LINEAR SHIFT INVARIANT (LSI) DISCRETE TIME SYSTEMS – LCCDE – Z-TRANSFORMS

LSI: Are those systems described by Linear Constant Coefficient Difference Equations (LCCDE) or Systems described by an "impulse response" $h(n)$.

$$x(n) \longrightarrow \boxed{\;h(n)\;} \longrightarrow y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$

DTFT: $H(\omega)$ $\quad Y(\omega) = H(\omega) \cdot X(\omega)$.

Z-Tran. $H(z)$ $\quad Y(z) = H(z) \cdot X(z)$

Causal model (system): $h(n) = 0 \;,\; n<0$

Causal LCCDE:

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k) \Rightarrow$$

$$H(\omega) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

$$h(n) = \text{IDTFT}[H(\omega)]$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Z-Transform: Given $x(n)$, $n = 0, \pm 1, \pm 2, \ldots$ and $z = re^{j\omega}$, $r \geq 0$, $-\pi \leq \omega < \pi$

Forward: $\boxed{X(z) = \sum_n x(n)\, z^{-n} \;,\; \text{R.O.C. (Region of Convergence): all } z: |X(z)| < \infty.}$

PROPERTIES:
- Linearity
- Linear shift $x(n-k) \longrightarrow X(z)\,z^{-k}$
- Linear Convolution: $x_1(n) * x_2(n) \longrightarrow X_1(z) X_2(z)$
- Time reversal : $x(-n) \longrightarrow X(z^{-1})$
- DTFT: $X(\omega) = X(z)\big|_{z=e^{j\omega}}$ i.e., on the unit circle $(r=1)$
- Differentiation: $n\,x[n] \longrightarrow -z\dfrac{dX(z)}{dz}$

- $X(z)$ might exist even if $X(\omega)$ does not exist (i.e. $|\sum_n x(n) z^{-n}| < \infty$ but $|\sum_n x(n) e^{-j\omega n}| \to \infty$)
- stability: BIBO stable system: ROC of $H(z)$ includes unit circle ($H(\omega)$ exists)

- Poles of $H(z)$: values of z so that $H(z) \to \infty$
- Zeros of $H(z)$: " " " " " $H(z) = 0$

- The R.O.C. of a z-transform cannot contain any poles, i.e., the poles determine the borders of the R.O.C.

Ex. Let $x(n) = a^n u(n)$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}}$$ provided $|az^{-1}|<1 \Rightarrow \underline{|a| < |z| : \text{ROC.}}$

(Let $x(n) = -a^n u(-n-1)$, Then $X(z) = \frac{1}{1-az^{-1}}$ but with R.O.C. $|z|<|a|$)

In general: Right sided sequences ($x(n)=0$, $n<K$), R.O.C: $|R|<|z|<\infty$, for some R
Left sided sequences ($x(n)=0$, $n>K$), R.O.C: $0<|z|<|R|$, for some R
Causal sequences ($x(n)=0$, $n<K, K>0$), R.O.C: $|R|<|z|$, for some R
Anti-Causal sequences ($x(n)=0$, $n>K$, $K<0$), R.O.C: $|z|<|R|$, ″
Two sided sequence ($x(n)\neq 0$, all n), R.O.C: $|R_1|<|z|<|R_2|$, ″ R_1, R_2
Finite length sequence ($x(n)=0$, $n>K_1$, $n<K_2$), R.O.C: $\begin{cases} 0<|z|<\infty, & K_1>0, K_2<0 \\ 0\le|z|<\infty, & K_1\le 0 \\ 0<|z|\le\infty, & K_2>0 \end{cases}$

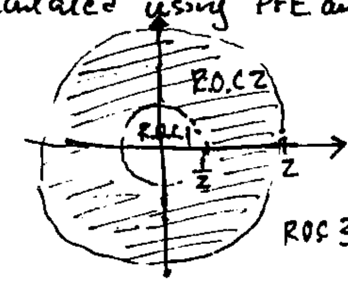
* An expression for $X(z) = \sum_n x(n) z^{-n}$ with different R.O.Cs correspond to different sequences $x(n)$.

Inverse z-transform

$$\boxed{x(n) = \frac{1}{2\pi j}\oint_C X(z)\, z^{n-1}\, dz, \quad C: \text{counterclockwise closed path within R.O.C. including origin}}$$

- For LCCDE systems the Inv. z-transform can be easily calculated using PFE and tables

Ex. Let $H(z) = \frac{z^2}{(z-\frac{1}{2})(z-2)}$ two poles $z=\frac{1}{2}, z=2$. two zeros at $z=0$



- Three possible R.O.Cs. Only R.O.C 2 corresponds to a stable system
- (is this system realizable or not?

$$H(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-2z^{-1}} = \frac{-\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{4/3}{1-2z^{-1}}$$

R.O.C 1: $h(n) = \frac{1}{3}(\frac{1}{2})^n u(-n-1) - \frac{4}{3} 2^n u(-n-1)$ .... anticausal system
R.O.C 2: $h(n) = -\frac{1}{3}(\frac{1}{2})^n u(n) - \frac{4}{3} 2^n u(-n-1)$ stable but two sided system
R.O.C 3: $h(n) = -\frac{1}{3}(\frac{1}{2})^n u(n) + \frac{4}{3} 2^n u(n)$ causal but unstable system