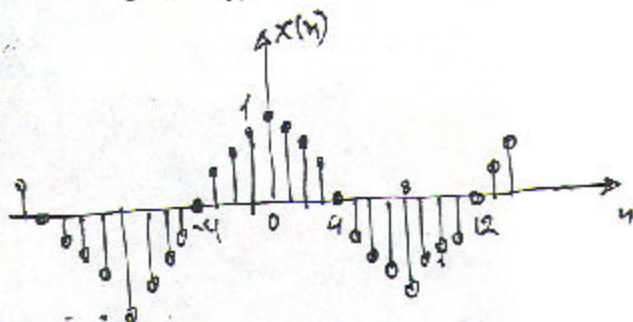
b) $\omega_0 = \pi/8$, $f = 1/16$, period $N = 16$.

zeros

$$\cos\left[\frac{\pi n}{8}\right] = 0 \Rightarrow \frac{\pi n}{8} = k\pi + \frac{\pi}{2} \Rightarrow n = 8k + 4$$

$$k = 0, 1, 2, 3, \dots$$

$$n = 4, 12, 20, \dots$$



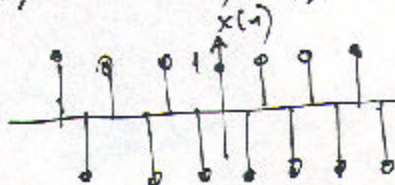
maxima

$$\cos\left[\frac{\pi n}{8}\right] = 1 \Rightarrow \frac{\pi n}{8} = k\pi \Rightarrow n = 8k$$

c)

Highest possible rate of oscillation

d) $\omega_0 = \pi$, $f = 1/2$, $N = 2$



zeros

$$\cos[\pi n] = 0$$

$$\pi n = k\pi + \pi/2 \Rightarrow n = k + 1/2$$

n : integer.

no solution

maxima

$$\pi n = k\pi \Rightarrow n = k$$

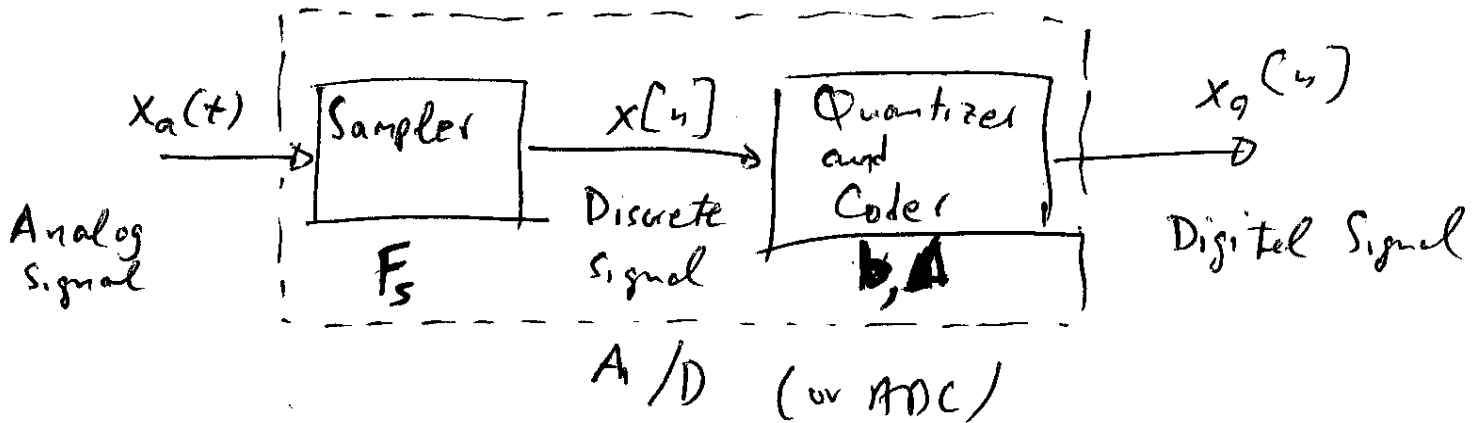
$$n = k$$

ECE 431, Lecture 2

(1)

Analog to Digital Conversion (ADC)

* Necessary in order to use digital processing methods on "real world signals"



IDEAL SAMPLING = [Uniform, i.e. take one sample every T_s seconds]

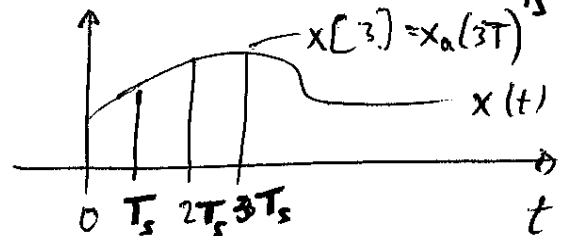
$$x[n] = x_a(nT), \quad -\infty < n < \infty$$

$n = \dots, -2, -1, 0, 1, 2, \dots$

[Sampling Period T_s
" frequency $F_s = \frac{1}{T_s}$]

$F_s = \frac{1}{T_s}$ samples/sec.

$x_a(t) \rightarrow x[n]$



* Let $x_a(t) = A \cos(\underbrace{2\pi f t}_{\Omega} + \theta)$

Then $x[n] = x_a(nT_s) = A \cos(2\pi f n T_s + \theta) = A \cos(\underbrace{(2\pi f T_s)}_{\omega} n + \theta)$

ω : Normalized Frequency (discrete domain frequency)

$\omega = \underbrace{2\pi f T_s}_{\text{true frequency}}$ $\frac{\text{rad}}{\text{sample}}$ or $f = \frac{F}{F_s} = \frac{\omega}{2\pi}$ cycles/sample

$= A \cos(\omega n + \theta)$

* $|F| < \infty$, $|f| \leq \frac{1}{2}$ (Can you figure out why?)

* f must be rational in order for $x[n]$ to be periodic
($f = \frac{k}{N}$) k, N , integers

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Also:

$$x_a(nT_s) = A \cos(\Omega nT_s + \theta) = A \cos\left(\left(\Omega + \frac{2\pi k}{T_s}\right)nT_s + \theta\right)$$

periodic in Ω with period $2\pi/T_s$

$$x[n] = A \cos(\omega n + \theta) = A \cos((\omega + 2\pi k)n + \theta) : \text{periodic in } \omega \text{ with period } 2\pi$$

Now let $x(t)$ be a bandlimited analog signal

Consider the following modeling of the sampling process

$x_a(t) \rightarrow \otimes \rightarrow x_s(t) = \sum_{l=-\infty}^{\infty} x_a(t) \delta(t - lT_s)$

$s(t) = \sum_{l=-\infty}^{\infty} \delta(t - lT_s)$ $\xrightarrow[\text{expansion}]{\text{Fourier Series}}$ $\sum_{l=-\infty}^{\infty} \frac{1}{T_s} e^{j l k \frac{2\pi}{T_s} t}$

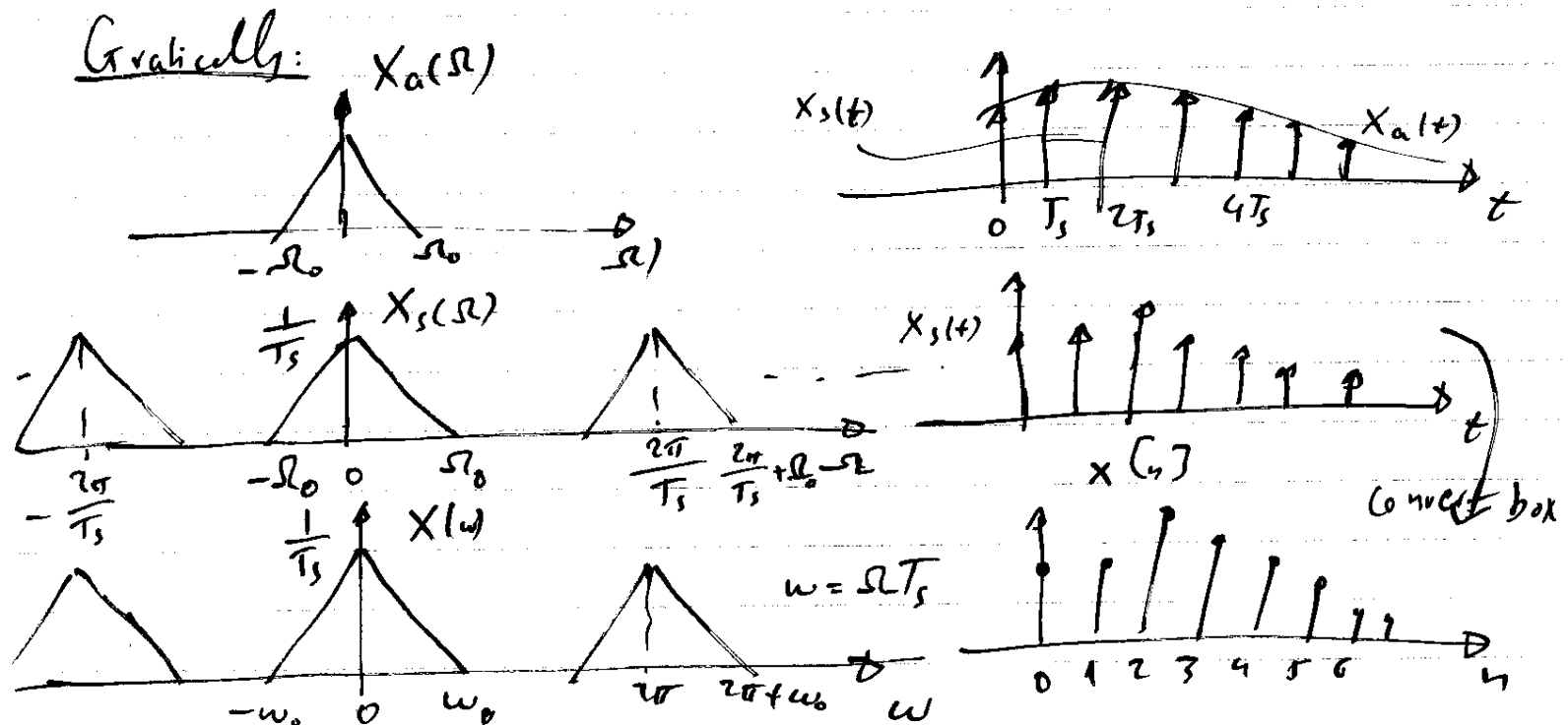
$x[n] = \sum_{l=-\infty}^{\infty} x_a(lT_s) \delta[n - l]$

$\xrightarrow{\text{Convert to discrete time sequence}}$

$\uparrow$ FS coefficient $\left(a_k = \frac{1}{T_s} \int_{T_s} s(t) e^{j k \frac{2\pi}{T_s} t} dt = \frac{1}{T_s} \right)$

Therefore note:

$$x_s(t) = \frac{1}{T_s} \sum_{l=-\infty}^{\infty} x_a(lT_s) e^{j k \frac{2\pi}{T_s} t} \xrightarrow[\text{Transforming}]{\text{Fourier}} X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a\left(\Omega \pm \frac{2\pi k}{T_s}\right)$$

Graphically:

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→ $X(\Omega)$ must be bandlimited ($X(\Omega) = 0$ $|\Omega| > \Omega_0$)

→ To avoid distortion from overlapping among replicas of the spectrum (named "aliasing") $\Omega_s \geq 2\Omega_0$

→ "Nyquist rate": $\Omega_s = 2\Omega_0$ (given Ω_0)

→ "Nyquist frequency": $\Omega_0 = \frac{\Omega_s}{2}$ (given Ω_s)

* Aliasing is non-reversible non-recoverable distortion

Assuming no aliasing occurs then $x_a(t)$ can be recovered (completely reconstructed) from $x[n]$ by means of ideal low pass (LP) filtering.