

LECTURE 7 ELE 431 S

The DTFT (discrete-time Fourier Transform, aperiodic discrete signals)

Let $x_a(t)$ aperiodic $\rightarrow X_s(\omega) = \sum_{n=-\infty}^{\infty} x_a(nT_s) \delta(\omega - n\omega_s)$ aperiodic

Then:

$$X_s(\omega) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_a(nT_s) \delta(\omega - n\omega_s) e^{-j\omega t} dt = \sum_{n=-\infty}^{\infty} x_a(nT_s) \int_{-\infty}^{\infty} \delta(t - nT_s) e^{-j\omega t} dt =$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT_s) e^{-j\omega nT_s} \quad \text{sampling Th.} \quad \frac{1}{T_s} \sum_{l=-\infty}^{\infty} X_a(\omega - l \frac{2\pi}{T_s})$$

or $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

Since $X_s(\omega)$: periodic $(\frac{2\pi}{T_s})$, ω continuous from $-\pi$ to π $\xrightarrow{\text{Fourier Series}}$

$$x(n) = X_a(nT_s) = \frac{1}{2\pi/T} \int_{-\pi/T}^{\pi/T} X_s(\omega) e^{j\omega nT_s} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

So:

DTFT pair:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$x(n)$: aperiodic discrete

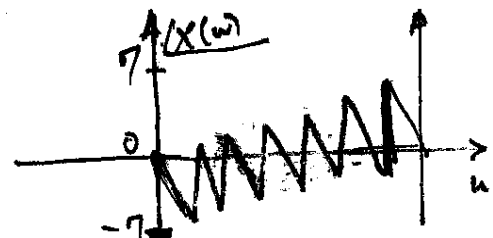
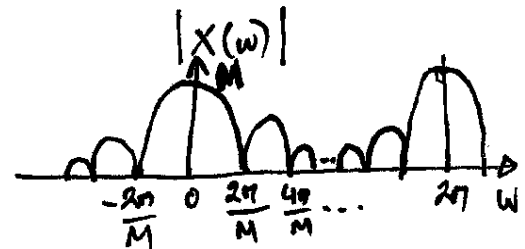
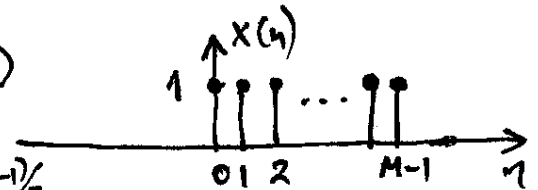
$X(\omega)$: continuous periodic (2π)

Note: $x(n)$ can be written as $x(n) = \sum_k x(k) f(n-k)$ ($T=1$, assumption)

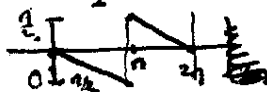
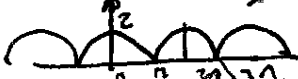
Ex Let $x(n) = u(n) - u(n-M) = \sum_{k=0}^{M-1} \delta(n-k)$

then, $X(\omega) = \sum_{n=0}^{M-1} e^{-j\omega n} = \frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}} = \frac{\sin(\omega M/2)}{\sin(\omega/2)} e^{-j\omega(M-1)/2}$

$$|X(\omega)| = \left| \frac{\sin(\omega M/2)}{\sin(\omega/2)} \right|, \quad \angle X(\omega) = -\omega \frac{(M-1)}{2} + \text{or } 0$$



$M=2$ $X(\omega) = \frac{\sin \omega}{\sin \frac{\omega}{2}} = 2 \cos \frac{\omega}{2} e^{-j\frac{\omega}{2}}$ $\angle X(\omega) = -\frac{\omega}{2} + \pi$



Section 8.1

$$\tilde{X}_a(t) = \tilde{X}_a(t + KT_0)$$

$$\tilde{X}_a(nT_s) = \tilde{X}_a(nT_s + KT_0) = \tilde{X}_a[(n+N)T_s] \Rightarrow$$

THE DTFS (or DFS) Discrete-time Fourier Series.

Let $\tilde{X}_a(t)$ be periodic (T_0): Then if $KT_0 = NT_s$, $\tilde{X}_s(t)$ also periodic (per. NT_s)
Then from Fourier Series

$$\tilde{X}_s(\Omega = \frac{2\pi k}{NT_s}) = \frac{1}{NT_s} \sum_{n=-\infty}^{\infty} \tilde{X}_a(nT_s) \delta(t - nT_s) e^{-j\frac{2\pi k t}{NT_s}} dt = \frac{1}{NT_s} \sum_{n=-\infty}^{\infty} \tilde{X}_a(nT_s) \int_0^{NT_s} \delta(t - nT_s) e^{-j\frac{2\pi k t}{NT_s}} dt =$$

$$= \frac{1}{NT_s} \sum_{n=0}^{N-1} \tilde{X}_a(nT_s) e^{-j\frac{2\pi k n}{N}} \quad \text{periodic with period } N \text{ in } k$$

or $\tilde{X}_s(\Omega) = \sum_{k=-\infty}^{\infty} \tilde{X}_s(\frac{2\pi k}{NT_s}) \delta(\Omega - \frac{2\pi k}{NT_s}) \xrightarrow{\text{Sampling Theorem}} \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a(\Omega - \frac{2\pi k}{T_s})$

N denotes periodicity

Since $\tilde{X}_s(\Omega)$ periodic ($\frac{2\pi}{T_s}$) Fourier Series $\rightarrow$

$$\tilde{X}_s(nT_s) = \tilde{X}_a(nT_s) = \frac{1}{2\pi/T_s} \int_0^{2\pi/T_s} \sum_{k=-\infty}^{\infty} \tilde{X}_s(\frac{2\pi k}{NT_s}) \delta(\Omega - \frac{2\pi k}{NT_s}) e^{j\Omega nT_s} d\Omega = \sum_{k=0}^{N-1} \tilde{X}_s(\frac{2\pi k}{NT_s}) e^{j\frac{2\pi k n}{N}}$$

[Also by placing $\frac{1}{N} \tilde{X}_s(nT_s) = x(n) \Rightarrow x(n) = \dots$

So in normalized freq. ($\omega = \Omega T$)
(or placing $T_s = 1$)

DTFS pair

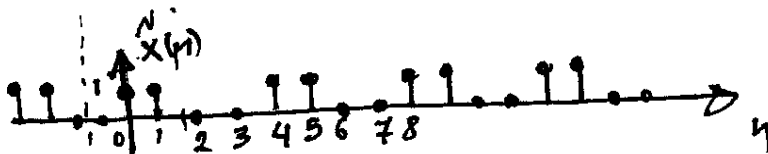
$\left. \begin{matrix} \tilde{X}_s(nT_s) \\ \tilde{X}_s(k) \end{matrix} \right\}$ Discrete and Periodic with period N .

$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{X}(n) e^{-j\frac{2\pi k n}{N}}$$

$$\tilde{X}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) e^{j\frac{2\pi k n}{N}}$$

$k, n = 0, \dots, N-1$ or over $\langle N \rangle$

Ex.



Let

$$Q(\omega) = \sum_{n=0}^{\infty} e^{-j\omega n} = \frac{\sin' \omega}{\sin \omega/2} e^{-j\omega/2} = 2 \cos \frac{\omega}{2} e^{-j\omega/2}$$

$$\tilde{X}(k) = Q(\omega = \frac{2\pi k}{N} = \frac{\pi k}{2})$$

